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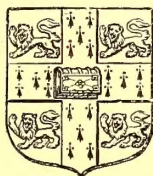
A JOURNAL FOR THE STATISTICAL STUDY OF
BIOLOGICAL PROBLEMS

FOUNDED BY
W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY
KARL PEARSON

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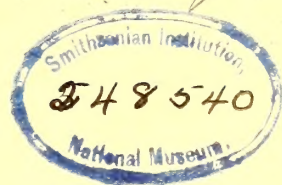
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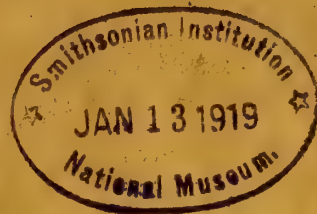
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ON THE STANDARD DEVIATIONS OF ADJUSTED AND INTERPOLATED VALUES OF AN OBSERVED *POLYNOMIAL FUNCTION* AND ITS CONSTANTS AND THE GUIDANCE THEY GIVE TOWARDS A PROPER CHOICE OF THE DISTRIBUTION OF OBSERVATIONS.

By KIRSTINE SMITH, Copenhagen.

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INTRODUCTION

In all sorts of experiments which are not simple repetitions but have at least one varying essential circumstance or indefinite variate the experimentalist is confronted with a choice in regard to the values of that variate. If the experiments be quite simple the question may be without great importance; but when their requirements as to time or expenditure come into account the problem arises, how the observations should be chosen in order that a limited number of them may give the maximum amount of knowledge. It clearly depends upon the relationship between the observed quantity, which we shall name the primary variate, and its essential circumstances, the secondary variates, and upon the variation of the errors of the observations.



When we deal with, for example, a linear function which it is possible to observe with the same accuracy for all values of the indefinite variate we should not hesitate to put the observations in two equally big groups as far apart from each other as feasible. But if the standard deviation of the observations be a function of the indefinite variate and increases with the distance from the middle of the range, where is then the point in which the advantage of removing the two groups of observations from each other just counterbalances the disadvantages of increasing the error of observations? The problem becomes very complicated for functions of higher degrees.

We shall in this memoir try to contribute to the solution in the case of polynomial functions by examining the standard deviations of the adjusted and more especially the interpolated values of such functions for different distributions of observations. Those values inside the working range of observations may be considered the sum of knowledge acquired by the experiments. The adjusted values outside the working range may probably in exceptional cases be of interest, but as only by some other type of experiment we can make sure that the form of function holds outside the range they are in ordinary cases without great value. We shall therefore aim at finding the distribution of observations which within the selected range gives the most satisfactory standard deviations of the adjusted values of the function.

To consider the standard deviations satisfactory we must of course demand that they shall be as small as possible, and since a greater accuracy in one part may be expected to be accompanied by a smaller accuracy in another part we want them in addition to be as near constant as possible. In other words the curve of standard deviation with the lowest possible maximum value within the working range of observations is what we shall attempt to find. It appears that the distribution of observations which fulfils this demand consists of specially placed groups in number just sufficient to determine the constants of the function. We shall accordingly pay attention also to the desirability usually present of ascertaining the form of function by means of the observations. As might be expected we find that the standard deviations obtained from a uniform continuous distribution of observations increase towards the ends of the range. By choosing a uniform continuous distribution with additional clusters at the ends of the range we shall try to find a compromise between the two desiderata of a low maximum of standard deviation and of a uniform distribution.

The indefinite variate is supposed to have a vanishing error of observation compared with that of the principal variate. This error may be constant or varying with the indefinite variate, but in either case it is supposed to follow the typical law so closely that the method of least squares may satisfactorily be applied to the observations. After having found first the most advantageous distributions for observations of functions up to the sixth degree with constant standard deviations we examine the case for observations of functions of the first and of the second degree which have standard deviations of the form $\sigma(1 + ax)$ and $\sigma(1 + ax^2)$. If it is profitable to use the whole of the working range the latter distributions

are practically found from the former by multiplying their frequencies by the squared standard deviations of the observations at the corresponding place. But in cases where extrapolation is of advantage, and the whole range therefore not to be used, the law of the frequencies has to be examined anew.

In Section VIII we find for the same two cases of varying error of observation the distributions which make each single constant of a function of the first and of the second degree a minimum.

I. *Adjustment of a polynomial function of one variable; general distribution of observations.*

(1) Let $y_1, y_2, \dots, y_p, \dots, y_N$ be N observations of a function of n th degree taken at the points $x_1, x_2, \dots, x_p, \dots, x_N$,

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \dots \dots \dots (1).$$

Let us assume that from earlier experience we know the standard deviation of an observation of y to be $\sigma\sqrt{f(x)}$. The method of least squares will then give us the following system of normal equations in which the sums are to be extended over all the observations:

$$\left. \begin{aligned} S \left\{ \frac{y_p}{f(x_p)} \right\} &= S \left\{ \frac{1}{f(x_p)} \right\} a_0 + S \left\{ \frac{x_p}{f(x_p)} \right\} a_1 + S \left\{ \frac{x_p^2}{f(x_p)} \right\} a_2 + \dots + S \left\{ \frac{x_p^n}{f(x_p)} \right\} a_n \\ S \left\{ \frac{y_p x_p}{f(x_p)} \right\} &= S \left\{ \frac{x_p}{f(x_p)} \right\} a_0 + S \left\{ \frac{x_p^2}{f(x_p)} \right\} a_1 + S \left\{ \frac{x_p^3}{f(x_p)} \right\} a_2 + \dots + S \left\{ \frac{x_p^{n+1}}{f(x_p)} \right\} a_n \\ S \left\{ \frac{y_p x_p^2}{f(x_p)} \right\} &= S \left\{ \frac{x_p^2}{f(x_p)} \right\} a_0 + S \left\{ \frac{x_p^3}{f(x_p)} \right\} a_1 + S \left\{ \frac{x_p^4}{f(x_p)} \right\} a_2 + \dots + S \left\{ \frac{x_p^{n+2}}{f(x_p)} \right\} a_n \\ &\vdots \\ S \left\{ \frac{y_p x_p^n}{f(x_p)} \right\} &= S \left\{ \frac{x_p^n}{f(x_p)} \right\} a_0 + S \left\{ \frac{x_p^{n+1}}{f(x_p)} \right\} a_1 + S \left\{ \frac{x_p^{n+2}}{f(x_p)} \right\} a_2 + \dots + S \left\{ \frac{x_p^{2n}}{f(x_p)} \right\} a_n \end{aligned} \right\} \dots \dots \dots (2).$$

If $f(x)$ is 1 the sums are the moment coefficients of the places of observations multiplied by N , and in the general case we shall for brevity put

$$S \left\{ \frac{x_p^r}{f(x_p)} \right\} = N \cdot m_r.$$

By elimination of the a 's between (1) and (2) we find

$$\left| \begin{array}{cccccc} N \cdot y & 1 & x & x^2 & \dots & x^n \\ S \left\{ \frac{y_p}{f(x_p)} \right\} & m_0 & m_1 & m_2 & \dots & m_n \\ S \left\{ \frac{y_p x_p}{f(x_p)} \right\} & m_1 & m_2 & m_3 & \dots & m_{n+1} \\ S \left\{ \frac{y_p x_p^2}{f(x_p)} \right\} & m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ S \left\{ \frac{y_p x_p^n}{f(x_p)} \right\} & m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{array} \right| = 0 \dots \dots \dots (3),$$

which determines the adjusted y corresponding to the variable x .

(2) To find the standard deviation σ_{y_r} of an adjusted y_r it will be easiest to start from the equations (2). If the first be multiplied by α_0 , the second by α_1 and so on before summing, and if we choose $\alpha_0, \alpha_1, \dots, \alpha_n$ so that

$$\left. \begin{array}{ccccccc} \alpha_0 m_0 + \alpha_1 m_1 & + \alpha_2 m_2 & + \dots & + \alpha_n m_n & = & 1 \\ \alpha_0 m_1 + \alpha_1 m_2 & + \alpha_2 m_3 & + \dots & + \alpha_n m_{n+1} & = & x_r \\ \alpha_0 m_2 + \alpha_1 m_3 & + \alpha_2 m_4 & + \dots & + \alpha_n m_{n+2} & = & x_r^2 \\ \vdots & \vdots & & \vdots & \vdots & \\ \alpha_0 m_n + \alpha_1 m_{n+1} & + \alpha_2 m_{n+2} & + \dots & + \alpha_n m_{2n} & = & x_r^n \end{array} \right\} \dots\dots\dots(4),$$

we find that $y_r = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [a_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \dots + \alpha_n x_p^n] \right\},$

and therefore $\sigma_{y_r}^2 = \frac{\sigma^2}{N^2} S \left\{ \frac{1}{f(x_p)} [a_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \dots + \alpha_n x_p^n]^2 \right\}.$

By multiplying out the square this may be written

$$\begin{aligned} \sigma_{y_r}^2 = \frac{\sigma^2}{N} \{ & \alpha_0 [a_0 m_0 + \alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_n m_n] \\ & + \alpha_1 [a_0 m_1 + \alpha_1 m_2 + \alpha_2 m_3 + \dots + \alpha_n m_{n+1}] \\ & + \alpha_2 [a_0 m_2 + \alpha_1 m_3 + \alpha_2 m_4 + \dots + \alpha_n m_{n+2}] \\ & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ & + \alpha_n [a_0 m_n + \alpha_1 m_{n+1} + \alpha_2 m_{n+2} + \dots + \alpha_n m_{2n}] \}, \end{aligned}$$

or applying (4) $\sigma_{y_r}^2 = \frac{\sigma^2}{N} (a_0 + \alpha_1 x_r + \alpha_2 x_r^2 + \dots + \alpha_n x_r^n) \dots\dots\dots(5).$

Hence $\sigma_{y_r}^2$ is found by elimination of the α 's between (4) and (5), which results in

$$\left| \begin{array}{cccccc} \sigma_{y_r}^2 \cdot \frac{N}{\sigma^2} & 1 & x_r & x_r^2 & \dots & x_r^n \\ 1 & m_0 & m_1 & m_2 & \dots & m_n \\ x_r & m_1 & m_2 & m_3 & \dots & m_{n+1} \\ x_r^2 & m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_r^n & m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{array} \right| = 0 \dots\dots\dots(6).$$

This determinant is of fundamental importance for all the following work and it will be useful at once to examine it more closely.

(3) First however it may be pointed out that the standard deviation of any other linear function $\phi = b_0 a_0 + b_1 a_1 + b_2 a_2 + \dots + b_n a_n$ of the constants of the function y may be determined in quite the same way by

$$\left| \begin{array}{cccccc} \sigma_\phi^2 \cdot \frac{N}{\sigma^2} & b_0 & b_1 & b_2 & \dots & b_n \\ b_0 & m_0 & m_1 & m_2 & \dots & m_n \\ b_1 & m_1 & m_2 & m_3 & \dots & m_{n+1} \\ b_2 & m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_n & m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{array} \right| = 0 \dots\dots\dots(7).$$

In particular $\sigma_{a_p}^2$ is found from

$$\begin{vmatrix} \sigma_{a_p}^2 \cdot \frac{N}{\sigma^2} & 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & m_0 & m_1 & m_2 & \dots & m_p & \dots & m_n \\ 0 & m_1 & m_2 & m_3 & \dots & m_{p+1} & \dots & m_{n+1} \\ 0 & m_2 & m_3 & m_4 & \dots & m_{p+2} & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p} & \dots & m_{n+p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & m_n & m_{n+1} & m_{n+2} & \dots & m_{p+n} & \dots & m_{2n} \end{vmatrix} = 0 \dots\dots\dots(8).$$

(4) Let us call a determinant, identical with that of (6) except that it has 0 instead of the element $\sigma_{y_r}^2 \cdot \frac{N}{\sigma^2}$, Δ , let $\Delta_{r,s}$ be its minor not containing the r th row and s th column, again let $\Delta_{r,s,p,q}$ be the minor of this not containing the p th row and the q th column of Δ . We then find from (8)

$$\sigma_{a_p}^2 = \frac{\sigma^2}{N} \frac{\Delta_{p+2,p+2,1,1}}{\Delta_{1,1}} \dots\dots\dots(9).$$

With this notation we obtain from (6)

$$\sigma_{y_r}^2 = \frac{\sigma^2}{N} \left(- \frac{\Delta}{\Delta_{1,1}} \right) \dots\dots\dots(10).$$

In the following we shall drop the index r and indicate by ${}_n\sigma_y$ the standard deviation of a y adjusted by means of a function of the n th degree.

If we were dealing with a function of $(n-1)$ st degree and retained the observations distributed as before we should find

$${}_{n-1}\sigma_y^2 = \frac{\sigma^2}{N} \left(- \frac{\Delta_{n+2,n+2}}{\Delta_{n+2,n+2,1,1}} \right),$$

and therefore

$${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{\Delta_{1,1} \cdot \Delta_{n+2,n+2} - \Delta \cdot \Delta_{n+2,n+2,1,1}}{\Delta_{1,1} \cdot \Delta_{n+2,n+2,1,1}},$$

but Δ is orthosymmetrical and therefore the numerator of this fraction equals $\Delta_{n+2,1}^2$, and

$${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{\Delta_{n+2,1}^2}{\Delta_{1,1} \cdot \Delta_{n+2,n+2,1,1}}.$$

It was shown before that

$$\sigma_{a_n}^2 = \frac{\sigma^2}{N} \cdot \frac{\Delta_{n+2,n+2,1,1}}{\Delta_{1,1}},$$

hence $\Delta_{1,1}$ and $\Delta_{n+2,n+2,1,1}$ have the same sign, and ${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2$ is therefore a square of a function of x . In the same way we can express ${}_{n-1}\sigma_y^2 - {}_{n-2}\sigma_y^2$ and thus further down all the differences till ${}_0\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{1}{m_0}$ by which means ${}_n\sigma_y^2$ is developed in a sum of squares and takes the shape

$$\begin{aligned}
 {}_n\sigma_y^2 = \frac{\sigma^2}{N} & \left\{ \frac{1}{m_0} + \frac{\begin{vmatrix} 1 & m_0 \\ x & m_1 \end{vmatrix}^2}{m_0 \cdot \begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix}} + \frac{\begin{vmatrix} 1 & m_0 & m_1 \\ x & m_1 & m_2 \\ x^2 & m_2 & m_3 \end{vmatrix}^2}{\begin{vmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix}} + \dots \right. \\
 & \left. + \frac{\begin{vmatrix} 1 & m_0 & m_1 & \dots & m_{n-1} \\ x & m_1 & m_2 & \dots & m_n \\ x^2 & m_2 & m_3 & \dots & m_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^n & m_n & m_{n+1} & \dots & m_{2n-1} \end{vmatrix}^2}{\begin{vmatrix} m_0 & m_1 & \dots & m_{n-1} \\ m_1 & m_2 & \dots & m_n \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \dots & m_{2n-2} \end{vmatrix} \cdot \begin{vmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_n & m_{n+1} & \dots & m_{2n} \end{vmatrix}} \right\} \dots\dots\dots(11).
 \end{aligned}$$

It will be seen that the squared standard deviation of an adjusted y is a function of the $2n$ th degree of x . The coefficient of x^{2n} is the square of $\frac{\Delta_{n+2, n+2, 1, 1}}{\Delta_{1, 1}}$ which, as was just seen, is the factor with which $\frac{\sigma^2}{N}$ should be multiplied in order to give $\sigma_{\sigma_n}^2$, it is therefore positive and can never vanish.

(5) If all the m 's with odd indices are zero it is seen from (6) that σ_y^2 is a function of x^2 . This is, at least in theory, a natural thing to aim at, since our general purpose is to find a curve for σ_y^2 giving as nearly as possible a constant value for σ_y^2 throughout the range.

Rearranging the order of rows and columns in (6) we get, when all $m_{2q+1} = 0$ and $n = 2p$,

$$\begin{vmatrix}
 {}_{2p}\sigma_y^2 \cdot \frac{N}{\sigma^2} & 1 & x^2 & x^4 & \dots & x^{2p} & x & x^3 & x^5 & \dots & x^{2p-1} \\
 1 & m_0 & m_2 & m_4 & \dots & m_{2p} & 0 & 0 & 0 & \dots & 0 \\
 x^2 & m_2 & m_4 & m_6 & \dots & m_{2p+2} & 0 & 0 & 0 & \dots & 0 \\
 x^4 & m_4 & m_6 & m_8 & \dots & m_{2p+4} & 0 & 0 & 0 & \dots & 0 \\
 \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 x^{2p} & m_{2p} & m_{2p+2} & m_{2p+4} & \dots & m_{4p} & 0 & 0 & 0 & \dots & 0 \\
 x & 0 & 0 & 0 & \dots & 0 & m_2 & m_4 & m_6 & \dots & m_{2p} \\
 x^3 & 0 & 0 & 0 & \dots & 0 & m_4 & m_6 & m_8 & \dots & m_{2p+2} \\
 x^5 & 0 & 0 & 0 & \dots & 0 & m_6 & m_8 & m_{10} & \dots & m_{2p+4} \\
 \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 x^{2p-1} & 0 & 0 & 0 & \dots & 0 & m_{2p} & m_{2p+2} & m_{2p+4} & \dots & m_{4p-2}
 \end{vmatrix} = 0 \dots\dots\dots(12),$$

from which we find

$$\begin{aligned}
 {}_{2p}\sigma_y^2 = & -\frac{\sigma^2}{N} \left\{ \begin{array}{c} \begin{array}{ccccc} 0 & 1 & x^2 & x^4 & \dots\dots x^{2p} \\ 1 & m_0 & m_2 & m_4 & \dots\dots m_{2p} \\ x^2 & m_2 & m_4 & m_6 & \dots\dots m_{2p+2} \\ x^4 & m_4 & m_6 & m_8 & \dots\dots m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \\ \hline \begin{array}{ccccc} x^{2p} & m_{2p} & m_{2p+2} & m_{2p+4} & \dots\dots m_{4p} \\ m_0 & m_2 & m_4 & \dots\dots m_{2p} \\ m_2 & m_4 & m_6 & \dots\dots m_{2p+2} \\ m_4 & m_6 & m_8 & \dots\dots m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots \\ m_{2p} & m_{2p+2} & m_{2p+4} & \dots\dots m_{4p} \end{array} \end{array} \right. \\
 & + x^2 \left\{ \begin{array}{c} \begin{array}{ccccc} 0 & 1 & x^2 & x^4 & \dots\dots x^{2p-2} \\ 1 & m_2 & m_4 & m_6 & \dots\dots m_{2p} \\ x^2 & m_4 & m_6 & m_8 & \dots\dots m_{2p+2} \\ x^4 & m_6 & m_8 & m_{10} & \dots\dots m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \\ \hline \begin{array}{ccccc} x^{2p-2} & m_{2p} & m_{2p+2} & m_{2p+4} & \dots\dots m_{4p-2} \\ m_2 & m_4 & m_6 & \dots\dots m_{2p} \\ m_4 & m_6 & m_8 & \dots\dots m_{2p+2} \\ m_6 & m_8 & m_{10} & \dots\dots m_{2p+4} \\ \vdots & \vdots & \vdots & \vdots \\ m_{2p} & m_{2p+2} & m_{2p+4} & \dots\dots m_{4p-2} \end{array} \end{array} \right\} \dots\dots (13).
 \end{aligned}$$

For a function of the degree $2p-1$ we get the same determinant as in (12) except that it does not contain the row and column in which x^{2p} is found. Hence we find

$${}_{2p-1}\sigma_y^2 = -\frac{\sigma^2}{N} \left\{ \begin{array}{c} \begin{array}{ccccc} 0 & 1 & x^2 & x^4 & \dots\dots x^{2p-2} \\ 1 & m_0 & m_2 & m_4 & \dots\dots m_{2p-2} \\ x^2 & m_2 & m_4 & m_6 & \dots\dots m_{2p} \\ x^4 & m_4 & m_6 & m_8 & \dots\dots m_{2p+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \\ \hline \begin{array}{ccccc} x^{2p-2} & m_{2p-2} & m_{2p} & m_{2p+2} & \dots\dots m_{4p-4} \\ m_0 & m_2 & m_4 & \dots\dots m_{2p-2} \\ m_2 & m_4 & m_6 & \dots\dots m_{2p} \\ m_4 & m_6 & m_8 & \dots\dots m_{2p+2} \\ \vdots & \vdots & \vdots & \vdots \\ m_{2p-2} & m_{2p} & m_{2p+2} & \dots\dots m_{4p-4} \end{array} \end{array} \right\}$$

$$\begin{array}{c}
 \begin{array}{cccccc}
 0 & 1 & x^2 & x^4 & \dots & x^{2p-2} \\
 1 & m_2 & m_4 & m_6 & \dots & m_{2p} \\
 x^2 & m_4 & m_6 & m_8 & \dots & m_{2p+2} \\
 x^4 & m_6 & m_8 & m_{10} & \dots & m_{2p+4} \\
 \vdots & \vdots & \vdots & \vdots & & \vdots \\
 x^{2p-2} & m_{2p} & m_{2p+2} & m_{2p+4} & \dots & m_{4p-2}
 \end{array} \\
 + x^2 \frac{\begin{array}{cccccc}
 m_2 & m_4 & m_6 & \dots & m_{2p} \\
 m_4 & m_6 & m_8 & \dots & m_{2p+2} \\
 m_6 & m_8 & m_{10} & \dots & m_{2p+4} \\
 \vdots & \vdots & \vdots & & \vdots \\
 m_{2p} & m_{2p+2} & m_{2p+4} & \dots & m_{4p-2}
 \end{array}}{\begin{array}{cccccc}
 m_2 & m_4 & m_6 & \dots & m_{2p} \\
 m_4 & m_6 & m_8 & \dots & m_{2p+2} \\
 m_6 & m_8 & m_{10} & \dots & m_{2p+4} \\
 \vdots & \vdots & \vdots & & \vdots \\
 m_{2p} & m_{2p+2} & m_{2p+4} & \dots & m_{4p-2}
 \end{array}} \dots \dots \dots (14).
 \end{array}$$

(6) The last two determinant ratios of (13) and (14) are identical, and when the numerator of the first fraction of (13) is indicated by δ we therefore find

$${}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N} \left(\frac{\delta_{p+2, p+2}}{\delta_{1, 1, p+2, p+2}} - \frac{\delta}{\delta_{1, 1}} \right),$$

or as δ is orthosymmetrical and therefore

$$\begin{aligned}
 \delta_{1, 1} \cdot \delta_{p+2, p+2} - \delta \cdot \delta_{1, 1, p+2, p+2} &= \delta_{p+2, 1}^2 \\
 {}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{\delta_{p+2, 1}^2}{\delta_{1, 1} \cdot \delta_{p+2, p+2, 1, 1}}.
 \end{aligned}$$

Comparing ${}_{2p-2}\sigma_y^2$ and ${}_{2p-1}\sigma_y^2$ we see that they have the first determinant ratio in common and that when γ stands for the numerator of the other fraction of ${}_{2p-1}\sigma_y^2$ we have

$${}_{2p-1}\sigma_y^2 - {}_{2p-2}\sigma_y^2 = \frac{\sigma^2}{N} x^2 \left(\frac{\gamma_{p+1, p+1}}{\gamma_{p+1, p+1, 1, 1}} - \frac{\gamma}{\gamma_{1, 1}} \right),$$

or again, since γ is orthosymmetrical,

$${}_{2p-1}\sigma_y^2 - {}_{2p-2}\sigma_y^2 = \frac{\sigma^2}{N} x^2 \frac{\gamma_{p+1, 1}^2}{\gamma_{1, 1} \cdot \gamma_{p+1, p+1, 1, 1}}.$$

The general formula (11) hence for any $m_{2q+1} = 0$ takes the shape

$$\begin{aligned}
 {}_{2p}\sigma_y^2 &= \frac{\sigma^2}{N} \left\{ \frac{1}{m_0} + x^2 \frac{1}{m_2} + \frac{\begin{vmatrix} 1 & m_0 \\ x^2 & m_2 \end{vmatrix}^2}{m_0 \cdot \begin{vmatrix} m_0 & m_2 \\ m_2 & m_4 \end{vmatrix}} + x^2 \frac{\begin{vmatrix} 1 & m_2 \\ x^2 & m_4 \end{vmatrix}^2}{m_2 \cdot \begin{vmatrix} m_2 & m_4 \\ m_4 & m_6 \end{vmatrix}} + \dots \right. \\
 &\quad + x^2 \frac{\begin{vmatrix} 1 & m_2 & m_4 & \dots & m_{2p-2} \\ x^2 & m_4 & m_6 & \dots & m_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & m_{2p} & m_{2p+2} & \dots & m_{4p-4} \end{vmatrix}^2}{\begin{vmatrix} m_2 & m_4 & \dots & m_{2p-2} \\ m_4 & m_6 & \dots & m_{2p} \\ \vdots & \vdots & & \vdots \\ m_{2p-2} & m_{2p} & \dots & m_{4p-6} \end{vmatrix} \cdot \begin{vmatrix} m_2 & m_4 & \dots & m_{2p} \\ m_4 & m_6 & \dots & m_{2p+2} \\ \vdots & \vdots & & \vdots \\ m_{2p} & m_{2p+2} & \dots & m_{4p-2} \end{vmatrix}} \Big\}.
 \end{aligned}$$

$$+ \left\{ \begin{array}{c|c|c} \begin{array}{cccc} 1 & m_0 & m_2 & \dots m_{2p-2} \\ x^2 & m_2 & m_4 & \dots m_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x^{2p} & m_{2p} & m_{2p+2} & \dots m_{4p-2} \end{array} & \begin{array}{c} 2 \\ \end{array} & \\ \hline \begin{array}{cccc} m_0 & m_2 & \dots m_{2p-2} \\ m_2 & m_4 & \dots m_{2p} \\ \vdots & \vdots & \vdots \\ m_{2p-2} & m_{2p} & \dots m_{4p-4} \end{array} & \begin{array}{cccc} m_0 & m_2 & \dots m_{2p} \\ m_2 & m_4 & \dots m_{2p+2} \\ \vdots & \vdots & \vdots \\ m_{2p} & m_{2p+2} & \dots m_{4p} \end{array} & \end{array} \right\} \dots\dots\dots (15).$$

(7) Before leaving the general case and treating special distributions of observations three auxiliary propositions shall be proved. *We shall first prove that the curve of ${}_n\sigma_y^2$ can never be entirely below $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$.* With that purpose ${}_n\sigma_y^2$ will be summed over all the places of observation with the weight $\frac{1}{f(x)}$, i.e. for a continuous distribution of observations, the expression $\frac{\psi(x)}{f(x)} {}_n\sigma_y^2 dx$, where $\psi(x)$ is the number of observations, will be integrated over the range of observations.

Looking first at the numerator of the last term of (11) we find that it can be expanded into

$$(-1)^{n+1} \left\{ \begin{array}{c|c} \begin{array}{cccc} 1 & m_0 & m_1 & \dots m_{n-1} \\ x & m_1 & m_2 & \dots m_n \\ x^2 & m_2 & m_3 & \dots m_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ x^n & m_n & m_{n+1} & \dots m_{2n-1} \end{array} & \times \Delta_{1, n+2, 2, 1} + \\ \hline \begin{array}{c|c} \begin{array}{cccc} x^n & m_0 & m_1 & \dots m_{n-1} \\ x^{n+1} & m_1 & m_2 & \dots m_n \\ x^{n+2} & m_2 & m_3 & \dots m_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ x^{2n} & m_n & m_{n+1} & \dots m_{2n-1} \end{array} & \times \Delta_{1, n+2, 3, 1} + \dots + \end{array} \right\} \\ \times \Delta_{1, n+2, n+2, 1} \end{array} \left. \right\}.$$

Now $\int \frac{\psi(x) x^r}{f(x)} dx$ integrated over all the observations is what we have called $N \cdot m_r$. When integrating the determinants we therefore find that the first n of them will vanish, two of their columns consisting of proportional elements, whereas the integral of the last determinant is

$$N \begin{vmatrix} m_n & m_0 & m_1 & \dots m_{n-1} \\ m_{n+1} & m_1 & m_2 & \dots m_n \\ m_{n+2} & m_2 & m_3 & \dots m_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ m_{2n} & m_n & m_{n+1} & \dots m_{2n-1} \end{vmatrix} = (-1)^n N \Delta_{1, 1}.$$

As $\Delta_{1,n+2,n+2,1} = -\Delta_{n+2,n+2,1,1}$, the integral of the last term of (11) equals N . The integration of the other terms, including the first, gives the same result so that

$$\int {}_n\sigma_y^2 \frac{\psi(x)}{f(x)} dx = \sigma^2 (n+1),$$

and as

$$\int \frac{\psi(x)}{f(x)} dx = Nm_0,$$

the mean value of ${}_n\sigma_y^2$ calculated in this special way is

$$\frac{\sigma^2}{N} \cdot \frac{(n+1)}{m_0}.$$

It is therefore clear either that ${}_n\sigma_y^2$ must at all the places of observation be equal to $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$ or ${}_n\sigma_y^2$ must at some of these places be greater. The first case cannot be realised by a distribution of which any part is continuous, as ${}_n\sigma_y^2$ is proved to be of the $2n$ th degree in x . If therefore we could find a distribution consisting of groups of observations for which at all the places of observation ${}_n\sigma_y^2$ was equal to $\frac{\sigma^2}{N} \cdot \frac{n+1}{m_0}$, and if further we could choose the places of observation so that ${}_n\sigma_y^2$ at all other places within the range of observations was smaller than that value, we should know that no other distribution of observations with that value for m_0 could provide a curve of standard deviation with a lower maximum.

If the standard deviation of the observations be constant and equal σ , $f(x)$ equals 1, and so does m_0 . After what we have just proved the maximum of the ${}_n\sigma_y^2$ curve cannot then be lower than $\frac{\sigma^2}{N} (n+1)$. Now when we choose to distribute our N observations in $(n+1)$ equally big groups the adjusted y at each of these $(n+1)$ places will be the mean of the observations and its squared standard deviation will be $\frac{\sigma^2}{N} (n+1)$. Hence our problem is reduced to find out how to arrange a table of $(n+1)$ values of a function of the n th degree to make the squared standard deviation of any interpolation result inside the range smaller than the squared standard deviation of the values of the table. It will be seen in what follows that this can up to n equal 6—that is so far as the problem here has been investigated—be obtained by one and only one form of grouping.

When the standard deviation of the observations varies over the range, m_0 varies with the different distributions, and we cannot use the same method for finding the best distribution. It even appears that the best distribution has not always its maxima at the places of observation.

(8) A second problem which we want to consider here is *the condition for two adjusted y 's being uncorrelated*. In the beginning of this section it has been shown that the adjusted y ,

$$y_r = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \dots + \alpha_n x_p^n] \right\},$$

when

$$\left. \begin{array}{ccccccc} \alpha_0 m_0 + \alpha_1 m_1 & + \alpha_2 m_2 & + \dots & + \alpha_n m_n & = & 1 \\ \alpha_0 m_1 + \alpha_1 m_2 & + \alpha_2 m_3 & + \dots & + \alpha_n m_{n+1} & = & x_r \\ \alpha_0 m_2 + \alpha_1 m_3 & + \alpha_2 m_4 & + \dots & + \alpha_n m_{n+2} & = & x_r^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_0 m_n + \alpha_1 m_{n+1} & + \alpha_2 m_{n+2} & + \dots & + \alpha_n m_{2n} & = & x_r^n \end{array} \right\} \dots\dots\dots(16).$$

Let y_s be another adjusted value, then

$$y_s = \frac{1}{N} S \left\{ \frac{y_p}{f(x_p)} [\gamma_0 + \gamma_1 x_p + \gamma_2 x_p^2 + \dots + \gamma_n x_p^n] \right\},$$

where

$$\left. \begin{array}{ccccccc} \gamma_0 m_0 + \gamma_1 m_1 & + \gamma_2 m_2 & + \dots & + \gamma_n m_n & = & 1 \\ \gamma_0 m_1 + \gamma_1 m_2 & + \gamma_2 m_3 & + \dots & + \gamma_n m_{n+1} & = & x_s \\ \gamma_0 m_2 + \gamma_1 m_3 & + \gamma_2 m_4 & + \dots & + \gamma_n m_{n+2} & = & x_s^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ \gamma_0 m_n + \gamma_1 m_{n+1} & + \gamma_2 m_{n+2} & + \dots & + \gamma_n m_{2n} & = & x_s^n \end{array} \right\} \dots\dots\dots(17).$$

Hence the condition that y_r and y_s are uncorrelated is, since the squared standard deviation of the observed y_p equals $\sigma^2 f(x_p)$,

$$S \left\{ \frac{1}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \dots + \alpha_n x_p^n] \cdot [\gamma_0 + \gamma_1 x_p + \gamma_2 x_p^2 + \dots + \gamma_n x_p^n] \right\} = 0,$$

$$\begin{aligned} \text{or } S \left\{ \frac{\gamma_0}{f(x_p)} [\alpha_0 + \alpha_1 x_p + \alpha_2 x_p^2 + \dots + \alpha_n x_p^n] \right\} \\ + S \left\{ \frac{\gamma_1}{f(x_p)} [\alpha_0 x_p + \alpha_1 x_p^2 + \alpha_2 x_p^3 + \dots + \alpha_n x_p^{n+1}] \right\} \\ + S \left\{ \frac{\gamma_2}{f(x_p)} [\alpha_0 x_p^2 + \alpha_1 x_p^3 + \alpha_2 x_p^4 + \dots + \alpha_n x_p^{n+2}] \right\} + \dots \\ + S \left\{ \frac{\gamma_n}{f(x_p)} [\alpha_0 x_p^n + \alpha_1 x_p^{n+1} + \alpha_2 x_p^{n+2} + \dots + \alpha_n x_p^{2n}] \right\} = 0. \end{aligned}$$

Remembering that $S \left\{ \frac{x_p^q}{f(x_p)} \right\} = N m_q$ and applying the relations (16) this reduces to

$$\gamma_0 + \gamma_1 x_r + \gamma_2 x_r^2 + \dots + \gamma_n x_r^n = 0,$$

from which the γ 's are eliminated by (17).

$$\begin{vmatrix} 0 & 1 & x_r & x_r^2 & \dots & x_r^n \\ 1 & m_0 & m_1 & m_2 & \dots & m_n \\ x_s & m_1 & m_2 & m_3 & \dots & m_{n+1} \\ x_s^2 & m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_s^n & m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{vmatrix} = 0 \dots\dots\dots(18)$$

is therefore the condition that y_r and y_s are uncorrelated.

(9) Returning to the formula (11) for σ_y^2 written as a sum of squares we shall now prove that *the* $(p+1)$ st term of this put equal to zero determines a set of p abscissae the adjusted y 's of which are mutually uncorrelated both for a function of the p th and the $(p-1)$ st degree.

The condition for y_1 and y_2 corresponding to the arguments x_1 and x_2 being uncorrelated is for a function of the $(p-1)$ st degree

$$\begin{vmatrix} 0 & 1 & x_1 & x_1^2 & \dots & x_1^{p-1} \\ 1 & m_0 & m_1 & m_2 & \dots & m_{p-1} \\ x_2 & m_1 & m_2 & m_3 & \dots & m_p \\ x_2^2 & m_2 & m_3 & m_4 & \dots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^{p-1} & m_{p-1} & m_p & m_{p+1} & \dots & m_{2p-2} \end{vmatrix} = 0,$$

and for the same distribution of observations and for a function of the p th degree the condition is

$$\begin{vmatrix} 0 & 1 & x_1 & x_1^2 & \dots & x_1^p \\ 1 & m_0 & m_1 & m_2 & \dots & m_p \\ x_2 & m_1 & m_2 & m_3 & \dots & m_{p+1} \\ x_2^2 & m_2 & m_3 & m_4 & \dots & m_{p+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^p & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p} \end{vmatrix} = 0.$$

Putting

$$\begin{vmatrix} m_0 & m_1 & m_2 & \dots & m_p \\ m_1 & m_2 & m_3 & \dots & m_{p+1} \\ m_2 & m_3 & m_4 & \dots & m_{p+2} \\ \vdots & \vdots & \vdots & & \vdots \\ m_p & m_{p+1} & m_{p+2} & \dots & m_{2p} \end{vmatrix} = D,$$

these conditions may be written

$$\sum_0^{p-1} \{x_1^r \cdot x_2^s D_{p+1, p+1, r+1, s+1}\} = 0 \dots\dots\dots (19)$$

and

$$\sum_0^p \{x_1^r \cdot x_2^s D_{r+1, s+1}\} = 0 \dots\dots\dots (20),$$

where the sums include all combinations of powers with r and s lying between 0 and $(p-1)$, and 0 and p respectively.

Now we have for an orthosymmetrical determinant Δ ,

$$\Delta_{ss} \cdot \Delta_{s's''} - \Delta \cdot \Delta_{ss's''} = \Delta_{ss'} \cdot \Delta_{ss''}.$$

If therefore (19) is multiplied by D and subtracted from (20) multiplied by $D_{p+1, p+1}$ the coefficient of $x_1^r \cdot x_2^s$ becomes

$$D_{p+1, p+1} \cdot D_{r+1, s+1} - D \cdot D_{p+1, p+1, r+1, s+1} = D_{p+1, r+1} \cdot D_{p+1, s+1},$$

as long as both r and s are smaller than p .

When one of them, for example s , equals p the term is

$$x_1^r \cdot x_2^p \cdot D_{p+1, p+1} \cdot D_{r+1, p+1},$$

which is of the same form and this also holds for $r = s = p$ when the term is

$$x_1^p \cdot x_2^p \cdot D_{p+1, p+1}^2.$$

The total result is thus

$$\sum_0^p x_1^r \cdot x_2^s D_{p+1, r+1} \cdot D_{p+1, s+1} = 0,$$

or in the form of determinants

$$\begin{vmatrix} 1 & m_0 & m_1 & m_2 & \dots & m_{p-1} \\ x_1 & m_1 & m_2 & m_3 & \dots & m_p \\ x_1^2 & m_2 & m_3 & m_4 & \dots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_1^p & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & m_0 & m_1 & m_2 & \dots & m_{p-1} \\ x_2 & m_1 & m_2 & m_3 & \dots & m_p \\ x_2^2 & m_2 & m_3 & m_4 & \dots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_2^p & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p-1} \end{vmatrix} = 0.$$

Hence x_1 and x_2 must be roots of

$$\begin{vmatrix} 1 & m_0 & m_1 & m_2 & \dots & m_{p-1} \\ x & m_1 & m_2 & m_3 & \dots & m_p \\ x^2 & m_2 & m_3 & m_4 & \dots & m_{p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^p & m_p & m_{p+1} & m_{p+2} & \dots & m_{2p-1} \end{vmatrix} = 0 \dots \dots \dots (21).$$

When x_1 is found from this and substituted in (19) or (20) we get since the coefficient of x_2^p in the latter is zero an equation of the $(p - 1)$ st degree to determine x_2 . It is therefore clear that any pair of roots of (21) determine a pair of uncorrelated y 's.

II. The "best" grouping of observations with constant standard deviation.

(1) It was shown in the last section under (7) that the mean of the squared standard deviations of the adjusted y taken over the places of observation and weighted with the number of observations at each place is equal to $\frac{\sigma^2}{N} (n + 1)$ and that therefore the curve of squared standard deviation can never be entirely below that value. And further, that since $(n + 1)$ equally big groups of observations at the places of observations give the squared standard deviation this minimum, there is the possibility, ${}_n\sigma_y^2$ being of the $2n$ th degree in x , that by placing the groups at special positions the curve of squared standard deviation could have those values $\frac{\sigma^2}{N} (n + 1)$ as its maxima within the range of observations.

Let $x_1, x_2 \dots x_p \dots x_{n+1}$ be the places of observations and $\bar{y}_{x_p}$ the mean of the observations at x_p , the interpolation formula of Lagrange is then

$$y = \Sigma \left\{ \frac{(x - x_1)(x - x_2) \dots (x - x_{n+1})}{(x_p - x_1)(x_p - x_2) \dots (x_p - x_{n+1})} \bar{y}_{x_p} \right\},$$

the sum taken over all the places of observation.

From this we find

$$\sigma_y^2 = \frac{\sigma^2}{N} (n+1) \Sigma \left\{ \frac{(x-x_1)(x-x_2)\dots(x-x_{n+1})}{(x_p-x_1)(x_p-x_2)\dots(x_p-x_{n+1})} \right\}^2 \dots\dots\dots (22),$$

which for $x = x_1, x_2 \dots x_{n+1}$ equals $\frac{\sigma^2}{N} (n+1)$, the n terms of the sum being zero and the $(n+1)$ st taking the value 1 as it ought to. If x_p be the greatest of the x 's it is hence clear that for $x > x_p$, since

$$\left\{ \frac{(x-x_1)(x-x_2)\dots(x-x_{n+1})}{(x_p-x_1)(x_p-x_2)\dots(x_p-x_{n+1})} \right\}^2 > 1,$$

$$\sigma_y^2 > \frac{\sigma^2}{N} (n+1).$$

The same applies to any x smaller than the smallest of the places of observation. Therefore as we want σ_y^2 to be $\frac{\sigma^2}{N} (n+1)$ at the ends of the range *we have to place two of our groups of observations there.*

Let us take *the half of the range within which it is possible to make observations as the unit of x so that the range goes from -1 to 1 .*

(2) Hence for a linear function there is no choice left, the two groups of observations must be at -1 and 1 .

According to (22) we have

$${}_1\sigma_y^2 = \frac{\sigma^2}{N} \cdot 2 \left\{ \frac{(x+1)^2}{4} + \frac{(x-1)^2}{4} \right\}$$

or

$${}_1\sigma_y^2 = \frac{\sigma^2}{N} \cdot 2 \left\{ 1 - \frac{1}{2} (1 - x^2) \right\},$$

which illustrate the well-known fact that *by simple interpolation between two equally good values of a table, we obtain interpolated values with less probable error than those of the table.*

(3) Investigating a function of the second degree we have a third group to place besides the two at -1 and 1 , that is if we do not beforehand suppose the distribution to be symmetrical. Let the third group be at α , then the interpolation gives

$$y = \frac{(x-1)(x-\alpha)}{2(1+\alpha)} \bar{y}_{-1} + \frac{(x+1)(x-\alpha)}{2(1-\alpha)} \bar{y}_1 + \frac{x^2-1}{\alpha^2-1} \bar{y}_\alpha,$$

from which

$${}_2\sigma_y^2 = \frac{\sigma^2}{N} \cdot 3 \left\{ \left[\frac{(x-1)(x-\alpha)}{2(1+\alpha)} \right]^2 + \left[\frac{(x+1)(x-\alpha)}{2(1-\alpha)} \right]^2 + \left[\frac{x^2-1}{\alpha^2-1} \right]^2 \right\}.$$

We want this to be a maximum for $x = \alpha$, but $\left(\frac{d\sigma_y^2}{dx} \right)_{x=\alpha}$ can only vanish for $\alpha = 0$, in which case σ_y^2 is reduced to

$${}_2\sigma_y^2 = \frac{\sigma^2}{N} \cdot 3 \left\{ 1 - \frac{3}{2} x^2 (1 - x^2) \right\},$$

which shows that we have succeeded in making σ_y^2 a maximum at $x = 0$ and obtained a standard deviation with the maximum value $\frac{\sigma^2}{N} 3$, as we desired.

(4) For a function of the third degree we find from four groups of observations at $-1, 1, \alpha$ and γ that

$$y = \frac{(x-1)(x-\alpha)(x-\gamma)}{-2(1+\alpha)(1+\gamma)} \bar{y}_{-1} + \frac{(x+1)(x-\alpha)(x-\gamma)}{2(1-\alpha)(1-\gamma)} \bar{y}_1 \\ + \frac{(x^2-1)(x-\gamma)}{(\alpha^2-1)(\alpha-\gamma)} \bar{y}_\alpha + \frac{(x^2-1)(x-\alpha)}{(\gamma^2-1)(\gamma-\alpha)} \bar{y}_\gamma$$

and
$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \cdot 4 \left\{ \left[\frac{(x-1)(x-\alpha)(x-\gamma)}{2(1+\alpha)(1+\gamma)} \right]^2 + \left[\frac{(x+1)(x-\alpha)(x-\gamma)}{2(1-\alpha)(1-\gamma)} \right]^2 \right. \\ \left. + \left[\frac{(x^2-1)(x-\gamma)}{(1-\alpha^2)(\alpha-\gamma)} \right]^2 + \left[\frac{(x^2-1)(x-\alpha)}{(1-\gamma^2)(\alpha-\gamma)} \right]^2 \right\}.$$

The condition

$$\left(\frac{d\sigma_y^2}{dx} \right)_{x=\alpha} = 0$$

requires

$$3\alpha^2 - 2\alpha\gamma - 1 = 0,$$

and

$$\left(\frac{d\sigma_y^2}{dx} \right)_{x=\gamma} = 0$$

requires

$$3\gamma^2 - 2\alpha\gamma - 1 = 0,$$

from which is got

$$\alpha^2 = \gamma^2,$$

and, since $\alpha \geq \gamma$,

$$\alpha^2 = \gamma^2 = \frac{1}{3}.$$

By introducing this value for α^2 and γ^2 in σ_y^2 we find

$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \cdot 4 \left\{ 1 - \frac{3 \cdot 5^2}{2^4} (x^2 - \frac{1}{3})^2 (1 - x^2) \right\},$$

which has the required maxima at $\pm \sqrt{\frac{1}{3}}$.

(5) For the functions of higher degree we shall at once assume that the distributions sought are symmetrical, since it is pretty clear from the symmetry of y and σ_y^2 with regard to the sought positions that it must be so.

To determine a function of the fourth degree let us put groups of observations at $\pm 1, \pm \alpha$ and 0 . The expression for σ_y^2 can be written down at once and is such that the terms arising from the groups at $+1$ and -1 can be put together as well as the terms from $+\alpha$ and $-\alpha$, then

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \cdot 5 \left[\left\{ \frac{(x^2-1)(x^2-\alpha^2)}{\alpha^2} \right\}^2 + \frac{1}{2} \left\{ \frac{x(x^2-\alpha^2)}{1-\alpha^2} \right\}^2 (x^2+1) + \frac{1}{2} \left\{ \frac{x(x^2-1)}{\alpha^2(1-\alpha^2)} \right\}^2 (x^2-\alpha^2) \right].$$

$\left(\frac{d\sigma_y^2}{dx} \right)_{x=\alpha} = 0$ provides the condition $\frac{3-7\alpha^2}{\alpha(1-\alpha^2)} = 0$ or

$\alpha^2 = \frac{3}{7},$

with which value the squared standard deviation becomes

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \cdot 5 \left\{ 1 - \frac{5 \cdot 7^2}{2^4} x^2 (x^2 - \frac{3}{7})^2 (1 - x^2) \right\},$$

which has the required characteristics.

(6) Adjusting by a function of the fifth degree six equally big groups of observations at the arguments ± 1 , $\pm a$ and $\pm \gamma$ the squared standard deviation of the adjusted y is

$${}_5\sigma_y^2 = \frac{\sigma^2}{N} \cdot 6 \left\{ \frac{1}{2} \left[\frac{(x^2 - a^2)(x^2 - \gamma^2)}{(1 - a^2)(1 - \gamma^2)} \right]^2 (x^2 + 1) + \frac{1}{2} \left[\frac{(x^2 - \gamma^2)(x^2 - 1)}{a(1 - a^2)(a^2 - \gamma^2)} \right]^2 (x^2 + a^2) \right. \\ \left. + \frac{1}{2} \left[\frac{(x^2 - 1)(x^2 - a^2)}{\gamma(1 - \gamma^2)(a^2 - \gamma^2)} \right]^2 (x^2 + \gamma^2) \right\}.$$

The condition for maximum at $x = \pm a$ is

$$9a^4 - 5a^2\gamma^2 - 5a^2 + \gamma^2 = 0,$$

which together with the condition for maximum at $x = \pm \gamma$

$$9\gamma^4 - 5a^2\gamma^2 - 5\gamma^2 + a^2 = 0,$$

since a^2 must be $\geq \gamma^2$ results in

$$a^2 + \gamma^2 = \frac{2}{3} \quad \text{and} \quad a^2\gamma^2 = \frac{1}{21}$$

or

$$\frac{a^2}{\gamma^2} = \frac{7 \pm 2\sqrt{7}}{21}.$$

When these values are substituted in the expression above for σ_y^2 this may by somewhat lengthy algebraic operations be brought into the form

$${}_5\sigma_y^2 = \frac{\sigma^2}{N} \cdot 6 \left\{ 1 - \frac{3^3 \cdot 5 \cdot 7^2}{2^7} (x^2 - a^2)^2 (x^2 - \gamma^2)^2 (1 - x^2) \right\}.$$

(7) For a function of the sixth degree the observations may be supposed to be at ± 1 , $\pm a$, $\pm \gamma$ and 0.

The expression for the squared standard deviation of an adjusted y becomes

$${}_6\sigma_y^2 = \frac{\sigma^2}{N} \cdot 7 \left\{ \left[\frac{(x^2 - a^2)(x^2 - \gamma^2)(x^2 - 1)}{a^2\gamma^2} \right]^2 + \frac{1}{2} \left[\frac{x(x^2 - \gamma^2)(x^2 - 1)}{a^2(a^2 - \gamma^2)(a^2 - 1)} \right]^2 (x^2 + a^2) \right. \\ \left. + \frac{1}{2} \left[\frac{x(x^2 - a^2)(x^2 - 1)}{\gamma^2(a^2 - \gamma^2)(\gamma^2 - 1)} \right]^2 (x^2 + \gamma^2) + \frac{1}{2} \left[\frac{x(x^2 - a^2)(x^2 - \gamma^2)}{(a^2 - 1)(\gamma^2 - 1)} \right]^2 (x^2 + 1) \right\}.$$

A maximum at $x = \pm a$ requires

$$11a^4 - 7a^2\gamma^2 - 7a^2 + 3\gamma^2 = 0,$$

and a maximum at $x = \pm \gamma$ requires

$$11\gamma^4 - 7a^2\gamma^2 - 7\gamma^2 + 3a^2 = 0,$$

which added and subtracted provide

$$11(a^2 + \gamma^2)^2 - 36a^2\gamma^2 - 4(a^2 + \gamma^2) = 0,$$

and

$$(a^2 - \gamma^2) \{11(a^2 + \gamma^2) - 10\} = 0.$$

Since we must have $a^2 \leq \gamma^2$,

$$a^2 + \gamma^2 = \frac{10}{11} \quad \text{and} \quad a^2\gamma^2 = \frac{5}{33},$$

or

$$\frac{a^2}{\gamma^2} = \frac{15 \pm 2\sqrt{15}}{33}.$$

The expression for σ_y^2 may after rather laborious operations be brought into the form

$${}_6\sigma_y^2 = \frac{\sigma^2}{N} \cdot 7 \left\{ 1 - \frac{3^3 \cdot 7 \cdot 11^2}{2^7} x^2 (x^2 - a^2)^2 (x^2 - \gamma^2)^2 (1 - x^2) \right\}.$$

(8) It is thus, as we aimed at, shown for functions up to the sixth degree that *by distributing the observations in $(n + 1)$ equally big groups and choosing the places of these groups in one special way we can manage to keep the standard deviation of any adjusted y within the possible range of observations less than the standard deviation at the places of observation.* There is every reason to believe that the rule holds for any degree of function, but as the general proof would be very complicated and as almost all practical cases will be covered by functions up to the sixth degree, the problem can therefore be left at this stage.

As we have proved, *any other distribution of observations leads to a curve of squared standard deviation that has a higher maximum value within the range.* This special set of $(n + 1)$ groups has therefore a very conspicuous advantage over all other distributions of observations. *The application of it is however limited in that it demands that the degree of the function must be known beforehand and thus the observations do not provide any justification for the form of function chosen.* If however the function has been fully investigated beforehand and there is no doubt about its form, $(n + 1)$ equally big groups of observations placed as indicated are the most desirable set of observations possible. The approximate values of the places of the groups are given in the table below.

TABLE I.

Degree of function	1st	2nd	3rd	4th	5th	6th
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Places of observation	—	.0000	.4472	.6547	.7651	.8302
	—	—	—	.0000	.2852	.4689
	—	—	—	—	—	.0000

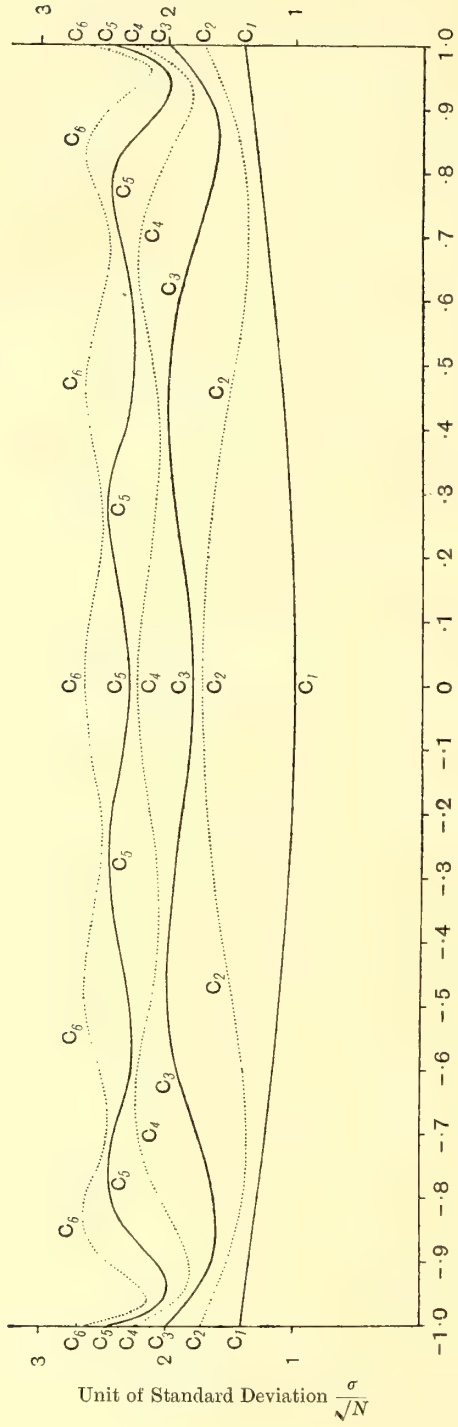
With rougher approximation the intervals between the observations, still expressed by the half range as unit, are as follows:

1st degree of function	2					
2nd	„	„	1		1	
3rd	„	„	$\frac{1}{2}$	1	$\frac{1}{2}$	
4th	„	„	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
5th	„	„	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
6th	„	„	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

The six curves of standard deviation are represented in Diagram 1. It will be seen that the minima of a curve, if it has more than two, are the lower the greater their distances from the middle of the range, so that the variation of the standard deviation is greatest in the outermost intervals of the range.

III. Uniform continuous distribution of observations with constant standard deviation. General formulae.

(1) As was pointed out in the last section the lumping up of observations in groups just necessary to determine the constants of the function in question has some drawbacks and cannot be recommended as a universal rule. In many cases it is through the observations themselves that we first get to know the form of the



Range of observations.

C ₁ C ₁ .	Function of First Degree
C ₂ C ₂ .	Second "
C ₃ C ₃ .	Third "
C ₄ C ₄ .	Fourth "
C ₅ C ₅ .	Fifth "
C ₆ C ₆ .	Sixth "

DIAGRAM 1. Curves of Standard Deviations. Equally big clusters at definite points.

function, and thus a full investigation may require more groups of observations than merely a number equal to the assumed number of constants in the formula. Besides, even when we believe we know on theoretical or other grounds beforehand the nature of the function *a priori* we may consider it prudent to distribute the observations so that they supply us with data whereby we may control our hypothesis that the assumed function is the right one.

It is therefore desirable to find other forms of distributions which, at the same time as they make the standard deviation of the adjusted function vary little inside the range of observations, are more uniformly spread over this range.

(2) A uniform continuous distribution at once recommends itself as the simplest assumption. As we suppose the observations to have constant standard deviations the elements of the determinants of (15) are the moment coefficients of the x 's at the places of observation.

When the N observations are uniformly spread between $x = -1$ and $x = 1$,

$$\mu_{2r} = \frac{1}{2r+1} \quad \text{and} \quad \mu_{2r+1} = 0,$$

and the expression for ${}_{2p}\sigma_y^2$ is, according to (15),

$$\begin{aligned} {}_{2p}\sigma_y^2 = \frac{\sigma^2}{N} & \left\{ 1 + \frac{x^2}{\mu_2} + \frac{\begin{vmatrix} 1 & 1 \\ x^2 & \mu_2 \end{vmatrix}^2}{1 \cdot \begin{vmatrix} 1 & \mu_2 \\ \mu_2 & \mu_4 \end{vmatrix}} + x^2 \frac{\begin{vmatrix} 1 & \mu_2 \\ x^2 & \mu_4 \end{vmatrix}^2}{\mu_2 \cdot \begin{vmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{vmatrix}} + \dots \right. \\ & + x^2 \frac{\begin{vmatrix} 1 & \mu_2 & \mu_4 & \dots & \mu_{2p-2} \\ x^2 & \mu_4 & \mu_6 & \dots & \mu_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-2} & \mu_{2p} & \mu_{2p+2} & \dots & \mu_{4p-4} \end{vmatrix}^2}{\begin{vmatrix} \mu_2 & \mu_4 & \dots & \mu_{2p-2} \\ \mu_4 & \mu_6 & \dots & \mu_{2p} \\ \vdots & \vdots & & \vdots \\ \mu_{2p-2} & \mu_{2p} & \dots & \mu_{4p-6} \end{vmatrix} \cdot \begin{vmatrix} \mu_2 & \mu_4 & \dots & \mu_{2p} \\ \mu_4 & \mu_6 & \dots & \mu_{2p+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2p} & \mu_{2p+2} & \dots & \mu_{4p-2} \end{vmatrix}} \\ & \left. + \frac{\begin{vmatrix} 1 & 1 & \mu_2 & \dots & \mu_{2p-2} \\ x^2 & \mu_2 & \mu_4 & \dots & \mu_{2p} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \mu_{2p} & \mu_{2p+2} & \dots & \mu_{4p-2} \end{vmatrix}^2}{\begin{vmatrix} 1 & \mu_2 & \dots & \mu_{2p-2} \\ \mu_2 & \mu_4 & \dots & \mu_{2p} \\ \vdots & \vdots & & \vdots \\ \mu_{2p-2} & \mu_{2p} & \dots & \mu_{4p-4} \end{vmatrix} \cdot \begin{vmatrix} 1 & \mu_2 & \dots & \mu_{2p} \\ \mu_2 & \mu_4 & \dots & \mu_{2p+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2p} & \mu_{2p+2} & \dots & \mu_{4p} \end{vmatrix}} \right\} \dots \dots \dots (23). \end{aligned}$$

By this formula we may evaluate successively ${}_1\sigma_y^2, {}_2\sigma_y^2 \dots {}_{2p}\sigma_y^2$ when we know the two general terms of which the sum consists.

(3) The determinant of the order p ,

$${}_p\Delta^q = \begin{vmatrix} \frac{1}{2q-1} & \frac{1}{2q+1} & \cdots & \frac{1}{2q+2p-3} \\ \frac{1}{2q+1} & \frac{1}{2q+3} & \cdots & \frac{1}{2q+2p-1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2q+2p-3} & \frac{1}{2q+2p-1} & \cdots & \frac{1}{2q+4p-5} \end{vmatrix},$$

which includes the two types of the denominators in (23), shall first be evaluated.

We find ${}_1\Delta^q = \frac{1}{2q-1}$ and ${}_2\Delta^q = \frac{2^2}{(2q-1)(2q+1)^2(2q+3)}$,

and it shall be proved that if

$${}_p\Delta^q = \{1^{p-1} \cdot 2^{p-2} \cdots (p-2)^2(p-1)\}^2 \cdot 2^{p(p-1)} \cdot {}_p\Pi^q \cdots \quad (24)$$

up to the order p , ${}_p\Pi^q$ being the product of the elements of ${}_p\Delta^q$, the rule holds for determinants of any order.

It is clear that

$${}_{p+1}\Delta_{1,1}^q = {}_p\Delta^{q+2}, \quad {}_{p+1}\Delta_{p+1,p+1}^q = {}_p\Delta^q, \quad {}_{p+1}\Delta_{1,p}^q (-1)^{p+1} = {}_p\Delta^{q+1}$$

and

$${}_{p+1}\Delta_{p+1,p+1,1,1}^q = {}_{p-1}\Delta^{q+2}.$$

If we therefore in the general relation for an orthosymmetrical determinant

$$\Delta = \frac{\Delta_{ss} \cdot \Delta_{s's'} - \Delta_{ss'}^2}{\Delta_{ss's's'}}$$

put $s = 1$ and $s' = p + 1$ and $\Delta = {}_{p+1}\Delta^q$, we find

$${}_{p+1}\Delta^q = \frac{{}_p\Delta^q \cdot {}_p\Delta^{q+2} - {}_p\Delta^{q+1}^2}{{}_{p-1}\Delta^{q+2}},$$

and, using (24),

$${}_{p+1}\Delta^q = \frac{\{1^{p-1} \cdot 2^{p-2} \cdots (p-2)^2(p-1)\}^4}{\{1^{p-2} \cdot 2^{p-3} \cdots (p-3)^2(p-2)\}^2} \cdot 2^{(p-1)(p+2)} \cdot \frac{{}_p\Pi^q \cdot {}_p\Pi^{q+2} - {}_p\Pi^{q+1}^2}{{}_{p-1}\Pi^{q+2}}.$$

Now, according to the definition of Π ,

$$\frac{{}_p\Pi^q}{{}_{p+1}\Pi^q} \cdot \frac{{}_p\Pi^{q+2}}{{}_{p+1}\Pi^q} = (2q-1)(2q+1)^2(2q+3)^2 \cdots (2q+4p-3)^2(2q+4p-1) \times (2q+2p-1)^2,$$

$$\frac{{}_p\Pi^{q+1}^2}{{}_{p+1}\Pi^q} = (2q-1)^2(2q+1)^2(2q+3)^2 \cdots (2q+4p-3)^2(2q+4p-1)^2$$

and

$$\frac{{}_{p-1}\Pi^{q+2}}{{}_p\Pi^q} = (2q-1)(2q+1)^2(2q+3)^2 \cdots (2q+4p-3)^2(2q+4p-1).$$

Hence

$${}_{p+1}\overset{q}{\Delta} = \{1^p \cdot 2^{p-1} \dots (p-2)^3 (p-1)^2\}^2 \cdot 2^{(p-1)(p+2)} \cdot {}_{p+1}\overset{q}{\Pi} \cdot [(2q+2p-1)^2 - (2q-1)(2q+4p-1)],$$

$${}_{p+1}\overset{q}{\Delta} = \{1^p \cdot 2^{p-1} \dots (p-1)^2 \cdot p\}^2 \cdot 2^{p(p+1)} \cdot {}_{p+1}\overset{q}{\Pi},$$

which agrees with (24).

(4) Next we have to evaluate the minors of ${}_p\overset{q}{\Delta}$ necessary for calculating the numerators in (23). For this purpose we only need the minors ${}_p\overset{q}{\Delta}_{s,p}$, but to carry through the proof by induction ${}_p\overset{q}{\Delta}_{s,r}$ for any values of s and r is needed.

For ${}_3\overset{q}{\Delta}_{2,3}$ we directly find,

$${}_3\overset{q}{\Delta}_{2,3} = \frac{-2^2 \cdot 2}{(2q-1)(2q+1)(2q+3)(2q+5)}$$

and

$${}_3\overset{q}{\Delta}_{2,2} = \frac{2^2 \cdot 2^2}{(2q-1)(2q+3)^2(2q+7)},$$

these both agree with the following formula which will be proved by induction,

$${}_p\overset{q}{\Delta}_{s,r} = (-1)^{r+s} \beta_{p-1,s-1} \cdot \beta_{p-1,r-1} \{1^{p-2} \cdot 2^{p-3} \dots (p-3)^2 (p-2)\}^2 \cdot 2^{(p-1)(p-2)} {}_p\overset{q}{\Pi}_{s,r} \dots (25),$$

$\beta_{p-1,s-1}$ is the binomial coefficient $\frac{|p-1|}{|s-1| |p-s|}$ and ${}_p\overset{q}{\Pi}_{s,r}$ the product of all the elements of ${}_p\overset{q}{\Delta}_{s,r}$.

The relation has to be proved first for $r=s$, then for $r=p$ and finally for any combination s and r .

For the first two proofs we use the relation between the minors of an orthosymmetrical determinant

$$\frac{\Delta_{ss}}{\Delta_{s's'}} = \frac{\Delta_{ss's'} \cdot \Delta_{ss's''} - \Delta_{ss's'}^2}{\Delta_{s's's''} \cdot \Delta_{ss's''} + \Delta_{s''s's'} \cdot \Delta_{s's's''}} \dots (26).$$

This is found from two relations given by Professor Pearson* by dividing one of them by the other.

(5) Let Δ be ${}_{p+1}\overset{q}{\Delta}$, $s'=1$ and $s''=p+1$, then

$$\frac{{}_{p+1}\overset{q}{\Delta}_{ss}}{{}_{p+1}\overset{q}{\Delta}_{1,p+1}} = \frac{{}_{p+1}\overset{q}{\Delta}_{ss11} \cdot {}_{p+1}\overset{q}{\Delta}_{s,s,p+1,p+1} - {}_{p+1}\overset{q}{\Delta}_{s,s,1,p+1}^2}{{}_{p+1}\overset{q}{\Delta}_{1,1,p+1,p+1} \cdot {}_{p+1}\overset{q}{\Delta}_{s,s,1,p+1} + {}_{p+1}\overset{q}{\Delta}_{p+1,p+1,s,1} \cdot {}_{p+1}\overset{q}{\Delta}_{1,1,s,p+1}} \dots (27).$$

Now

$$\begin{aligned} {}_{p+1}\overset{q}{\Delta}_{ss11} &= {}_p\overset{q+2}{\Delta}_{s-1,s-1}, \\ {}_{p+1}\overset{q}{\Delta}_{s,s,p+1,p+1} &= {}_p\overset{q}{\Delta}_{ss}, \\ {}_{p+1}\overset{q}{\Delta}_{1,1,p+1,p+1} &= (-1)^{p+1} {}_p\overset{q+1}{\Delta}_{1,p}, \\ {}_{p+1}\overset{q}{\Delta}_{s,s,1,p+1} &= (-1)^p {}_p\overset{q+1}{\Delta}_{s-1,s}, \end{aligned}$$

* *Biometrika*, Vol. XI, pp. 232-3.

$$\begin{aligned} {}_{p+1}\Delta_{p+1, p+1, s, 1}^q &= (-1)^{p+1} {}_p\Delta_{p, s}^{q+1}, \\ {}_{p+1}\Delta_{1, 1, s, p+1}^q &= (-1)^{p+1} {}_p\Delta_{s-1, 1}^{q+1}, \end{aligned}$$

so that all the determinants on the right side in (27) can be evaluated by (25).

They all have the factor

$$\{1^{p-2} \cdot 2^{p-3} \dots (p-3)^2 (p-2)\}^2 \cdot 2^{(p-1)(p-2)}$$

in common, when that is divided out there remains

$$\frac{{}_{p+1}\Delta_{ss}^q}{{}_{p+1}\Delta_{1, p+1}^q} = \frac{(-1)^p \beta_{p-1, s-2}^2 \cdot \beta_{p-1, s-1}^2 ({}_p\Pi_{s-1, s-1}^{q+2} \cdot {}_p\Pi_{ss}^q - {}_p\Pi_{s-1, s}^{q+1})}{\beta_{p-1, s-2} \cdot \beta_{p-1, s-1} ({}_p\Pi_{s-1, 1}^{q+1} \cdot {}_p\Pi_{p, s}^{q+1} - {}_p\Pi_{p, 1}^{q+1} \cdot {}_p\Pi_{s-1, s}^{q+1})} \dots (28).$$

Now indicating by C_r the product of the elements of the r th column or r th row in ${}_{p+1}\Delta$ and by e_{rs} the element of the ${}_{p+1}\Delta$ common for the r th row and s th column we find

$$\begin{aligned} {}_{p+1}\Pi_{ss}^q &= {}_p\Pi_{s-1, s-1}^{q+2} \cdot \frac{C_1^2}{e_{11} \cdot e_{s1}^2}, \\ {}_{p+1}\Pi_{ss}^q &= {}_p\Pi_{ss}^q \cdot \frac{C_{p+1}^2}{e_{p+1, p+1} \cdot e_{p+1, s}^2}, \\ {}_{p+1}\Pi_{ss}^q &= {}_p\Pi_{s-1, s}^{q+1} \cdot \frac{C_1 C_{p+1}}{e_{1, p+1} \cdot e_{s, 1} \cdot e_{s, p+1}}. \end{aligned}$$

Hence the factor of the numerator in (28) is reduced to

$${}_{p+1}\Pi_{ss}^q \frac{e_{s, 1}^2 \cdot e_{s, p+1}^2}{C_1^2 \cdot C_{p+1}^2} \{e_{11} \cdot e_{p+1, p+1} - e_{1, p+1}^2\}.$$

For the Π 's of the denominator we find

$$\begin{aligned} {}_{p+1}\Pi_{1, p+1}^q &= {}_p\Pi_{s-1, 1}^{q+1} \cdot \frac{C_1 \cdot C_s}{e_{s1} \cdot e_{s, p+1} \cdot e_{1, 1}}, \\ {}_{p+1}\Pi_{1, p+1}^q &= {}_p\Pi_{p, s}^{q+1} \cdot \frac{C_{p+1} \cdot C_s}{e_{s, p+1} \cdot e_{p+1, p+1} \cdot e_{1, s}}, \\ {}_{p+1}\Pi_{1, p+1}^q &= {}_p\Pi_{p, 1}^{q+1} \cdot \frac{C_{p+1} \cdot C_1}{e_{1, p+1} \cdot e_{11} \cdot e_{p+1, p+1}}, \\ {}_{p+1}\Pi_{1, p+1}^q &= {}_p\Pi_{s-1, s}^{q+1} \cdot \frac{C_s^2}{e_{ss} \cdot e_{s1} \cdot e_{s, p+1}}, \end{aligned}$$

the factor containing Π 's of the denominator of (28) is therefore equal to

$${}_{p+1}\Pi_{1, p+1}^q \frac{e_{1s} \cdot e_{p+1, s} \cdot e_{1, 1} \cdot e_{p+1, p+1}}{C_1 \cdot C_{p+1} \cdot C_s^2} \{e_{1s} \cdot e_{p+1, s} - e_{1, p+1} \cdot e_{ss}\}.$$

Introducing these two expressions in (28) and substituting for the one factor

$$\frac{{}_{p+1}\Pi_{ss}^q}{{}_{p+1}\Pi_{1, p+1}^q} \text{ the value } \frac{C_1 \cdot C_{p+1}}{C_s^2} \cdot \frac{e_{ss}}{e_{1, p+1}} \text{ we hence find}$$

$$\frac{{}_{p+1}\Delta_{ss}^q}{{}_{p+1}\Delta_{1, p+1}^q} = (-1)^p \beta_{p-1, s-2} \beta_{p-1, s-1} \frac{\frac{1}{e_{ss} \cdot e_{1, p+1}} - \frac{1}{e_{1, s} \cdot e_{p+1, s}}}{\frac{e_{1, p+1}^2}{1} - \frac{e_{11} \cdot e_{p+1, p+1}}{1}} \cdot \frac{{}_{p+1}\Pi_{ss}^q}{{}_{p+1}\Pi_{1, p+1}^q}.$$

The fraction containing e 's equals

$$\frac{(2q+2p-1)^2 - (2q-1)(2q+4p-1)}{(2q+4s-5)(2q+2p-1) - (2q+2s-3)(2q+2s+2p-3)} = \frac{p^2}{(s-1)(p-s+1)},$$

hence

$$\frac{\frac{q}{p+1}\Delta_{ss}}{\frac{q}{p+1}\Delta_{1,p+1}} = (-1)^p \beta_{p,s-1}^2 \cdot \frac{\frac{q}{p+1}\Pi_{ss}}{\frac{q}{p+1}\Pi_{1,p+1}}.$$

As

$$\frac{q}{p+1}\Delta_{1,p+1} = (-1)^p \frac{q+1}{p}\Delta = (-1)^p \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} \frac{q}{p+1}\Pi_{1,p+1},$$

we therefore find

$$\frac{q}{p+1}\Delta_{ss} = \beta_{p,s-1}^2 \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} \frac{q}{p+1}\Pi_{ss},$$

agreeing with (25).

(6) To evaluate $\frac{q}{p+1}\Delta_{s,p+1}$ we shall in (26) put $\Delta = \frac{q}{p+1}\Delta$, $s = 1$, $s' = s$ and $s'' = p+1$. Reversing the fractions we then get

$$\frac{\frac{q}{p+1}\Delta_{s,p+1}}{\frac{q}{p+1}\Delta_{11}} = \frac{\frac{q}{p+1}\Delta_{s,s,p+1} \cdot \frac{q}{p+1}\Delta_{1,1,s,p+1} + \frac{q}{p+1}\Delta_{p+1,p+1,1,s} \cdot \frac{q}{p+1}\Delta_{s,s,1,p+1}}{\frac{q}{p+1}\Delta_{1,1,s,s} \cdot \frac{q}{p+1}\Delta_{1,1,p+1,p+1} - \frac{q}{p+1}\Delta_{1,1,s,p+1}^2} \dots (29).$$

As

$$\begin{aligned} \frac{q}{p+1}\Delta_{s,s,p+1,p+1} &= \frac{q}{p}\Delta_{ss}, \\ \frac{q}{p+1}\Delta_{1,1,s,p+1} &= \frac{q+2}{p}\Delta_{s-1,p} = (-1)^{p+1} \frac{q+1}{p}\Delta_{1,s-1}, \\ \frac{q}{p+1}\Delta_{p+1,p+1,1,s} &= (-1)^{p+1} \frac{q+1}{p}\Delta_{s,p}, \\ \frac{q}{p+1}\Delta_{s,s,1,p+1} &= (-1)^p \frac{q+1}{p}\Delta_{s-1,s}, \\ \frac{q}{p+1}\Delta_{1,1,s,s} &= \frac{q+2}{p}\Delta_{s-1,s-1}, \\ \frac{q}{p+1}\Delta_{1,1,p+1,p+1} &= \frac{q}{p}\Delta_{1,1}, \end{aligned}$$

the right side of (29) can be evaluated by (25).

We thus get

$$\frac{\frac{q}{p+1}\Delta_{s,p+1}}{\frac{q}{p+1}\Delta_{1,1}} = \frac{(-1)^{s+p-1} \beta_{p-1,s-1} \cdot \beta_{p-1,s-2} \cdot \beta_{p-1,s-1} \left(-\frac{q+1}{p}\Pi_{s,s-1} \cdot \frac{q+1}{p}\Pi_{s,p} + \frac{q}{p}\Pi_{ss} \cdot \frac{q+2}{p}\Pi_{s-1,p} \right)}{\beta_{p-1,s-2}^2 \left(\frac{q+2}{p}\Pi_{s-1,s-1} \cdot \frac{q}{p}\Pi_{1,1} - \frac{q+1}{p}\Pi_{1,s-1}^2 \right)} \dots (30).$$

We want here to express the Π 's of the numerator by $\frac{q}{p+1}\Pi_{s,p+1}$ and those of the denominator by $\frac{q}{p+1}\Pi_{1,1}$ and we find the following relations

$$\begin{aligned} \frac{q}{p+1}\Pi_{s,p+1} &= \frac{q+1}{p}\Pi_{s,s-1} \cdot \frac{C_1 \cdot C_s}{e_{1s} \cdot e_{1,p+1} \cdot e_{ss}}, \\ \frac{q}{p+1}\Pi_{s,p+1} &= \frac{q+1}{p}\Pi_{sp} \cdot \frac{C_{p+1} \cdot C_1}{e_{1,p+1} \cdot e_{p+1,p+1} \cdot e_{1,s}}, \\ \frac{q}{p+1}\Pi_{s,p+1} &= \frac{q}{p}\Pi_{ss} \cdot \frac{C_s \cdot C_{p+1}}{e_{s,p+1} \cdot e_{ss} \cdot e_{p+1,p+1}}, \end{aligned}$$

$$\begin{aligned} {}_{p+1}\Pi_{s, p+1}^q &= {}_p\Pi_{s-1, p}^{q+2} \cdot \frac{C_1 \cdot C_1}{e_{1,1} \cdot e_{1s} \cdot e_{1, p+1}}, \\ \text{and} \quad {}_{p+1}\Pi_{11}^q &= {}_p\Pi_{s-1, s-1}^{q+2} \cdot \frac{C_s \cdot C_s}{e_{ss} \cdot e_{1s}^2}, \\ {}_{p+1}\Pi_{11}^q &= {}_p\Pi_{11}^q \cdot \frac{C_{p+1}^2}{e_{p+1, p+1} \cdot e_{1, p+1}^2}, \\ {}_{p+1}\Pi_{11}^q &= {}_p\Pi_{1, s-1}^{q+1} \cdot \frac{C_s \cdot C_{p+1}}{e_{s, p+1} \cdot e_{1, p+1} \cdot e_{1s}}. \end{aligned}$$

Substituting the Π 's found from these relations into (30) and eliminating the one factor $\frac{{}_{p+1}\Pi_{s, p+1}^q}{{}_{p+1}\Pi_{11}^q}$ by

$$\frac{{}_{p+1}\Pi_{s, p+1}^q}{{}_{p+1}\Pi_{11}^q} = \frac{C_1^2}{C_s \cdot C_{p+1}} \cdot \frac{e_{s, p+1}}{e_{1,1}}$$

we get

$$\frac{{}_{p+1}\Delta_{s, p+1}^q}{{}_{p+1}\Delta_{11}^q} = \frac{(-1)^{s+p+1} \beta_{p-1, s-1}^2}{\beta_{p-1, s-2}} \cdot \frac{1}{\frac{1}{e_{s, p+1}^2} - \frac{1}{e_{ss} \cdot e_{p+1, p+1}}} \cdot \frac{1}{\frac{1}{e_{1,1} \cdot e_{s, p+1}} - \frac{1}{e_{1,1} \cdot e_{1s} \cdot e_{1, p+1}}} \cdot \frac{{}_{p+1}\Pi_{s, p+1}^q}{{}_{p+1}\Pi_{1,1}^q},$$

or introducing the values of the e 's

$$\begin{aligned} &\frac{{}_{p+1}\Delta_{s, p+1}^q}{{}_{p+1}\Delta_{1,1}^q} \\ &= \frac{(-1)^{s+p+1} \beta_{p-1, s-2}^2}{\beta_{p-1, s-2}} \cdot \frac{(2q+2s-3)(2q+2p-1) - (2q-1)(2q+2p+2s-3)}{(2q+2p+2s-3)^2 - (2q+4s-5)(2q+4p-1)} \cdot \frac{{}_{p+1}\Pi_{s, p+1}^q}{{}_{p+1}\Pi_{1,1}^q} \\ &= (-1)^{s+p+1} \cdot \beta_{p, s-1} \cdot \frac{{}_{p+1}\Pi_{s, p+1}^q}{{}_{p+1}\Pi_{1,1}^q}. \end{aligned}$$

Now

$${}_{p+1}\Delta_{1,1}^q = {}_p\Delta = \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} {}_{p+1}\Pi_{1,1}^q,$$

and hence

$${}_{p+1}\Delta_{s, p+1}^q = (-1)^{s+p+1} \cdot \beta_{p, s-1} \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} {}_{p+1}\Pi_{s, p+1}^q$$

in agreement with (25).

(7) It now remains to prove that (25) holds for ${}_{p+1}\Delta_{s,r}^q$ when both s and r are different from 1 and $p+1$, and r different from s .

For this shall be used the relation

$$\Delta \cdot \Delta_{sss' s'} = \Delta_{ss} \cdot \Delta_{s' s''} - \Delta_{ss'} \cdot \Delta_{ss''}$$

between an orthosymmetrical determinant and its minors.

Putting $\Delta = {}_{p+1}\Delta$, $s = p+1$, $s' = r$ and $s'' = s$ and solving the equation with regard to ${}_{p+1}\Delta_{r,s}$ we have

$${}_{p+1}\Delta_{r,s} = \frac{1}{{}_{p+1}\Delta_{p+1, p+1}} ({}_{p+1}\Delta \cdot {}_{p+1}\Delta_{p+1, p+1, r, s} + {}_{p+1}\Delta_{p+1, r} \cdot {}_{p+1}\Delta_{p+1, s}),$$

where

$${}_{p+1}\Delta_{p+1, p+1, r, s} = {}_p\Delta_{r, s}.$$

Evaluating this by (24) and (25) we get

$${}_{p+1}\overset{q}{\Delta}_{r,s} = \frac{(-1)^{r+s}}{{}_p\overset{q}{\prod}} \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)^2 \cdot 2^{p(p-1)} \\ \times [4p^2\beta_{p-1,s-1} \cdot \beta_{p-1,r-1} \cdot {}_{p+1}\overset{q}{\prod} \cdot {}_p\overset{q}{\prod}_{r,s} + \beta_{p,s-1} \cdot \beta_{p,r-1} \cdot {}_{p+1}\overset{q}{\prod}_{p+1,r} \cdot {}_{p+1}\overset{q}{\prod}_{p+1,s}] \dots (31).$$

But

$${}_p\overset{q}{\prod}_{r,s} = {}_{p+1}\overset{q}{\prod}_{r,s} \frac{e_{p+1,p+1} \cdot e_{r,p+1} \cdot e_{s,p+1}}{C_{p+1}^2},$$

$${}_{p+1}\overset{q}{\prod}_{p+1,r} = {}_{p+1}\overset{q}{\prod}_{r,s} \frac{C_s}{C_{p+1}} \cdot \frac{e_{r,p+1}}{e_{r,s}},$$

and

$${}_{p+1}\overset{q}{\prod} = {}_p\overset{q}{\prod} \frac{C_{p+1}^2}{e_{p+1,p+1}},$$

$${}_{p+1}\overset{q}{\prod}_{p+1,s} = {}_p\overset{q}{\prod} \frac{C_{p+1}}{C_s} \cdot \frac{e_{s,p+1}}{e_{p+1,p+1}}.$$

Substituting these values in (31) we find

$${}_{p+1}\overset{q}{\Delta}_{r,s} = (-1)^{r+s} \cdot 1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)^2 \cdot 2^{p(p-1)} \cdot \beta_{p-1,s-1} \cdot \beta_{p-1,r-1} \cdot p^2 \cdot {}_{p+1}\overset{q}{\prod}_{r,s} \\ \times \frac{\left[4 + \frac{1}{(p-s+1)(p-r+1)e_{r,s} \cdot e_{p+1,p+1}} \right]}{\frac{1}{e_{r,p+1}} \cdot \frac{1}{e_{s,p+1}}},$$

and as the last fraction equals

$$\frac{1}{(p-s+1)(p-r+1)},$$

$${}_{p+1}\overset{q}{\Delta}_{r,s} = (-1)^{r+s} \beta_{p,s-1} \cdot \beta_{p,r-1} \{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)^2 \cdot 2^{p(p-1)} {}_{p+1}\overset{q}{\prod}_{r,s},$$

with which the proof by induction for (25) is carried through.

(8) We shall now return to (23). It consists of $2p+1$ terms of which the $(2r+1)$ st originally was found as $({}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2)$ so that

$${}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N} \left[\begin{array}{c|c} \begin{array}{cccc} 1 & 1 & \frac{1}{3} & \dots \frac{1}{2r-1} \\ x^2 & \frac{1}{3} & \frac{1}{5} & \dots \frac{1}{2r+1} \\ \vdots & \vdots & \vdots & \vdots \\ x^{2r} & \frac{1}{2r+1} & \frac{1}{2r+3} & \dots \frac{1}{4r-1} \end{array} & \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \\ \hline \begin{array}{ccc|ccc} 1 & \frac{1}{3} & \dots \frac{1}{2r-1} & 1 & \frac{1}{3} & \dots \frac{1}{2r+1} \\ \frac{1}{3} & \frac{1}{5} & \dots \frac{1}{2r+1} & \frac{1}{3} & \frac{1}{5} & \dots \frac{1}{2r+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2r-1} & \frac{1}{2r+1} & \dots \frac{1}{4r-3} & \frac{1}{2r+1} & \frac{1}{2r+3} & \dots \frac{1}{4r+1} \end{array} \end{array} \right]$$

and

$${}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N} \begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2r-1} \\ x^2 & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2r-2} & \frac{1}{2r+1} & \frac{1}{2r+3} & \cdots & \frac{1}{4r-3} \end{vmatrix}^2$$

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2r-1} \\ \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2r+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r-1} & \frac{1}{2r+1} & \cdots & \frac{1}{4r-5} \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2r+1} \\ \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2r+3} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2r+1} & \frac{1}{2r+3} & \cdots & \frac{1}{4r-1} \end{vmatrix}.$$

With the notations later adopted we therefore find

$${}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N} \frac{\left\{ \sum_{s=0}^{s=r} [x^{2s} \cdot {}_{r+1}\Delta_{s+1, r+1}] \right\}^2}{{}_r\Delta \cdot {}_{r+1}\Delta}$$

and

$${}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N} \frac{\left\{ \sum_{s=0}^{s=r-1} [x^{2s} \cdot {}_r\Delta_{s+1, r}] \right\}^2}{{}_{r-1}\Delta \cdot {}_r\Delta}.$$

Substituting the values for Δ 's from (24) and (25) we get

$${}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N \cdot 2^{2r} \cdot (|r|)^2} \left\{ \sum_{s=0}^{s=r} \left[(-1)^s \beta_{r,s} x^{2s} \frac{{}_{r+1}\prod_{s+1, r+1}}{\sqrt{{}_r\prod \cdot {}_{r+1}\prod}} \right] \right\}^2$$

and

$${}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 x^2}{N \cdot 2^{2r-2} \cdot (|r-1|)^2} \left\{ \sum_{s=0}^{s=r-1} \left[(-1)^s \beta_{r-1, s} x^{2s} \frac{{}_r\prod_{s+1, r}}{\sqrt{{}_{r-1}\prod \cdot {}_r\prod}} \right] \right\}^2,$$

or, as

$$\frac{{}_{r+1}\prod_{s+1, r+1}}{\sqrt{{}_r\prod \cdot {}_{r+1}\prod}} = \frac{e_{s+1, r+1}}{C_{s+1} \cdot \sqrt{e_{r+1, r+1}}} = \sqrt{4r+1} \cdot (2s+1)(2s+3) \cdots (2s+2r-1)$$

and

$$\frac{{}_r\prod_{s+1, r}}{\sqrt{{}_{r-1}\prod \cdot {}_r\prod}} = \frac{e_{s+1, r}}{C_{s+1} \cdot \sqrt{e_{r, r}}} = \sqrt{4r-1} \cdot (2s+3)(2s+5) \cdots (2s+2r-1)*,$$

$${}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2 (4r+1)}{N (|r|)^2 \cdot 2^{2r}} \left\{ \sum_{s=0}^{s=r} [(-1)^s \beta_{r,s} x^{2s} (2s+1)(2s+3) \cdots (2s+2r-1)] \right\}^2$$

.....(32),

* The e 's and C do not of course have the same value in the two equations as they represent columns and elements in two different determinants.

and

$${}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2 (4r-1) x^2}{N (r-1)^2 \cdot 2^{2r-2}} \left\{ \sum_{s=0}^{s=r-1} [(-1)^s \beta_{r-1,s} x^{2s} (2s+3)(2s+5) \dots (2s+2r-1)]^2 \right\} \dots \dots \dots (33),$$

which enables us to form ${}_n\sigma_y^2$ by successive summations from ${}_0\sigma_y^2 = \frac{\sigma^2}{N}$.

Before investigating the curve for ${}_n\sigma_y^2$ for a special n we shall first look at ${}_n\sigma_y^2$ for $x=0$ and $x=\pm 1$.

(9) From (33) we see that when $x=0$

$${}_{2r-1}\sigma_y^2 = {}_{2r-2}\sigma_y^2,$$

${}_{2p}\sigma_y^2$ is for $x=0$ most easily evaluated from the formula (13).

Remembering that in our case $m_{2r} = \frac{1}{2r+1}$ we find from this

$${}_{2p}\sigma_y^2 = \frac{\sigma^2}{N} \frac{\Delta_{1,1}^1}{\Delta_{p+1}^1},$$

and hence by (24) and (25)

$${}_{2p+1}\sigma_y^2 = {}_{2p}\sigma_y^2 = \frac{\sigma^2}{N} \left\{ \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \dots \frac{2p+1}{2p} \right\}^2 \dots \dots \dots (34).$$

(10) To evaluate ${}_n\sigma_y^2$ for $x=\pm 1$ we use (32) and (33). The sum in (32) may be considered as

$$S_{2r} = \frac{d}{dx} \frac{1}{x} \frac{d}{dx} \dots \frac{1}{x} \frac{d}{dx} \{x^{2r-1} (x^2-1)^r\}$$

with a number r of differentiations. If these operations are undertaken directly upon $x^{2r-1} (x^2-1)^r$ the result is

$$\alpha_r (x^2-1)^r + \alpha_{r-1} (x^2-1)^{r-1} + \dots \alpha_1 (x^2-1) + \alpha_0,$$

of which only

$$\alpha_0 = 2r(2r-2) \dots 4 \cdot 2 = \underline{r \cdot 2^r}$$

remains for $x=\pm 1$.

Corresponding to this the sum in (33) comes out from

$$S_{2r-1} = \frac{d}{dx} \frac{1}{x} \frac{d}{dx} \dots \frac{d}{dx} \frac{1}{x} \frac{d}{dx} \{x^{2r-1} (x^2-1)^{r-1}\}$$

by taking $(r-1)$ differentiations and therefore

$$S_{2r-1} \text{ equals, for } x=\pm 1, (2r-2)(2r-4) \dots 4 \cdot 2 = \underline{r-1 \cdot 2^{r-1}}.$$

Hence

$${}_{2r}\sigma_y^2 - {}_{2r-1}\sigma_y^2 = \frac{\sigma^2}{N} (4r+1)$$

and

$${}_{2r-1}\sigma_y^2 - {}_{2r-2}\sigma_y^2 = \frac{\sigma^2}{N} (4r-1),$$

or since

$${}_0\sigma_y^2 = \frac{\sigma^2}{N},$$

$${}_n\sigma_y^2 = \frac{\sigma^2}{N} \{1 + 3 + 5 + \dots (2n+1)\},$$

$${}_n\sigma_y^2 = \frac{\sigma^2}{N} (n+1)^2 \dots \dots \dots (35).$$

(11) In Section I under (7) it was found that

$$\int \frac{\psi(x)}{f(x)} {}_n\sigma_y^2 dx = \sigma^2 (n+1)$$

when the integration was taken over the places of observation. For the present distribution $f(x)$ is 1, $\psi(x)$ constant and $\int \psi(x) dx = N$, hence the mean of ${}_n\sigma_y^2$ in the range of observations is for a uniform continuous distribution

$$\frac{\sigma^2}{N} (n+1).$$

For the grouped observations in Section II we find by integration of the formulae for functions from the first to the sixth degree that

$$\frac{1}{2} \int_{-1}^1 {}_n\sigma_y^2 dx = \frac{\sigma^2}{N} (n+1) \left(1 - \frac{1}{2n+1}\right).$$

IV. *Uniform continuous distribution of observations with constant standard deviation. Special formulae.*

(1) Let ${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2$ be indicated by S_n , then the formulae (32) and (33) give us

$$\left. \begin{aligned} S_1 &= \frac{\sigma^2}{N} \cdot 3x^2 \\ S_2 &= \frac{\sigma^2}{N} \cdot \frac{5}{4} (1 - 3x^2)^2 \\ S_3 &= \frac{\sigma^2}{N} \cdot \frac{7}{4} x^2 (3 - 5x^2)^2 \\ S_4 &= \frac{\sigma^2}{N} \cdot \frac{9}{64} (3 - 30x^2 + 35x^4)^2 \\ S_5 &= \frac{\sigma^2}{N} \cdot \frac{11x^2}{64} (15 - 70x^2 + 63x^4)^2 \\ S_6 &= \frac{\sigma^2}{N} \cdot \frac{13}{9 \times 256} (15 - 315x^2 + 945x^4 - 693x^6)^2 \end{aligned} \right\} \dots\dots\dots(36),$$

from which we form ${}_n\sigma_y^2$ beginning with

$$\left. \begin{aligned} {}_0\sigma_y^2 &= \frac{\sigma^2}{N} \\ {}_1\sigma_y^2 &= \frac{\sigma^2}{N} (1 + 3x^2) \\ {}_2\sigma_y^2 &= \frac{\sigma^2}{N} (1 + 3x^2 + \frac{5}{4} (1 - 3x^2)^2) = \frac{\sigma^2}{N} \cdot \frac{9}{4} (1 - 2x^2 + 5x^4), \\ \text{and further in the same way} \\ {}_3\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{1}{4} (9 + 45x^2 - 165x^4 + 175x^6) \\ {}_4\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{25}{64} (9 - 36x^2 + 294x^4 - 644x^6 + 441x^8) \\ {}_5\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{9}{64} (25 + 175x^2 - 1750x^4 + 6510x^6 - 9555x^8 + 4851x^{10}) \\ {}_6\sigma_y^2 &= \frac{\sigma^2}{N} \cdot \frac{7}{256} (175 - 1050x^2 + 17325x^4 - 93660x^6 + 225225x^8 \\ &\quad - 245322x^{10} + 99099x^{12}) \end{aligned} \right\} \dots\dots\dots(37).$$

(2) Since ${}_n\sigma_y^2 = {}_{n-1}\sigma_y^2 + S_n$ the curve for ${}_n\sigma_y^2$ is entirely above the ${}_{n-1}\sigma_y^2$ curve except where $S_n = 0$.

Solving the equations $S_n = 0$ the following roots are found:

$$\begin{array}{ll} \text{For } S_1 = 0 & x = 0 \\ \text{,, } S_2 = 0 & x = \pm \sqrt{\frac{1}{3}} = \pm .5773 \\ \text{,, } S_3 = 0 & x = 0 \quad x = \pm \sqrt{\frac{3}{5}} = \pm .7746 \\ \text{,, } S_4 = 0 & x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}} = \pm \begin{cases} .8611 \\ .3400 \end{cases} \\ \text{,, } S_5 = 0 & x = 0 \quad x = \pm \sqrt{\frac{35 \pm 2\sqrt{70}}{63}} = \pm \begin{cases} .9030 \\ .5438 \end{cases} \\ \text{,, } S_6 = 0 & x = \pm \begin{cases} .2386 \\ .6612 \\ .9325 \end{cases} \end{array}$$

Since all the roots are rational and all lie between -1 and $+1$, ${}_n\sigma_y^2$ therefore equals ${}_{n-1}\sigma_y^2$ for n values of x all of which are inside the range of the observations.

The adjusted values of the functions at these abscissae appear to be of special interest since they are uncorrelated as was shown in Section I under (9).

(3) Looking at Diagram 2, representing the curves of ${}_n\sigma_y$ up to $n = 6$, it is seen, as was also clear from the formula for σ_y^2 $\overset{x=0}{}$ and σ_y^2 $\overset{x^2=1}{}$ given in the last section, that while the standard deviation in the middle of the range increases slowly with the degree of function it increases very rapidly at the ends of the range. At $x = 0$ the curve has a minimum when the degree of function is odd and a maximum when it is even. Besides that the curve has $(2n - 2)$ maxima and minima between -1 and 1 . As the curve for ${}_n\sigma_y^2$ is of the $2n$ th degree, ${}_n\sigma_y^2$ is therefore increasing for x increasing above 1 or for x decreasing below -1 .

The abscissae of the maxima and minima are given in the following table.

Degree of function	Abscissae of maxima	Abscissae of minima
1		0
2	0	$\pm \sqrt{\frac{1}{5}} = \pm .4472$
3	$\pm \sqrt{\frac{1}{5}} = \pm .4472$	$\begin{cases} 0 \\ \pm \sqrt{\frac{3}{5}} = \pm .6547 \end{cases}$
4	$\begin{cases} 0 \\ \pm \sqrt{\frac{3}{7}} = \pm .6547 \end{cases}$	$\pm \sqrt{\frac{7 \pm 2\sqrt{7}}{21}} = \pm \begin{cases} .7651 \\ .2852 \end{cases}$
5	$\pm \sqrt{\frac{7 \pm 2\sqrt{7}}{21}} = \pm \begin{cases} .7651 \\ .2852 \end{cases}$	$\begin{cases} 0 \\ \pm \sqrt{\frac{15 \pm 2\sqrt{15}}{33}} = \pm \begin{cases} .8302 \\ .4689 \end{cases} \end{cases}$
6	$\pm \sqrt{\frac{15 \pm 2\sqrt{15}}{33}} = \pm \begin{cases} .8302 \\ .4689 \end{cases}$	$\pm \begin{cases} .8718 \\ .5917 \\ .2093 \end{cases}$

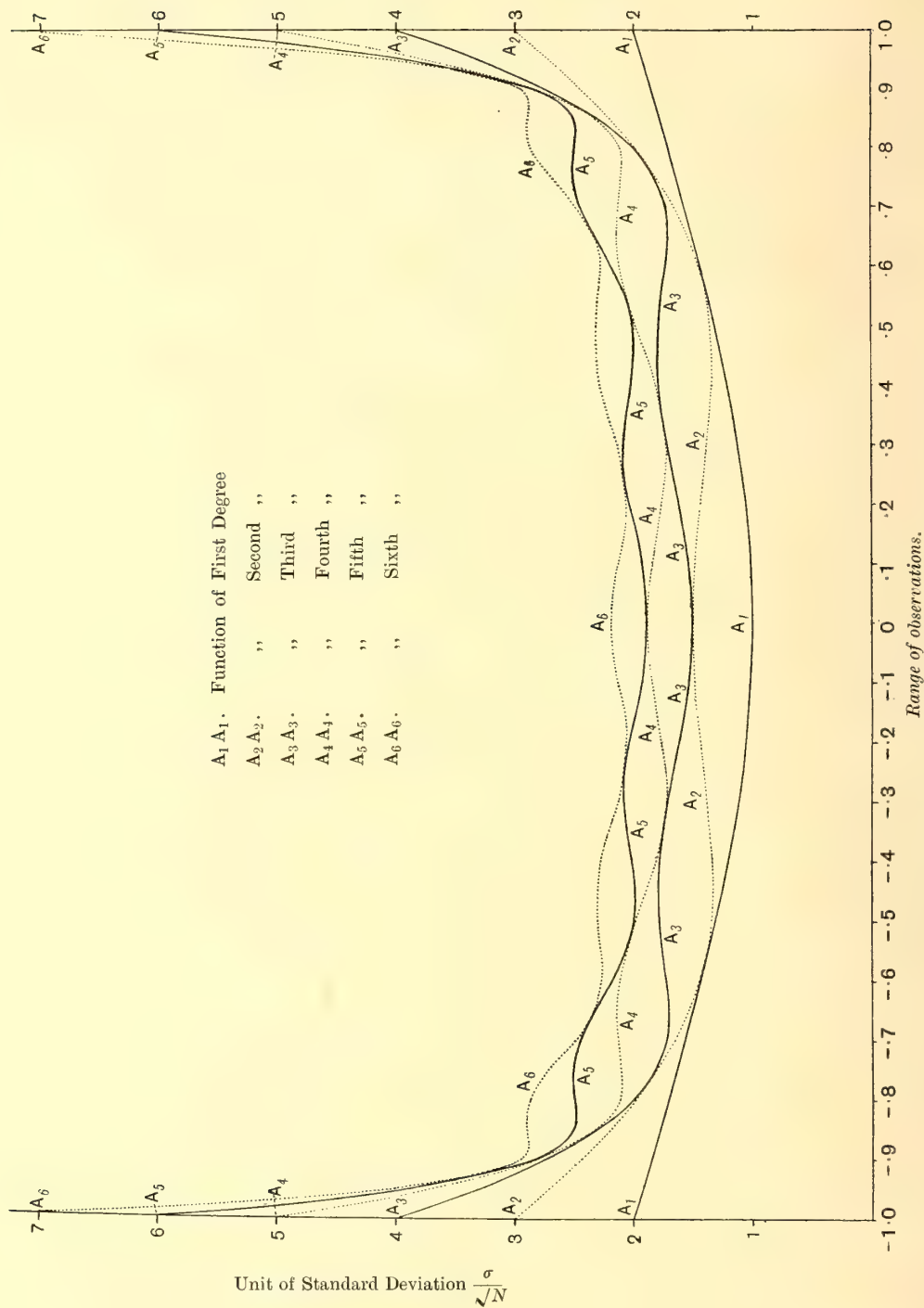


DIAGRAM 2. Curves of Standard Deviations. Uniform Distribution.

Hence the curve for ${}_{2p+1}\sigma_y^2$ has a maximum for the abscissae at which ${}_{2p}\sigma_y^2$ has a minimum. A comparison with the results in Section II shows that the abscissae of the maxima found here are the same as those of the best places of observation for $(n+1)$ equally big groups of observations of a function of the n th degree. These places tally with the places where ${}_n\sigma_y^2$ was a maximum. Thus if we imagine that we had started the investigations with a uniform distribution of observations, and to lower the maxima of the curve of standard deviation had put clusters of observations at those maxima and at the ends of the range we should not get the best curve of standard deviation till all the observations of the continuous distribution had been distributed at the $n-1$ places of maxima and at 1 and -1 .

The minima of the standard deviations obtained from a uniform continuous distribution and the $(n+1)$ best groups of observations do not fall at the same abscissae.

(4) The curves are very far from our ideal of a constant standard deviation throughout the range. To obtain the same maximum of standard deviation as $(n+1)$ groups could give us we should have to limit the part of the range used to the following fractions of the range:

for 1st degree	·58
„ 2nd „	·73
„ 3rd „	·80
„ 4th „	·84
„ 5th „	·83
„ 6th „	·73

It is not likely that the range of values of the function which we investigate would only be of interest inside a range so much smaller than that within which we might actually observe; further it seems likely that observations all of which were taken inside the smaller part of the range would give better information for that special interval. I shall therefore examine in the following sections if a uniform distribution of observations to which is added clusters of observations at the ends of the range will not possibly give a more satisfactory curve of standard deviations.

V. *Uniform continuous distribution of observations with additional observations clustered at the ends of the range; constant standard deviation of observations. General formulae.*

(1) Suppose we have $N \cdot \frac{1}{1+\alpha}$ observations uniformly distributed from -1 to 1 and besides $\frac{N}{2} \cdot \frac{\alpha}{1+\alpha}$ observations at -1 and the same number at 1. We then have

$$\mu_{2r} = \frac{1}{N} \left\{ \int_{-1}^1 \frac{Nx^{2r}}{2(1+\alpha)} dx + \frac{Na}{1+\alpha} \right\}$$

or

$$\mu_{2r} = \frac{1}{1+a} \left(\frac{1}{2r+1} + a \right)$$

and

$$\mu_{2r+1} = 0.$$

According to (13) and (14) we find,

$${}_{2p}\sigma_y^2 = \frac{-\sigma^2(1+a)}{N} \left\{ \begin{array}{c} \begin{array}{cccc} 0 & 1 & x^2 & \dots\dots x^{2p} \\ 1 & 1+a & \frac{1}{3}+a & \dots\dots \frac{1}{2p+1}+a \\ x^2 & \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots & \vdots \\ x^{2p} & \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p+1}+a \end{array} \\ \hline \begin{array}{cccc} 1+a & \frac{1}{3}+a & \dots\dots \frac{1}{2p+1}+a \\ \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots \\ \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p+1}+a \end{array} \end{array} \right. \\ + x^2 \left\{ \begin{array}{cccc} 0 & 1 & x^2 & \dots\dots x^{2p-2} \\ 1 & \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ x^2 & \frac{1}{5}+a & \frac{1}{7}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots & \vdots \\ x^{2p-2} & \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p-1}+a \end{array} \right\} \dots (38) \\ \begin{array}{cccc} \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ \frac{1}{5}+a & \frac{1}{7}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots \\ \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p-1}+a \end{array} \end{array}$$

and

$$\begin{aligned}
{}_{2p-1}\sigma_y^2 = & -\frac{\sigma^2(1+a)}{N} \left\{ \begin{array}{cccc} 0 & 1 & x^2 & \dots\dots x^{2p-2} \\ 1 & 1+a & \frac{1}{3}+a & \dots\dots \frac{1}{2p-1}+a \\ x^2 & \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ \vdots & \vdots & \vdots & \vdots \\ x^{2p-2} & \frac{1}{2p-1}+a & \frac{1}{2p+1}+a & \dots\dots \frac{1}{4p-3}+a \end{array} \right. \\
& \left. \begin{array}{cccc} 1+a & \frac{1}{3}+a & \dots\dots \frac{1}{2p-1}+a \\ \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ \vdots & \vdots & \vdots \\ \frac{1}{2p-1}+a & \frac{1}{2p+1}+a & \dots\dots \frac{1}{4p-3}+a \end{array} \right\} \\
& + x^2 \left\{ \begin{array}{cccc} 0 & 1 & x^2 & \dots\dots x^{2p-2} \\ 1 & \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ x^2 & \frac{1}{5}+a & \frac{1}{7}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots & \vdots \\ x^{2p-2} & \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p-1}+a \end{array} \right. \\
& \left. \begin{array}{cccc} \frac{1}{3}+a & \frac{1}{5}+a & \dots\dots \frac{1}{2p+1}+a \\ \frac{1}{5}+a & \frac{1}{7}+a & \dots\dots \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots \\ \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots\dots \frac{1}{4p-1}+a \end{array} \right\} \\
& \dots\dots\dots(39),
\end{aligned}$$

and, according to I under (6),

$${}_{2p}\sigma_{ij}^2 - {}_{2p-1}\sigma_{ij}^2 = \frac{\sigma^2}{N}(1+a) \times$$

$$\begin{vmatrix} 1 & 1+a & \frac{1}{3}+a & \dots & \frac{1}{2p-1}+a \\ x^2 & \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+1}+a \\ x^4 & \frac{1}{5}+a & \frac{1}{7}+a & \dots & \frac{1}{2p+3}+a \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots & \frac{1}{4p-1}+a \end{vmatrix}^2$$

$$\begin{vmatrix} 1+a & \frac{1}{3}+a & \dots & \frac{1}{2p-1}+a \\ \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+1}+a \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p-1}+a & \frac{1}{2p+1}+a & \dots & \frac{1}{4p-3}+a \end{vmatrix} \cdot \begin{vmatrix} 1+a & \frac{1}{3}+a & \dots & \frac{1}{2p+1}+a \\ \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+3}+a \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots & \frac{1}{4p+1}+a \end{vmatrix} \dots \dots \dots (40)$$

and

$${}_{2p+1}\sigma^2 - {}_{2p}\sigma_{ij}^2 = \frac{\sigma^2(1+a)}{N} x^2 \times$$

$$\begin{vmatrix} 1 & \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+1}+a \\ x^2 & \frac{1}{5}+a & \frac{1}{7}+a & \dots & \frac{1}{2p+3}+a \\ x^4 & \frac{1}{7}+a & \frac{1}{9}+a & \dots & \frac{1}{2p+5}+a \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2p+3}+a & \frac{1}{2p+5}+a & \dots & \frac{1}{4p+1}+a \end{vmatrix}^2$$

$$\begin{vmatrix} \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+1}+a \\ \frac{1}{5}+a & \frac{1}{7}+a & \dots & \frac{1}{2p+3}+a \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+1}+a & \frac{1}{2p+3}+a & \dots & \frac{1}{4p-1}+a \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{3}+a & \frac{1}{5}+a & \dots & \frac{1}{2p+3}+a \\ \frac{1}{5}+a & \frac{1}{7}+a & \dots & \frac{1}{2p+5}+a \\ \vdots & \vdots & & \vdots \\ \frac{1}{2p+3}+a & \frac{1}{2p+5}+a & \dots & \frac{1}{4p+3}+a \end{vmatrix} \dots \dots \dots (41).$$

(2) For the reduction of these formulae we have to evaluate the determinant of p th order

$${}_p\delta^q = \begin{vmatrix} \frac{1}{2q-1} + \alpha & \frac{1}{2q+1} + \alpha & \cdots & \frac{1}{2q+2p-3} + \alpha \\ \frac{1}{2q+1} + \alpha & \frac{1}{2q+3} + \alpha & \cdots & \frac{1}{2q+2p-1} + \alpha \\ \vdots & \vdots & & \vdots \\ \frac{1}{2q+2p-3} + \alpha & \frac{1}{2q+2p-1} + \alpha & \cdots & \frac{1}{2q+4p-5} + \alpha \end{vmatrix}.$$

By subtracting from the elements of each row the elements of the proceeding and leaving the first row as it is, it is transformed to

$${}_p\delta^q = (-1)^{p-1} \times \begin{vmatrix} \frac{1}{2q-1} + \alpha & \frac{1}{2q+1} + \alpha & \cdots & \frac{1}{2q+2p-3} + \alpha \\ \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\ \vdots & \vdots & & \vdots \\ \frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2}{(2q+2p-3)(2q+2p-1)} & \cdots & \frac{2}{(2q+4p-7)(2q+4p-5)} \end{vmatrix},$$

which when the columns undergo the same process takes the form

$${}_p\delta^q = \begin{vmatrix} \frac{1}{2q-1} + \alpha & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-5)(2q+2p-3)} \\ \frac{2}{(2q-1)(2q+1)} & \frac{2 \cdot 4}{(2q-1)(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2 \cdot 4}{(2q+2p-5) \dots (2q+2p-1)} \\ \frac{2}{(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \frac{2 \cdot 4}{(2q+3)(2q+5)(2q+7)} & \cdots & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2 \cdot 4}{(2q+2p-5) \dots (2q+2p-1)} & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} & \cdots & \frac{2 \cdot 4}{(2q+4p-9) \dots (2q+4p-5)} \end{vmatrix}.$$

Let us introduce the notation

$${}_p\bar{D}^q = \begin{vmatrix} \frac{1}{(2q-1)(2q+1)(2q+3)} & \frac{1}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{1}{(2q+2p-3) \dots (2q+2p+1)} \\ \frac{1}{(2q+1)(2q+3)(2q+5)} & \frac{1}{(2q+3)(2q+5)(2q+7)} & \cdots & \frac{1}{(2q+2p-1) \dots (2q+2p+3)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(2q+2p-3) \dots (2q+2p+1)} & \frac{1}{(2q+2p-1) \dots (2q+2p+3)} & \cdots & \frac{1}{(2q+4p-5) \dots (2q+4p-1)} \end{vmatrix}.$$

Then, since for $\alpha = 0$ ${}_p\delta^q$ equals the determinant ${}_p\bar{D}^q$, we have

$${}_p\delta^q = {}_p\bar{D}^q + \alpha \cdot 2^{3(p-1)} \cdot {}_{p-1}\bar{D}^q \dots \dots \dots (42),$$

and the problem is reduced to the evaluation of ${}_p\bar{D}^q$.

(3) It shall be proved by induction that

$${}_p^q D = \frac{\{1^p \cdot 2^{p-1} \dots (p-1)^2 \cdot p^{\frac{1}{2}} \cdot 2^{p(p-2)} (p+1)\}}{(2q-1)(2q+1)^2(2q+3)^3 \dots (2q+2p-5)^{p-1}(2q+2p-3)^p(2q+2p-1)^p(2q+2p+1)^p(2q+2p+3)^{p-1} \dots (2q+4p-3)^2(2q+4p-1)} \dots (43).$$

It contains the $2p+1$ different factors of the elements with indices increasing from 1 at the extreme to p in the middle so that the three factors of which the one diagonal line of the determinant consists occur with the index p .

For $p=1$ the formula gives

$${}_1^q D = \frac{1}{(2q-1)(2q+1)(2q+3)}$$

as it ought to.

As the determinant is orthosymmetrical the relation

$$\Delta = \frac{\Delta_{ss} \cdot \Delta_{s's'} - \Delta_{ss'}^2}{\Delta_{ss's'}} \text{ holds.}$$

Applied on ${}_{p+1}^q D$ for $s=1$ and $s'=p+1$ it may be written

$${}_{p+1}^q D = \frac{{}_p^q D \cdot {}_p^{q+2} D - {}_p^{q+1} D^2}{{}_{p-1}^{q+2} D} \dots (44).$$

Looking first at the numerator of (43) we see that it has the same value for the two terms of the numerator of (44), and divided by the corresponding factor of ${}_{p-1}^{q+2} D$ it becomes

$$\left\{ \frac{1^{2p} \cdot 2^{2(p-1)} \dots (p-2)^6 (p-1)^4 p^2}{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)} \right\}^2 \frac{(p+1)^2}{p} \cdot 2^{2p(p-2)-(p-1)(p-3)} \\ = \{1^{p+1} \cdot 2^p \dots (p-2)^4 (p-1)^3 p^2 (p+1)\}^2 \frac{1}{p} \cdot 2^{p^2-3}.$$

To evaluate the factor in ${}_{p+1}^q D$ arising from the denominator of (43) we shall give a table of the indices with which the different factors occur in the D 's and their ratios.

	$2q-1$	$2q+1$	$2q+3$	...	$2q+2p-5$	$2q+2p-3$	$2q+2p-1$	$2q+2p+1$	$2q+2p+3$	$2q+2p+5$	$2q+2p+7$	...	$2q+4p-1$	$2q+4p+1$	$2q+4p+3$
${}_p^q D$	1	2	3	...	$p-1$	p	p	p	$p-1$	$p-2$	$p-3$	...	1	—	—
${}_p^{q+2} D$	—	—	1	...	$p-3$	$p-2$	$p-1$	p	p	p	$p-1$	...	3	2	1
${}_{p-1}^{q+2} D$	—	—	1	...	$p-3$	$p-2$	$p-1$	$p-1$	$p-1$	$p-2$	$p-3$	...	1	—	—
${}_p^{q+1} D^2$	—	2	4	...	$2(p-2)$	$2(p-1)$	$2p$	$2p$	$2p$	$2(p-1)$	$2(p-2)$	...	4	2	—
$\frac{{}_p^q D \cdot {}_p^{q+2} D}{{}_{p-1}^{q+2} D}$	1	2	3	...	$p-1$	p	p	$p+1$	p	p	$p-1$	...	3	2	1
$\frac{{}_p^{q+1} D^2}{{}_{p-1}^{q+2} D}$	—	2	3	...	$p-1$	p	$p+1$	$p+1$	$p+1$	p	$p-1$	...	3	2	—

Hence the factor arising from the denominator of (43) is

$$\frac{(2q+2p-1)(2q+2p+3)-(2q-1)(2q+4p+3)}{(2q-1)(2q+1)^2 \dots (2q+2p-3)^p (2q+2p-1)^{p+1} (2q+2p+1)^{p+1} (2q+2p+3)^{p+1} (2q+2p+5)^p \dots (2q+4p+1)^2 (2q+4p+3)}.$$

The numerator of this equals $4p(p+2)$,

multiplying with the factor previously found we therefore get

$$\frac{1}{p+1} \frac{1}{D} = \frac{\{1^{p+1} \cdot 2^p \dots (p-1)^3 p^2 (p+1)\}^2 \cdot 2^{(p+1)(p-1)} \cdot (p+2)}{(2q-1)(2q+1)^2 \dots (2q+2p-3)^p (2q+2p-1)^{p+1} (2q+2p+1)^{p+1} (2q+2p+3)^{p+1} (2q+2p+5)^p \dots (2q+4p+1)^2 (2q+4p+3)},$$

which is what we wanted to prove.

(4) When the values of Δ and D are introduced in (42) we get

$$\frac{1}{p\delta} = \frac{\{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)}}{(2q-1)(2q+1)^2 \dots (2q+2p-5)^{p-1} (2q+2p-3)^p (2q+2p-1)^{p-1} \dots (2q+4p-7)^2 (2q+4p-5)} \\ + \alpha \cdot 2^{3(p-1)} \times \frac{\{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{(p-1)(p-3)} \cdot p}{(2q-1)(2q+1)^2 \dots (2q+2p-7)^{p-2} (2q+2p-5)^{p-1} (2q+2p-3)^p (2q+2p-1)^{p-1} (2q+2p+1)^{p-2} \dots (2q+4p-7)^2 (2q+4p-5)} \\ \text{or } \frac{1}{p\delta} = \frac{\{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1)\}^2 \cdot 2^{p(p-1)} [1 + \alpha p (2q+2p-3)]}{(2q-1)(2q+1)^2 \dots (2q+2p-5)^{p-1} (2q+2p-3)^p (2q+2p-1)^{p-1} \dots (2q+4p-7)^2 (2q+4p-5)} \dots (45).$$

The denominators of the formulae (38)–(41) for ${}_n\sigma_y^2$ are now known since they only consist of the factors $\frac{1}{x}\delta$ and $\frac{2}{x}\delta$. To be able to write down the general expression for ${}_n\sigma_y^2$ we should have to evaluate the minors of δ , but their form is so complicated that a direct calculation of the determinants for the degrees of function in question appears to be simpler. With the material in hand we are however able to determine ${}_n\sigma_y^2$ for $x=0$ and $x^2=1$.

(5) From (38) and (39) we see that

$$\frac{x=0}{2p}\sigma_y^2 = \frac{x=0}{2p+1}\sigma_y^2 = \frac{\sigma^2}{N} \frac{x\delta^3}{x+1\delta} (1+\alpha), \text{ and with the } \delta\text{'s as given by (45)}$$

$$\frac{x=0}{2p}\sigma_y^2 = \frac{x=0}{2p+1}\sigma_y^2 = \frac{\sigma^2 (1+\alpha) [1 + \alpha p (2p+3)] 1 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 9^5 \dots (2p-1)^p (2p+1)^{p+1} (2p+3)^p (2p+5)^{p-1} \dots (4p-1)^2 (4p+1)}{N \{1 \cdot 2 \cdot 3 \dots p\}^2 \cdot 2^{2p} \cdot [1 + \alpha (p+1) (2p+1)] 5 \cdot 7^2 \cdot 9^3 \dots (2p-1)^{p-2} (2p+1)^{p-1} (2p+3)^p (2p+5)^{p-1} \dots (4p-7)^2 (4p+1)} \\ = \frac{\sigma^2 (1+\alpha) 3^2 \cdot 5^2 \dots (2p-1)^2 (2p+1)^2 \cdot [1 + \alpha p (2p+3)]}{N \{1 \cdot 2 \cdot 3 \dots p\}^2 \cdot 2^{2p} \cdot [1 + \alpha (p+1) (2p+1)]} \\ \text{or } \frac{x=0}{2p}\sigma_y^2 = \frac{x=0}{2p+1}\sigma_y^2 = \frac{\sigma^2 \left\{ \frac{3}{2} \cdot \frac{5}{4} \dots \frac{2p-1}{2p-2} \cdot \frac{2p+1}{2p} \right\}^2 (1+\alpha) [1 + \alpha p (2p+3)]}{N [1 + \alpha (p+1) (2p+1)]} \dots (46).$$

(6) To find ${}_n\sigma_y^2$ we have to evaluate the determinant of $(p+1)$ st order,

$$\begin{vmatrix} 0 & 1 & & 1 & & & & 1 \\ 1 & & \frac{1}{2q-1} + \alpha & & \frac{1}{2q+1} + \alpha & & & \frac{1}{2q+2p-3} + \alpha \\ 1 & & \frac{1}{2q+1} + \alpha & & \frac{1}{2q+3} + \alpha & & & \frac{1}{2q+2p-1} + \alpha \\ \vdots & & \vdots & & \vdots & & & \vdots \\ 1 & & \frac{1}{2q+2p-3} + \alpha & & \frac{1}{2q+2p-1} + \alpha & & \dots & \frac{1}{2q+4p-5} + \alpha \end{vmatrix}.$$

Treating it as ${}_p\delta^q$ was treated under (2) of this section, except that now two rows or columns are left unaltered, it takes the form

$$\begin{array}{cccccc}
 0 & 1 & 0 & 0 & \dots & 0 \\
 1 & \frac{1}{2q-1} + \alpha & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \dots & \frac{2}{(2q+2p-5)(2q+2p-3)} \\
 0 & \frac{2}{(2q-1)(2q+1)} & \frac{2 \cdot 4}{(2q-1)(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \dots & \frac{2 \cdot 4}{(2q+2p-5) \dots (2q+2p-1)} \\
 0 & \frac{2}{(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \frac{2 \cdot 4}{(2q+3)(2q+5)(2q+7)} & \dots & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} \\
 \vdots & \vdots & \vdots & \vdots & & \vdots \\
 0 & \frac{2}{(2q+2p-5)(2q+2p-3)} & \frac{2 \cdot 4}{(2q+2p-5) \dots (2q+2p-1)} & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} & \dots & \frac{2 \cdot 4}{(2q+4p-9) \dots (2q+4p-5)}
 \end{array}$$

$$= -2^{3(p-1)} {}_{p-1}\bar{D}.$$

Hence we find from (38),

$$\frac{x^2=1}{2p\sigma_y^2} = \frac{\sigma^2}{N} (1 + \alpha) \left\{ \frac{2^{3p} {}_p\bar{D}}{{}_{p+1}\delta} + \frac{2^{3(p-1)} {}_{p-1}\bar{D}}{{}_p\delta} \right\}.$$

Now from (43) and (45) we get

$$\frac{2^{3p} {}_p\bar{D}}{{}_{p+1}\delta} = \frac{(p+1)(2q+2p-1)}{[1 + \alpha(p+1)(2q+2p-1)]} \dots \dots \dots (47),$$

and therefore

$$\begin{aligned}
 \frac{x^2=1}{2p\sigma_y^2} &= \frac{\sigma^2}{N} (1 + \alpha) \left\{ \frac{(p+1)(2p+1)}{1 + \alpha(p+1)(2p+1)} + \frac{p(2p+1)}{1 + \alpha p(2p+1)} \right\} \\
 \text{or } \frac{x^2=1}{2p\sigma_y^2} &= \frac{\sigma^2}{N} (1 + \alpha) (2p+1) \left\{ \frac{p+1}{1 + \alpha(p+1)(2p+1)} + \frac{p}{1 + \alpha p(2p+1)} \right\} \dots (48).
 \end{aligned}$$

In the same way we get from (39),

$$\frac{x^2=1}{2p-1\sigma_y^2} = \frac{\sigma^2}{N} (1 + \alpha) \left\{ \frac{2^{3(p-1)} {}_{p-1}\bar{D}}{{}_p\delta} + \frac{2^{3(p-1)} {}_{p-1}\bar{D}}{{}_p\delta} \right\},$$

which by the relation between ${}_p\bar{D}$ and ${}_{p+1}\delta$ just found is reduced to

$$\frac{x^2=1}{2p-1\sigma_y^2} = \frac{\sigma^2}{N} (1 + \alpha) \left\{ \frac{p(2p-1)}{1 + \alpha p(2p-1)} + \frac{p(2p+1)}{1 + \alpha p(2p+1)} \right\} \dots \dots \dots (49).$$

Both (48) and (49) are covered by the formula

$$\frac{x^2=1}{n\sigma_y^2} = \frac{\sigma^2}{N} (1 + \alpha) (n+1) \left\{ \frac{n}{2 + \alpha n(n+1)} + \frac{n+2}{2 + \alpha(n+1)(n+2)} \right\} \dots (50).$$

(7) The evaluation of ${}_n\sigma_y^2$ for special values of n can be made easier by a transformation of the determinant

$${}_{x+1}d^q = \begin{vmatrix} 1 & \frac{1}{2q-1} + \alpha & \frac{1}{2q+1} + \alpha & \cdots & \frac{1}{2q+2p-3} + \alpha \\ x^2 & \frac{1}{2q+1} + \alpha & \frac{1}{2q+3} + \alpha & \cdots & \frac{1}{2q+2p-1} + \alpha \\ x^4 & \frac{1}{2q+3} + \alpha & \frac{1}{2q+5} + \alpha & \cdots & \frac{1}{2q+2p+1} + \alpha \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p} & \frac{1}{2q+2p-1} + \alpha & \frac{1}{2q+2p+1} + \alpha & \cdots & \frac{1}{2q+4p-3} + \alpha \end{vmatrix}.$$

Leaving the first row unaltered and subtracting from each of the others the proceeding we get a determinant the first column of which is

$$1, x^2 - 1, x^2(x^2 - 1) \dots x^{2p-2}(x^2 - 1),$$

while the other columns are identical with those of the determinant δ previously treated in the same way. When next the two first rows are left as they are and from each of the others is subtracted the proceeding one the result is

$${}_{x+1}d^q = (-1)^{2p-1} \times \begin{vmatrix} 1 & \frac{1}{2q-1} + \alpha & \frac{1}{2q+1} + \alpha & \cdots & \frac{1}{2q+2p-3} + \alpha \\ 1 - x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\ (1 - x^2)^2 & \frac{2 \cdot 4}{(2q-1)(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} \\ \vdots & \vdots & \vdots & & \vdots \\ x^{2p-4}(1 - x^2)^2 & \frac{2 \cdot 4}{(2q+2p-5) \dots (2q+2p-1)} & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} & \cdots & \frac{2 \cdot 4}{(2q+4p-7) \dots (2q+4p-3)} \end{vmatrix}.$$

Leaving now three rows unaltered, next time four and so on, it is clear that we shall at last after p of these sets of operations get

$${}_{x+1}d^q = (-1)^{\frac{p(p+1)}{2}} \times \begin{vmatrix} 1 & \frac{1}{2q-1} + \alpha & \frac{1}{2q+1} + \alpha & \cdots & \frac{1}{2q+2p-3} + \alpha \\ 1 - x^2 & \frac{2}{(2q-1)(2q+1)} & \frac{2}{(2q+1)(2q+3)} & \cdots & \frac{2}{(2q+2p-3)(2q+2p-1)} \\ (1 - x^2)^2 & \frac{2 \cdot 4}{(2q-1)(2q+1)(2q+3)} & \frac{2 \cdot 4}{(2q+1)(2q+3)(2q+5)} & \cdots & \frac{2 \cdot 4}{(2q+2p-3) \dots (2q+2p+1)} \\ \vdots & \vdots & \vdots & & \vdots \\ (1 - x^2)^p & \frac{2 \cdot 4 \dots 2p}{(2q-1) \dots (2q+2p-1)} & \frac{2 \cdot 4 \dots 2p}{(2q+1) \dots (2q+2p+1)} & \cdots & \frac{2 \cdot 4 \dots 2p}{(2q+2p-3) \dots (2q+4p-3)} \end{vmatrix}.$$

By treating the columns in the same way, leaving first two then three and so on unaltered, we find after the first set of operations

$${}_{p+1}d^q = (-1)^{\frac{p(p+1)}{2} + (p-1)} \times$$

1	$\frac{1}{2q-1} + a$	$\frac{2}{(2q-1)(2q+1)}$	$\cdots \frac{2}{(2q+2p-5)(2q+2p-3)}$
$1-x^2$	$\frac{2}{(2q-1)(2q+1)}$	$\frac{2.4}{(2q-1)(2q+1)(2q+3)}$	$\cdots \frac{2.4}{(2q+2p-5)\dots(2q+2p-1)}$
$(1-x^2)^2$	$\frac{2.4}{(2q-1)(2q+1)(2q+3)}$	$\frac{2.4.6}{(2q-1)(2q+1)(2q+3)(2q+5)}$	$\cdots \frac{2.4.6}{(2q+2p-5)\dots(2q+2p+1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1-x^2)^p$	$\frac{2.4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)}$	$\frac{2.4\dots 2p(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)}$	$\cdots \frac{2.4\dots 2p(2p+2)}{(2q+2p-5)\dots(2q+4p-3)}$

and after $(p-1)$ sets of operations

$${}_{p+1}d^q = (-1)^{p^2} \times$$

1	$\frac{1}{2q-1} + a$	$\frac{2}{(2q-1)(2q+1)}$	$\cdots \frac{2.4\dots(2p-2)}{(2q-1)(2q+1)\dots(2q+2p-3)}$
$1-x^2$	$\frac{2}{(2q-1)(2q+1)}$	$\frac{2.4}{(2q-1)(2q+1)(2q+3)}$	$\cdots \frac{2.4\dots(2p-2)2p}{(2q-1)(2q+1)\dots(2q+2p-1)}$
$(1-x^2)^2$	$\frac{2.4}{(2q-1)(2q+1)(2q+3)}$	$\frac{2.4.6}{(2q-1)(2q+1)(2q+3)(2q+5)}$	$\cdots \frac{2.4\dots 2p(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1-x^2)^p$	$\frac{2.4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)}$	$\frac{2.4\dots 2p(2p+2)}{(2q-1)(2q+1)\dots(2q+2p+1)}$	$\cdots \frac{2.4\dots(4p-4)(4p-2)}{(2q-1)(2q+1)\dots(2q+4p-3)}$

since $(-1)^{\frac{p(p+1)}{2} + \frac{(p-1)p}{2}} = (-1)^{p^2}$.

Here the first element of the last $p-1$ columns is seen to occur as factor for the whole column so that we can put outside the factor

$$\frac{2^{p-1} \cdot 4^{p-2} \dots (2p-4)^2 (2p-2)}{(2q-1)^{p-1} (2q+1)^{p-1} (2q+3)^{p-2} (2q+5)^{p-3} \dots (2q+2p-5)^2 (2q+2p-3)}$$

$$= \frac{1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1) 2^{\frac{p(p-1)}{2}}}{(2q-1)^{p-1} (2q+1)^{p-1} (2q+3)^{p-2} (2q+5)^{p-3} \dots (2q+2p-5)^2 (2q+2p-3)},$$

the resulting expression being

$${}_{p+1}d^q = \frac{(-1)^{p^2} \cdot 1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1) 2^{\frac{p(p-1)}{2}}}{(2q-1)^{p-1} (2q+1)^{p-1} (2q+3)^{p-2} \dots (2q+2p-5)^2 (2q+2p-3)} \times$$

1	$\frac{1}{2q-1} + a$	1	$\cdots$	1
$1-x^2$	$\frac{2}{(2q-1)(2q+1)}$	$\frac{4}{2q+3}$	$\cdots$	$\frac{2p}{2q+2p-1}$
$(1-x^2)^2$	$\frac{2.4}{(2q-1)(2q+1)(2q+3)}$	$\frac{4.6}{(2q+3)(2q+5)}$	$\cdots$	$\frac{2p(2p+2)}{(2q+2p-1)(2q+2p+1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(1-x^2)^p$	$\frac{2.4\dots 2p}{(2q-1)(2q+1)\dots(2q+2p-1)}$	$\frac{4.6\dots(2p+2)}{(2q+3)\dots(2q+2p+1)}$	$\cdots$	$\frac{2p\dots(4p-2)}{(2q+2p-1)\dots(2q+4p-3)}$

In our formulae the two cases $q = 1$ or $q = 2$ only occur for which according to this we find

$${}_{p+1}d^{\frac{1}{2}} = \frac{(-1)^{p^2} \cdot 1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1) 2^{\frac{p(p-1)}{2}}}{3^{p-1} \cdot 5^{p-2} \cdot 7^{p-3} \dots (2p-3)^2 (2p-1)} \times$$

1	$1 + \alpha$	1	1	...	1
$1 - x^2$	$\frac{2}{1 \cdot 3}$	$\frac{4}{5}$	$\frac{6}{7}$	...	$\frac{2p}{2p+1}$
$(1 - x^2)^2$	$\frac{2 \cdot 4}{1 \cdot 3 \cdot 5}$	$\frac{4 \cdot 6}{5 \cdot 7}$	$\frac{6 \cdot 8}{7 \cdot 9}$	...	$\frac{2p(2p+2)}{(2p+1)(2p+3)}$
$(1 - x^2)^3$	$\frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7}$	$\frac{4 \cdot 6 \cdot 8}{5 \cdot 7 \cdot 9}$	$\frac{6 \cdot 8 \cdot 10}{7 \cdot 9 \cdot 11}$	...	$\frac{2p(2p+2)(2p+4)}{(2p+1)(2p+3)(2p+5)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(1 - x^2)^p$	$\frac{2 \cdot 4 \dots 2p}{1 \cdot 3 \dots (2p+1)}$	$\frac{4 \cdot 6 \dots (2p+2)}{5 \cdot 7 \dots (2p+3)}$	$\frac{6 \cdot 8 \dots (2p+4)}{7 \cdot 9 \dots (2p+5)}$	...	$\frac{2p(2p+2) \dots (4p-2)}{(2p+1)(2p+3) \dots (4p-1)}$

.....(51)

and

$${}_{p+1}d^{\frac{2}{2}} = \frac{(-1)^{p^2} \cdot 1^{p-1} \cdot 2^{p-2} \dots (p-2)^2 (p-1) 2^{\frac{p(p-1)}{2}}}{3^{p-1} \cdot 5^{p-1} \cdot 7^{p-2} \dots (2p-1)^2 (2p+1)} \times$$

1	$\frac{1}{3} + \alpha$	1	1	...	1
$1 - x^2$	$\frac{2}{3 \cdot 5}$	$\frac{4}{7}$	$\frac{6}{9}$	...	$\frac{2p}{2p+3}$
$(1 - x^2)^2$	$\frac{2 \cdot 4}{3 \cdot 5 \cdot 7}$	$\frac{4 \cdot 6}{7 \cdot 9}$	$\frac{6 \cdot 8}{9 \cdot 11}$	...	$\frac{2p(2p+2)}{(2p+3)(2p+5)}$
$(1 - x^2)^3$	$\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 9}$	$\frac{4 \cdot 6 \cdot 8}{7 \cdot 9 \cdot 11}$	$\frac{6 \cdot 8 \cdot 10}{9 \cdot 11 \cdot 13}$	...	$\frac{2p(2p+2)(2p+4)}{(2p+3)(2p+5)(2p+7)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(1 - x^2)^p$	$\frac{2 \cdot 4 \cdot 6 \dots 2p}{3 \cdot 5 \cdot 7 \dots (2p+3)}$	$\frac{4 \cdot 6 \dots (2p+2)}{7 \cdot 9 \dots (2p+5)}$	$\frac{6 \cdot 8 \dots (2p+4)}{9 \cdot 11 \dots (2p+7)}$	...	$\frac{2p(2p+2) \dots (4p-2)}{(2p+3)(2p+5) \dots (4p+1)}$

.....(52).

VI. Uniform continuous distribution of observations with additional clusters at the ends of the range; constant standard deviation of observations. Special formulae.

(1) Our first task shall be to work out the formulae for ${}_n\sigma_y^2 - {}_{n-1}\sigma_y^2$ for values of n up to 6, the next to find what values should be given to α in order to make ${}_n\sigma_y^2$ as flat a curve as possible within the range of observations.

With the notations just introduced (40) and (41) take the form

$$S_{2p} = {}_{2p}\sigma_y^2 - {}_{2p-1}\sigma_y^2 = \frac{\sigma^2}{N} (1 + \alpha) \frac{{}_{p+1}d^{\frac{1}{2}}}{\frac{1}{p\delta} \cdot \frac{1}{p+1}\delta}$$

and
$$S_{2p+1} = {}_{2p+1}\sigma_y^2 - {}_{2p}\sigma_y^2 = \frac{\sigma^2}{N} (1 + a) x^2 \frac{{}_2^2 d^2}{{}_p \delta \cdot {}_{p+1} \delta}.$$

From these formulae we find, after applying (45), (51) and (52),

$$S_1 = \frac{\sigma^2}{N} \frac{3(1+a)x^2}{1+3a} \dots\dots\dots(53),$$

$$S_2 = \frac{\sigma^2}{N} (1+a) \frac{1}{1+1 \cdot 1a} \cdot \frac{1 \cdot 3^2 \cdot 5}{2^2(1+2 \cdot 3a)} \left| \begin{array}{cc} 1 & 1+a \\ 1-x^2 & \frac{2}{3} \end{array} \right|^2 \\ = \frac{\sigma^2}{N} \cdot \frac{5}{2^2} \frac{[2+3(1+a)(x^2-1)]^2}{1+6a} \dots\dots\dots(54),$$

$$S_3 = \frac{\sigma^2}{N} (1+a) x^2 \frac{3}{1+1 \cdot 3a} \cdot \frac{3 \cdot 5^2 \cdot 7}{2^2(1+2 \cdot 5a)} \left| \begin{array}{cc} 1 & \frac{1}{3}+a \\ 1-x^2 & \frac{2}{3 \cdot 5} \end{array} \right|^2 \\ = \frac{\sigma^2}{N} \cdot \frac{7}{2^2} \frac{(1+a)x^2[2+5(1+3a)(x^2-1)]^2}{(1+3a)(1+10a)} \dots\dots\dots(55),$$

$$S_4 = \frac{\sigma^2}{N} (1+a) \frac{1 \cdot 3^2 \cdot 5}{2^2(1+2 \cdot 3a)} \cdot \frac{1 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 9}{2^2 \cdot 2^6(1+3 \cdot 5a)} \left(\frac{2}{3} \right)^2 \left| \begin{array}{ccc} 1 & 1+a & 1 \\ 1-x^2 & \frac{2}{3} & \frac{4}{5} \\ (1-x^2)^2 & \frac{2 \cdot 4}{3 \cdot 5} & \frac{4 \cdot 6}{5 \cdot 7} \end{array} \right|^2 \\ = \frac{\sigma^2}{N} \cdot \frac{9}{2^6} \frac{(1+a)[8+20(2+9a)(x^2-1)+35(1+6a)(x^2-1)^2]^2}{(1+6a)(1+15a)} \dots\dots\dots(56),$$

$$S_5 = \frac{\sigma^2}{N} (1+a) \frac{3 \cdot 5^2 \cdot 7}{2^2(1+2 \cdot 5a)} \cdot \frac{3 \cdot 5^2 \cdot 7^3 \cdot 9^2 \cdot 11}{2^2 \cdot 2^6(1+3 \cdot 7a)} \left(\frac{2}{3 \cdot 5} \right)^2 x^2 \left| \begin{array}{ccc} 1 & \frac{1}{3}+a & 1 \\ 1-x^2 & \frac{2}{3 \cdot 5} & \frac{4}{7} \\ (1-x^2)^2 & \frac{2 \cdot 4}{3 \cdot 5 \cdot 7} & \frac{4 \cdot 6}{7 \cdot 9} \end{array} \right|^2 \\ = \frac{\sigma^2}{N} \cdot \frac{11}{2^6} \frac{(1+a)x^2[8+28(2+15a)(x^2-1)+63(1+10a)(x^2-1)^2]^2}{(1+10a)(1+21a)} \dots\dots\dots(57),$$

$$S_6 = \frac{\sigma^2}{N} (1+a) \frac{1 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 9}{2^2 \cdot 2^6(1+3 \cdot 5a)} \cdot \frac{1 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 9^3 \cdot 11^2 \cdot 13}{(2^2 \cdot 3)^2 \cdot 2^{12}(1+4 \cdot 7a)} \left(\frac{2 \cdot 2^3}{3^2 \cdot 5} \right)^2 \left| \begin{array}{cccc} 1 & 1+a & 1 & 1 \\ 1-x^2 & \frac{2}{3} & \frac{4}{5} & \frac{6}{7} \\ (1-x^2)^2 & \frac{2 \cdot 4}{3 \cdot 5} & \frac{4 \cdot 6}{5 \cdot 7} & \frac{6 \cdot 8}{7 \cdot 9} \\ (1-x^2)^3 & \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} & \frac{4 \cdot 6 \cdot 8}{5 \cdot 7 \cdot 9} & \frac{6 \cdot 8 \cdot 10}{7 \cdot 9 \cdot 11} \end{array} \right|^2 \\ = \frac{\sigma^2}{N} \cdot \frac{13}{2^8} \frac{(1+a)[16+168(1+10a)(x^2-1)+126(3+40a)(x^2-1)^2+231(1+15a)(x^2-1)^3]^2}{(1+15a)(1+28a)} \dots\dots\dots(58).$$

(2) We shall now look at ${}_n\sigma_y^2$ for special values of n and as a first attempt at finding a flat curve for ${}_n\sigma_y^2$ try to make ${}_{x=0}^{}{}_n\sigma_y^2 = {}_{x^2=1}^{}{}_n\sigma_y^2$.

For a *linear function* we find, since

$${}_1\sigma_y^2 = {}_0\sigma_y^2 + S_1, \\ {}_1\sigma_y^2 = \frac{\sigma^2}{N} \left(1 + \frac{3(1+a)}{1+3a} x^2 \right) \dots\dots\dots (59).$$

As a is positive it is obvious that we cannot make ${}_{x=0}^{}{}_1\sigma_y^2 = {}_{x^2=1}^{}{}_1\sigma_y^2$ which indeed we knew beforehand. This follows because we have proved that ${}_n\sigma_y^2$ is of $2n$ th degree and never lower.

For $x = 0$ we find

$${}_{x=0}^{}{}_1\sigma_y^2 = \frac{\sigma^2}{N},$$

which holds for any symmetrical distribution of observations with constant standard deviation. a is the ratio between the number of observations at the ends of the range and the number uniformly distributed through the range, it may therefore vary from 0 to ∞ . As $\frac{3(1+a)}{1+3a}$ decreases when a increases we get the flattest possible curve when $a = \infty$, that is when the distribution of observations consists of two groups at the ends of the range. Then the curve is, as already shown in Section II,

$${}_1\sigma_y^2 = \frac{\sigma^2}{N} (1 + x^2).$$

To get a check on the degree of the function and at the same time a flatter curve of σ_y^2 than that obtained from a uniform distribution we may choose something between the two extreme cases and take for example $\frac{1}{4}N$ observations at each end of the range and $\frac{1}{2}N$ uniformly distributed through the range.

Then $a = 1$ and, according to (59),

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + \frac{3}{2} x^2),$$

with the maximum

$${}_{x^2=1}^{}{}_{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 1.581.$$

(3) For a *function of the second degree* we find, from (46),

$${}_{x=0}^{}{}_2\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{9}{4} \frac{(1+a)(1+5a)}{1+6a},$$

and from (50), ${}_{x^2=1}^{}{}_2\sigma_y^2 = \frac{\sigma^2}{N} 3(1+a) \left\{ \frac{1}{1+3a} + \frac{2}{1+6a} \right\}$.

We want to make these equal and this requires

$$3(1+5a)(1+3a) = 4\{1+6a+2(1+3a)\}$$

or

$$15a^2 - 8a - 3 = 0.$$

This has only one positive root $a = .7873500$.

For this value $\frac{a}{2(1+a)}$, which is the ratio between the number of observations at one end of the range and the total number of observations, is .2202562.

As ${}_2\sigma_y^2 = {}_1\sigma_y^2 + S_2$ we find, from (59) and (54),

$${}_2\sigma_y^2 = \frac{\sigma^2}{N} \left(1 + \frac{3(1+a)}{1+3a} x^2 + \frac{5[2+3(1+a)(x^2-1)]^2}{4(1+6a)} \right) \dots\dots\dots(60),$$

for $a = .7873500$ the curve is

$${}_2\sigma_y^2 = \frac{\sigma^2}{N} \{3.46837 - 6.27862x^2 + 6.27862x^4\},$$

which has minima at $x = \pm \frac{1}{\sqrt{2}}.$

The extreme values in the range of observations are therefore

$$\sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.8624 \text{ for } x = \begin{cases} 0 \\ \pm 1 \end{cases}$$

and

$$\sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.3779 \text{ for } x = \pm .70711.$$

(4) For a function of the third degree we have, from (46),

$$\stackrel{x=0}{_3\sigma_y^2} = \frac{\sigma^2}{N} \cdot \frac{9(1+a)(1+5a)}{4(1+6a)},$$

and from (50), $\stackrel{x^2=1}{_3\sigma_y^2} = \frac{\sigma^2}{N} 2(1+a) \left\{ \frac{3}{1+6a} + \frac{5}{1+10a} \right\} \dots\dots\dots(61).$

Hence the condition that they are equal is

$$9(1+5a)(1+10a) = 32(2+15a)$$

or

$$90a^2 - 69a - 11 = 0,$$

with one positive root $a = .9021461$.

From (60) and (55) we find

$$\begin{aligned} {}_3\sigma_y^2 = \frac{\sigma^2}{N} & \left(1 + \frac{3(1+a)}{1+3a} x^2 + \frac{5[2+3(1+a)(x^2-1)]^2}{4(1+6a)} \right. \\ & \left. + \frac{7(1+a)x^2[2+5(1+3a)(x^2-1)]^2}{4(1+3a)(1+10a)} \right) \dots\dots\dots(62), \end{aligned}$$

which for $a = .9021461$ becomes

$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \{3.67775 + 17.78799x^2 - 48.56651x^4 + 30.77852x^6\}.$$

Besides the minimum for $x = 0$ this curve has other minima for $x^2 = .815820$ and maxima for $x^2 = .2361366$.

The maxima and minima are as follows:

$$\begin{aligned} \text{For } x = \begin{cases} \pm 1 \\ 0 \end{cases} & \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.9177, \\ ,, \quad x = \pm .48594 & \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.3612, \\ ,, \quad x = \pm .90323 & \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.6055. \end{aligned}$$

By choosing $\alpha = .9021461$, that is by taking $.237139 \times N$ observations at each end of the range, we seem therefore to have overshoot our aim since the result is that we have got inside the range a maximum for σ_y greater than the value obtained for $x = \pm 1$.

(5) Our next attempt shall be to make

$$\begin{matrix} x=1 & x=0 \\ {}_3\sigma_y^2 = 2 & {}_3\sigma_y^2. \end{matrix}$$

It requires $9(1 + 5\alpha)(1 + 10\alpha) = 16(2 + 15\alpha)$

or $450\alpha^2 - 105\alpha - 23 = 0$.

The only positive root is $\alpha = .3710723$ which gives the curve

$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \{2.730117 + 12.89741x^2 - 37.07612x^4 + 26.90882x^6\}.$$

The maxima and minima are:

$$\begin{array}{ll} \text{For } x = .0000 & \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.652, \\ ,, \quad x = \pm .4828 & \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.016, \\ ,, \quad x = \pm .8279 & \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.678, \\ ,, \quad x = \pm 1.0000 & \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.337. \end{array}$$

This distribution of observations makes σ_y for $x = \pm 1$ greater than the maximum at $x = \pm .4828$. By interpolation between these two cases we shall now try to find an α , lying between those of our two trials, for which σ_y for $x = \pm 1$ equals the maximum value of σ_y which still may be expected at about $x = .48$.

(6) In our first attempt we found $\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 1.918$ and its difference from the maximum $\frac{\sigma}{\sqrt{N}} \cdot .444$, in the second attempt $\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 2.337$ and its difference from the maximum $-\frac{\sigma}{\sqrt{N}} \cdot .321$.

If the relation were linear this difference would be zero for

$$\overset{x=1}{\sigma_y} = \frac{\sigma}{\sqrt{N}} \cdot 2.161.$$

The α for which $\overset{x=1}{\sigma_y}$ takes this value is found by (61) which leads to

$$8(1 + \alpha)(2 + 15\alpha) = 2.161^2(1 + 6\alpha)(1 + 10\alpha)$$

or $160.2\alpha^2 - 61.28\alpha - 11.330 = 0$,

with the positive root $\alpha = .519$.

For this value (62) becomes

$${}_3\sigma_y^2 = \frac{\sigma^2}{N} \{2.9866 + 14.2364x^2 - 40.0058x^4 + 27.4521x^6\}.$$

The maxima and minima are:

$$\text{For } x = \cdot 0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.728,$$

$$,, \quad x = \pm \cdot 4843 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.116,$$

$$,, \quad x = \pm \cdot 8585 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.655,$$

$$,, \quad x = \pm 1.0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.161,$$

and this distribution which has $\cdot 1708 \times N$ observations at each end of the range may be considered satisfactory.

(7) From (46) and (50) we find, for a function of the fourth degree,

$${}^x_{4\sigma_y^2} = \frac{\sigma^2}{N} \cdot \frac{225}{64} \frac{(1+a)(1+14a)}{1+15a}$$

and

$${}^{x^2=1}_{4\sigma_y^2} = \frac{\sigma^2}{N} 5(1+a) \left\{ \frac{2}{1+10a} + \frac{3}{1+15a} \right\},$$

which are equal when

$$9(1+14a)(1+10a) = 64(1+12a)$$

or

$$1260a^2 - 552a - 55 = 0,$$

that is when

$$a = \cdot 5217564.$$

The formula for ${}_4\sigma_y^2$, found from (62) and (56), is

$$\begin{aligned} {}_4\sigma_y^2 = \frac{\sigma^2}{N} \left\{ 1 + \frac{3(1+a)}{1+3a} x^2 + \frac{5}{4} \frac{[2+3(1+a)(x^2-1)]^2}{1+6a} + \frac{7}{4} \frac{(1+a)x^2[2+5(1+3a)(x^2-1)]^2}{(1+3a)(1+10a)} \right. \\ \left. + \frac{9}{64} \frac{(1+a)[8+20(2+9a)(x^2-1)+35(1+6a)(x^2-1)^2]^2}{(1+6a)(1+15a)} \right\} \dots\dots(63). \end{aligned}$$

For $a = \cdot 5217564$ it becomes

$${}_4\sigma_y^2 = \frac{\sigma^2}{N} \{ 5.03367 - 19.72772x^2 + 133.01711x^4 - 235.96817x^6 + 122.67868x^8 \}.$$

The maxima and minima are as follows:

$$\text{For } x = \begin{cases} 0 \\ \pm 1 \end{cases} \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.244,$$

$$,, \quad x = \pm \cdot 3130 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.041,$$

$$,, \quad x = \pm \cdot 6844 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.575,$$

$$,, \quad x = \pm \cdot 9361 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.856.$$

We have again as for the function of the third degree brought $\sigma_y^{x^2=1}$ down below one of the maxima of ${}_4\sigma_y$, although since ${}_4\sigma_y$ has a maximum at $x = 0$ the demand that $\sigma_y^{x=0} = \sigma_y^{x^2=1}$ is not so exacting as for ${}_3\sigma_y$ which has a minimum at $x = 0$.

(8) We shall next make $\overset{x^2=1}{4}\sigma_y^2 = 1.2671861 \overset{x=0}{4}\sigma_y^2$ *

The condition obtained from (46) and (50) is

$$9 \times 1.2671861 (1 + 10a) (1 + 14a) = 64 (1 + 12a)$$

or
$$a^2 - .3095773a - .032940969 = 0,$$

with the only positive root
$$a = .3933269.$$

Introducing this value of a in (63) we get

$$\overset{4}{\sigma}_y^2 = \frac{\sigma^2}{N} \{4.61918 - 18.02388x^2 + 122.71833x^4 - 220.34099x^6 + 116.8807x^8\}.$$

The maxima and minima for this curve are:

$$\text{At } x = 0 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.149,$$

$$,, \quad x = \pm .3116 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.958,$$

$$,, \quad x = \pm .6839 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.467,$$

$$,, \quad x = \pm .9214 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 1.913,$$

$$,, \quad x = \pm 1.0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.419.$$

We have thus for $a = .3933269$, that is by taking $.141147 \times N$ observations at each end of the range, succeeded in bringing $\overset{x^2=1}{4}\sigma_y^2$ down to be approximately equal to the highest of the maxima of the curve, thus fulfilling our purpose.

(9) After our experiences in the cases of the functions of the third and fourth degree we cannot expect for a *function of the fifth degree* by making

$$\overset{x^2=1}{5}\sigma_y^2 = \overset{x=0}{5}\sigma_y^2$$

to find a curve which has not a greater maximum than that value. We shall therefore start with the attempt

$$\overset{x^2=1}{5}\sigma_y^2 = 2 \overset{x=0}{5}\sigma_y^2.$$

The condition found from (46) and (50) is

$$25 (1 + 14a) (1 + 21a) = 64 (2 + 35a)$$

or
$$7350a^2 - 1365a - 103 = 0,$$

with the only positive root
$$a = .2433100.$$

* The ratio 1.2671861 results from consideration of a special $\overset{2}{\sigma}_y^2$ curve. It was determined as that curve obtained from three groups of observations for which the standard deviation of $\overset{2}{\sigma}_y^2$'s within the range of observations was a minimum. It is not mentioned elsewhere in this memoir as it does not seem to have the interest I at first assumed it to have.

For ${}_5\sigma_y^2$ we find, from (63) and (57),

$${}_5\sigma_y^2 = \frac{\sigma^2}{N} \left\{ 1 + \frac{3(1+a)}{1+3a} x^2 + \frac{5[2+3(1+a)(x^2-1)]^2}{4(1+6a)} + \frac{7(1+a)x^2[2+5(1+3a)(x^2-1)]^2}{4(1+3a)(1+10a)} \right. \\ \left. + \frac{9(1+a)[8+20(2+9a)(x^2-1)+35(1+6a)(x^2-1)^2]^2}{64(1+6a)(1+15a)} \right. \\ \left. + \frac{11(1+a)x^2[8+28(2+15a)(x^2-1)+63(1+10a)(x^2-1)^2]^2}{64(1+10a)(1+21a)} \right\} \dots (64).$$

Introducing $\alpha = .2433100$ we get

$${}_5\sigma_y^2 = \frac{\sigma^2}{N} \{ 4.14228 + 28.47030x^2 - 258.05238x^4 + 853.0448x^6 - 1095.921x^8 \\ + 476.5990x^{10} \},$$

from which we find the maxima and minima:

$$\text{At } x = 0 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.035,$$

$$,, \quad x = \pm .2953 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.273,$$

$$,, \quad x = \pm .5004 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.155,$$

$$,, \quad x = \pm .7853 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.762,$$

$$,, \quad x = \pm .9418 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.231,$$

$$,, \quad x = \pm 1.0000 \quad \sigma_y = \frac{\sigma}{\sqrt{N}} \cdot 2.878.$$

$x^2=1$

σ_y does not differ much from the greatest maximum and we may thus consider the distribution with $.097848 \times N$ observations at each end of the range for which $\alpha = .2433100$ as satisfying fairly well our aim.

(10) Considering our previous results we must assume that for a *function of the sixth degree* $\frac{\sigma_y^{x^2=1}}{\sigma_y^{x=0}}$ ought to be made somewhat smaller than 2 which was the value that gave a satisfying result for a function of the fifth degree.

Let us assume $\sigma_y^{x^2=1} = 1.75 \sigma_y^{x=0}$ or, substituting from (46) and (50),

$$256(1+24\alpha) = 1.75 \times 25(1+21\alpha)(1+27\alpha)$$

from which

$$567\alpha^2 - 92.43430\alpha - 4.851429 = 0$$

and

$$\alpha = .2048019$$

are found.

For ${}_6\sigma_y^2$ we get, from (64) and (58),

$${}_6\sigma_y^2 = \frac{\sigma^2}{N} \left\{ 1 + \frac{3(1+\alpha)}{1+3\alpha} x^2 + \frac{5[2+3(1+\alpha)(x^2-1)]^2}{4(1+6\alpha)} \right. \\ + \frac{7(1+\alpha)x^2[2+5(1+3\alpha)(x^2-1)]^2}{4(1+3\alpha)(1+10\alpha)} \\ + \frac{9}{64} \frac{1+\alpha}{(1+6\alpha)(1+15\alpha)} [8+20(2+9\alpha)(x^2-1)+35(1+6\alpha)(x^2-1)^2]^2 \\ + \frac{11}{64} \frac{1+\alpha}{(1+10\alpha)(1+21\alpha)} [8+28(2+15\alpha)(x^2-1)+63(1+10\alpha)(x^2-1)^2]^2 \\ + \frac{13}{256} \frac{1+\alpha}{(1+15\alpha)(1+28\alpha)} [16+168(1+10\alpha)(x^2-1) \\ \left. + 126(3+40\alpha)(x^2-1)^2 + 231(1+15\alpha)(x^2-1)^3] \right\},$$

which for $\alpha = .2048019$ becomes

$${}_6\sigma_y^2 = \frac{\sigma^2}{N} \{ 5.58984 - 33.14234x^2 + 504.4523x^4 - 2512.673x^6 + 5524.186x^8 + \\ - 5452.650x^{10} + 1974.020x^{12} \}.$$

The maxima and minima are:

$$\begin{aligned} \text{At } x = 0 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 2.364, \\ ,, \quad x = \pm .2216 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 2.216, \\ ,, \quad x = \pm .4826 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 2.515, \\ ,, \quad x = \pm .6194 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 2.427, \\ ,, \quad x = \pm .8445 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 3.149, \\ ,, \quad x = \pm .9615 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 2.485, \\ ,, \quad x = \pm 1.0000 \quad \sigma_y &= \frac{\sigma}{\sqrt{N}} \cdot 3.128. \end{aligned}$$

It thus appears that this distribution which has $.08499 \times N$ observations at each end of the range fulfils our demand that $\sigma_y^{x^2=1}$ shall be approximately equal to the greatest of the maxima.

(11) We bring together our final results in the following table. It gives the distribution of observations, the maximum of σ_y within the range, the value of $\sqrt{n+1}$ or the lowest maximum of $\frac{\sigma_y \sqrt{N}}{\sigma}$ possible, which can only be obtained by distributing the observations of the function of the n th degree into $(n+1)$ groups, and the value of $n+1$ which is the maximum of $\frac{\sigma_y \sqrt{N}}{\sigma}$ for a uniform distribution.

TABLE II.

Degree of function	Ratio of number of observations at each end of the range to the total number	Maximum of $\frac{\sigma_y \sqrt{N}}{\sigma}$	$\sqrt{n+1}$	$n+1$
1	.2500	1.581	1.414	2
2	.2203	1.862	1.732	3
3	.1708	2.161	2.000	4
4	.1411	2.467	2.236	5
5	.0978	2.878	2.449	6
6	.0850	3.149	2.646	7

A comparison between our maximum and $\sqrt{n+1}$ shows the price we have to pay for information about the degree of the function. For lower degrees the maximum only differs quite insignificantly from $\sqrt{n+1}$, but with increasing degree the difference grows relatively greater for the sixth degree, being about one-fifth of $\sqrt{n+1}$.

The curves of standard deviation for the three sets of distributions are given in Diagrams 3—8, while Diagram 9 represents the six curves just reached. It seems likely from the form of the σ_y curves that two clusters of observations placed at the outermost of the maxima besides the two clusters at the ends of the range would produce a σ_y curve with a lower maximum than the one we have succeeded in getting for the functions from the fourth to the sixth degree. But then again the position of these new clusters would depend on the degree of the function and thus make the proceedings more complicated; and what is more at the same time as the maximum of the curve approached $\sqrt{n+1}$ the distribution of observations would incur the disadvantages of the grouping in $(n+1)$ clusters. On the whole the distribution arrived at seems to be satisfactory and certainly marks a great progress from the uniform distribution.

VII. Observations with varying standard deviation.

(1) In Section I we have already given the formula for the standard deviation σ_y of an adjusted y when the standard deviation s_y of an observation is $\sigma \sqrt{f(x)}$.

It is

$$\left| \begin{array}{cccccc} \sigma_y^2 \cdot \frac{N}{\sigma^2} & 1 & x & x^2 & \dots & x^n \\ 1 & m_0 & m_1 & m_2 & \dots & m_n \\ x & m_1 & m_2 & m_3 & \dots & m_{n+1} \\ x^2 & m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^n & m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{array} \right| = 0,$$

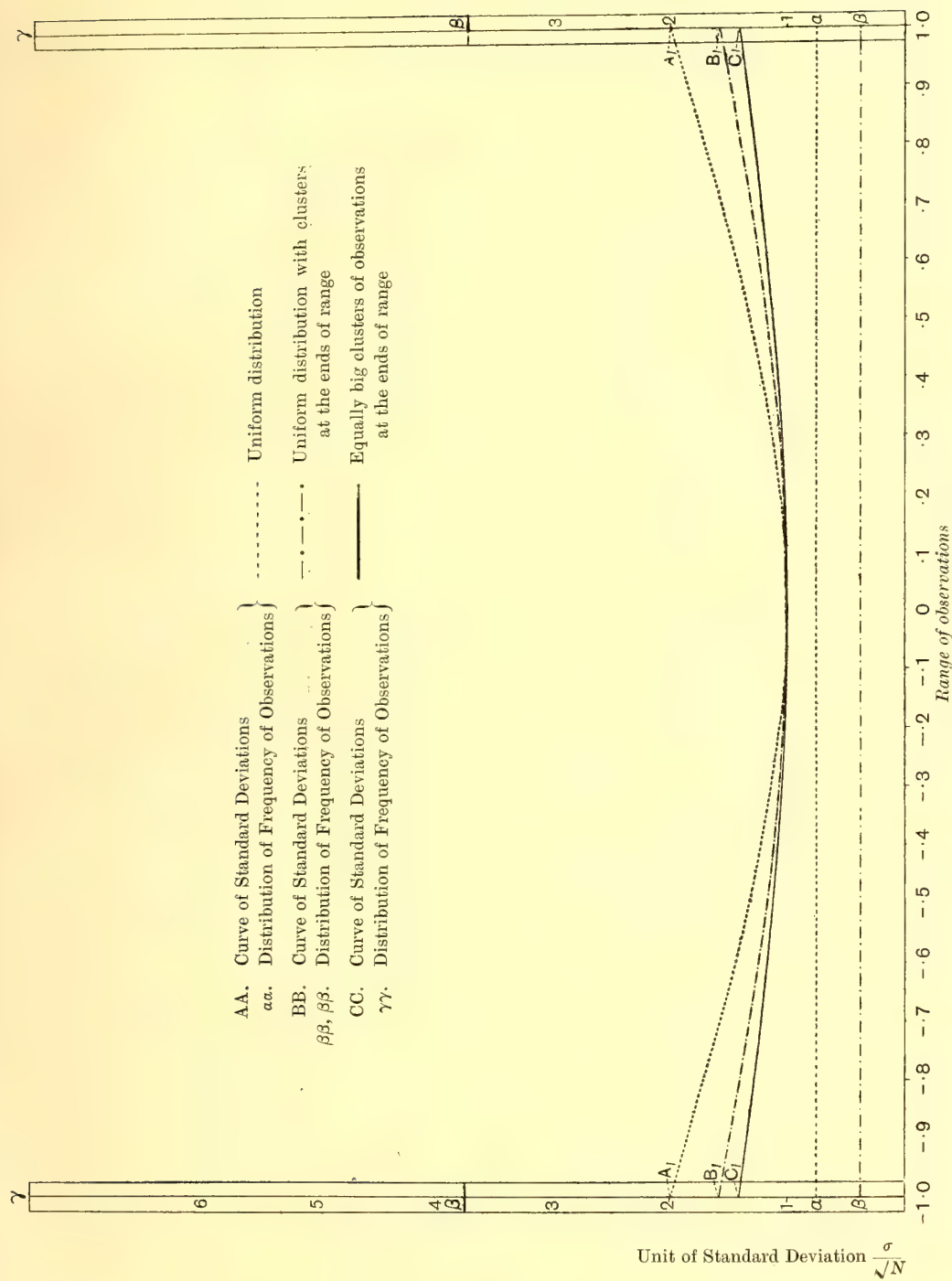


DIAGRAM 3. Linear Function.

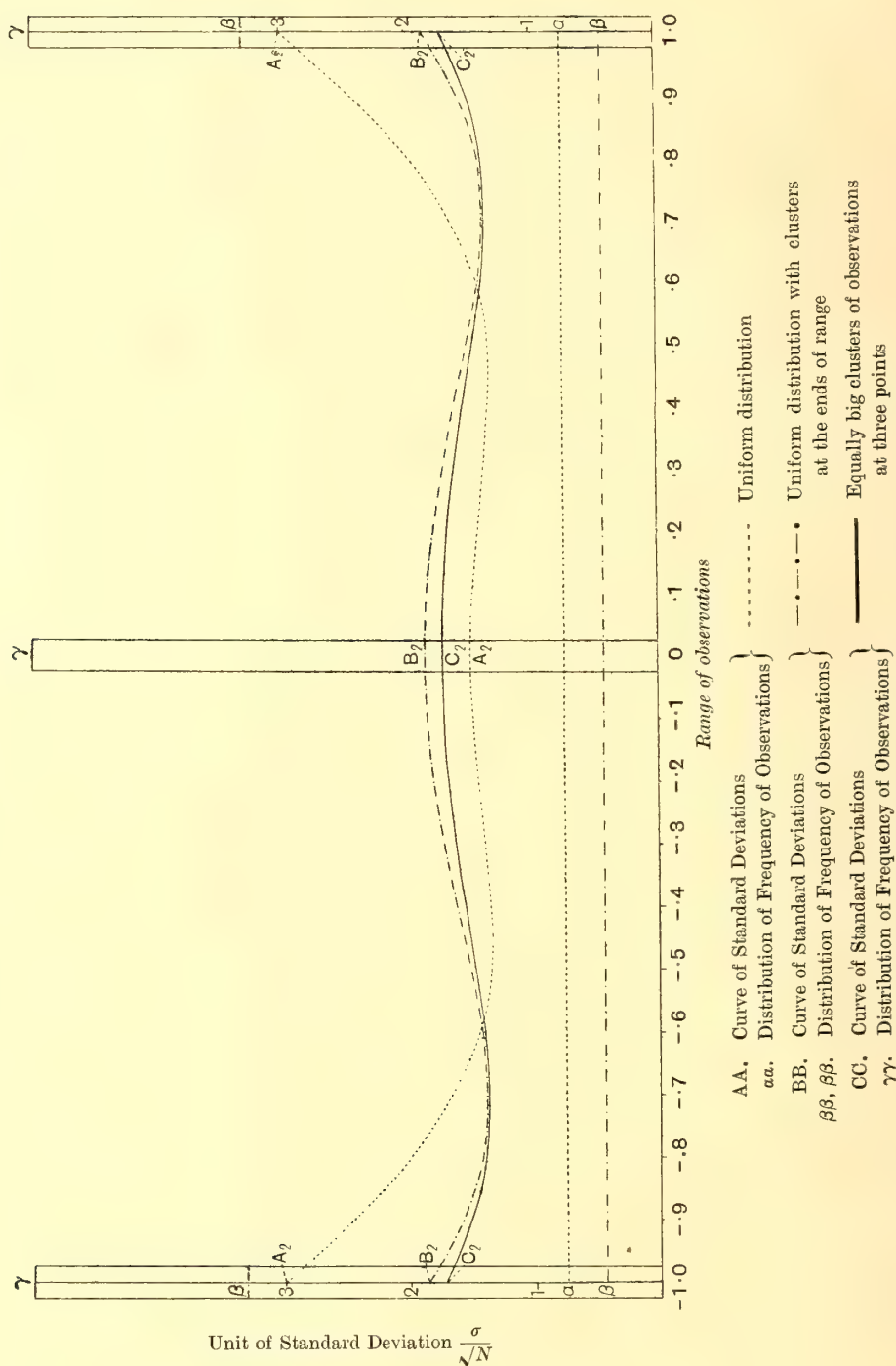


DIAGRAM 4. Function of Second Degree.

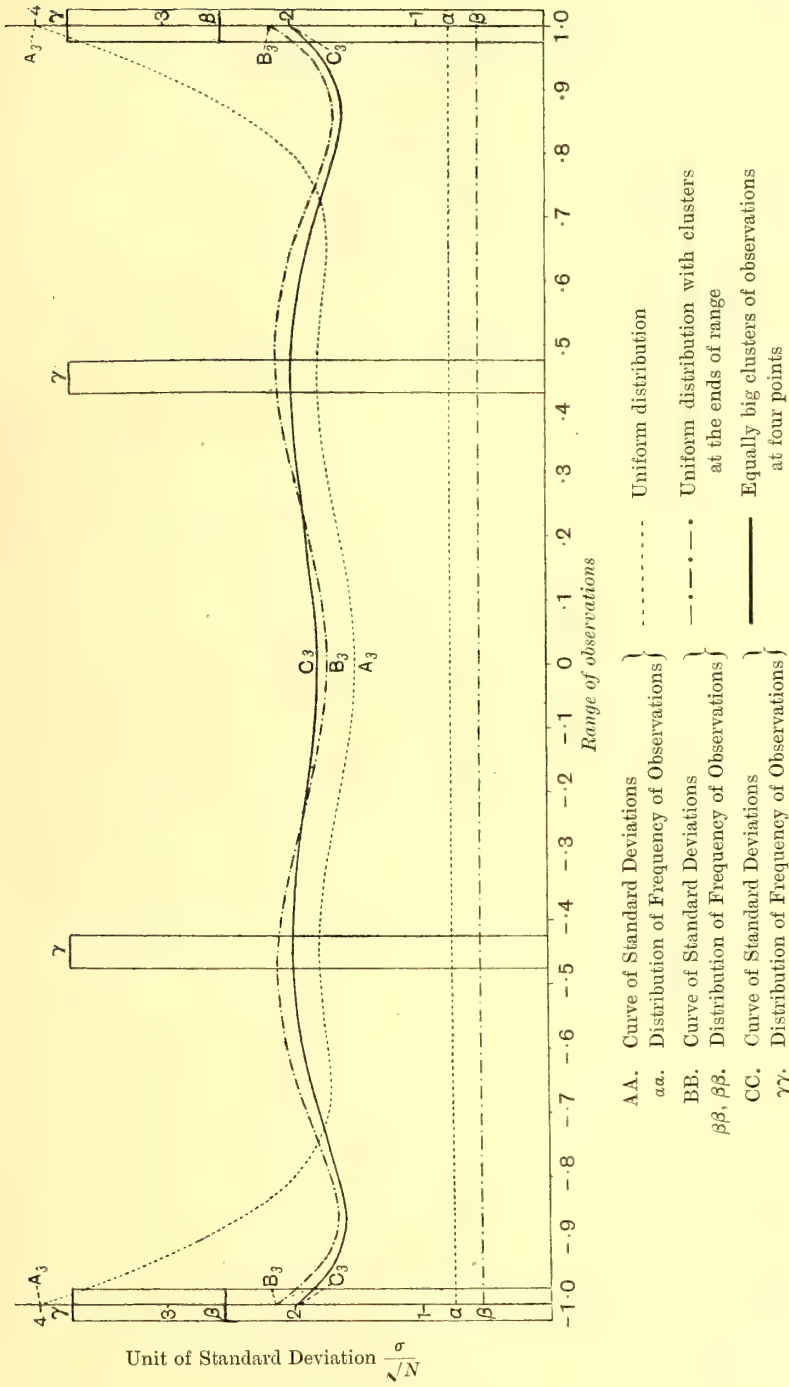
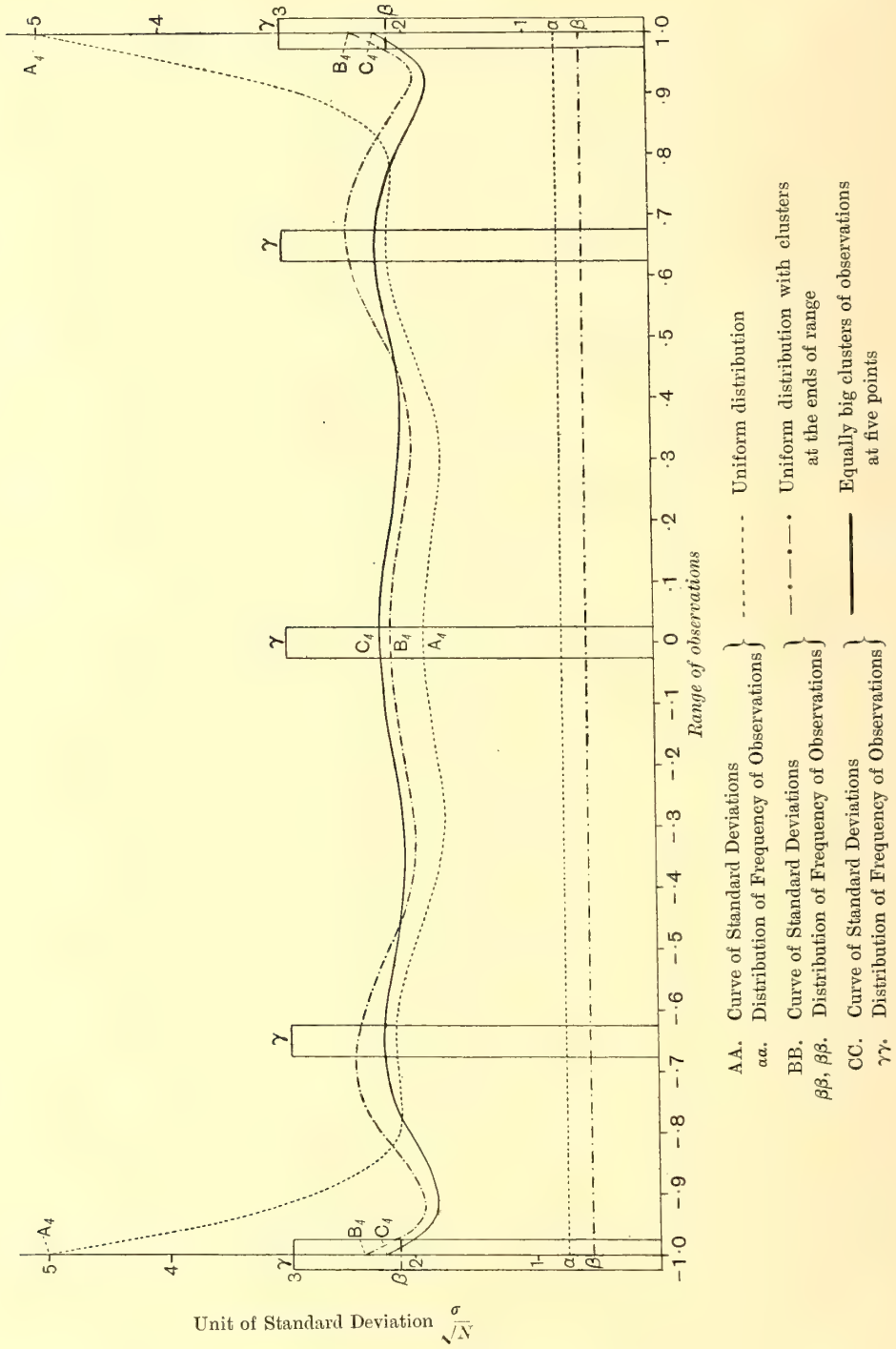


DIAGRAM 5. Function of Third Degree.



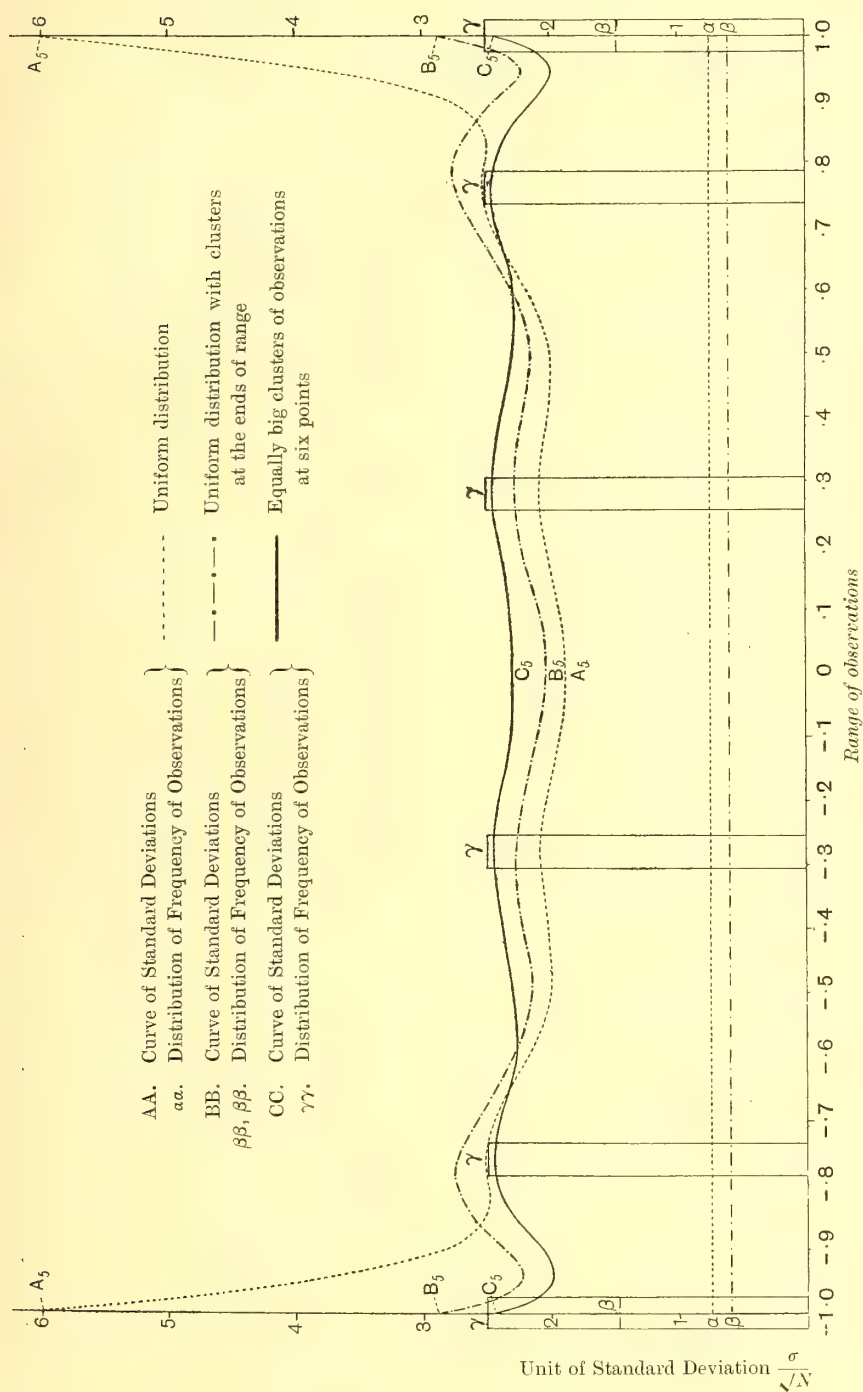


DIAGRAM 7. Function of Fifth Degree.

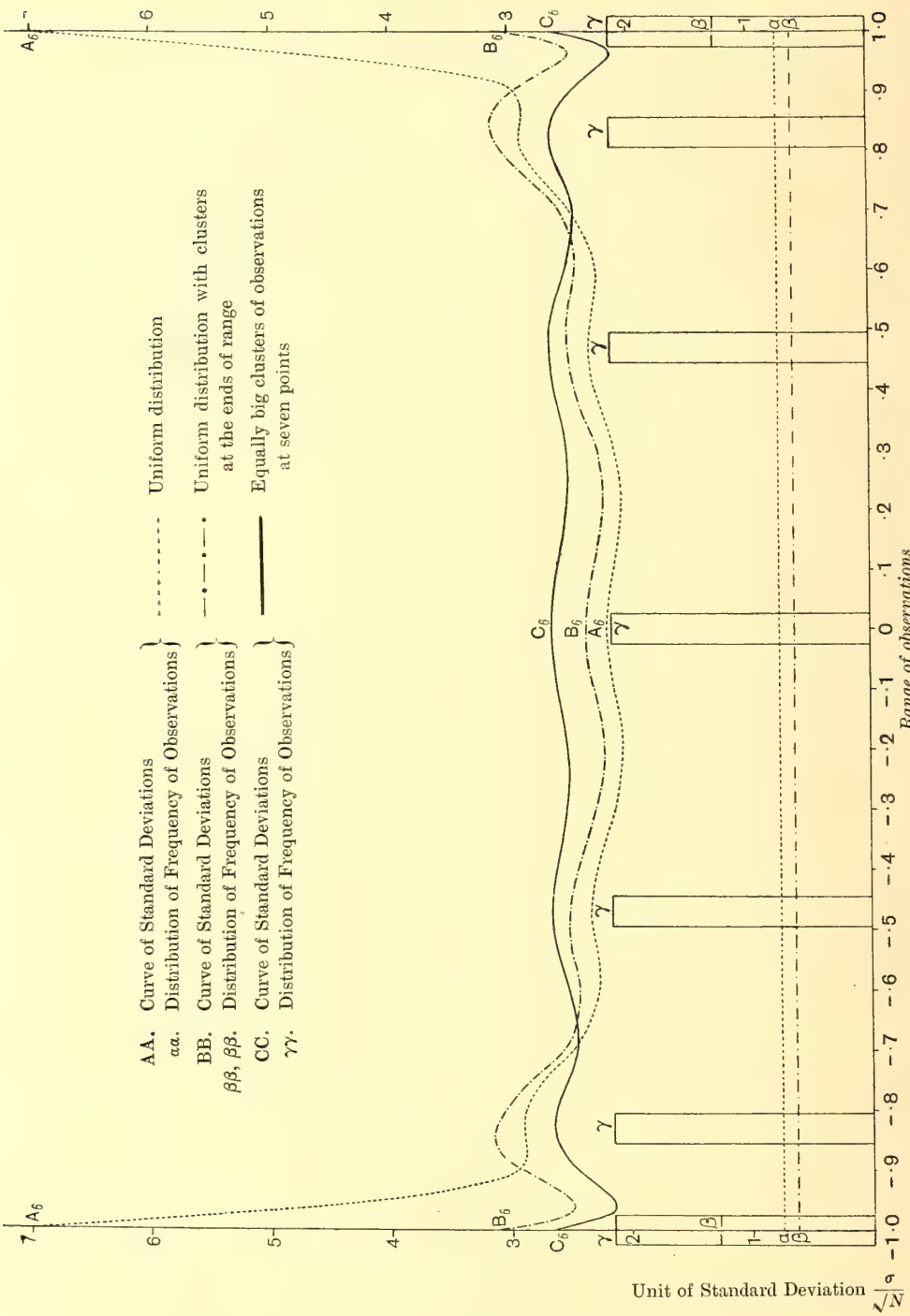
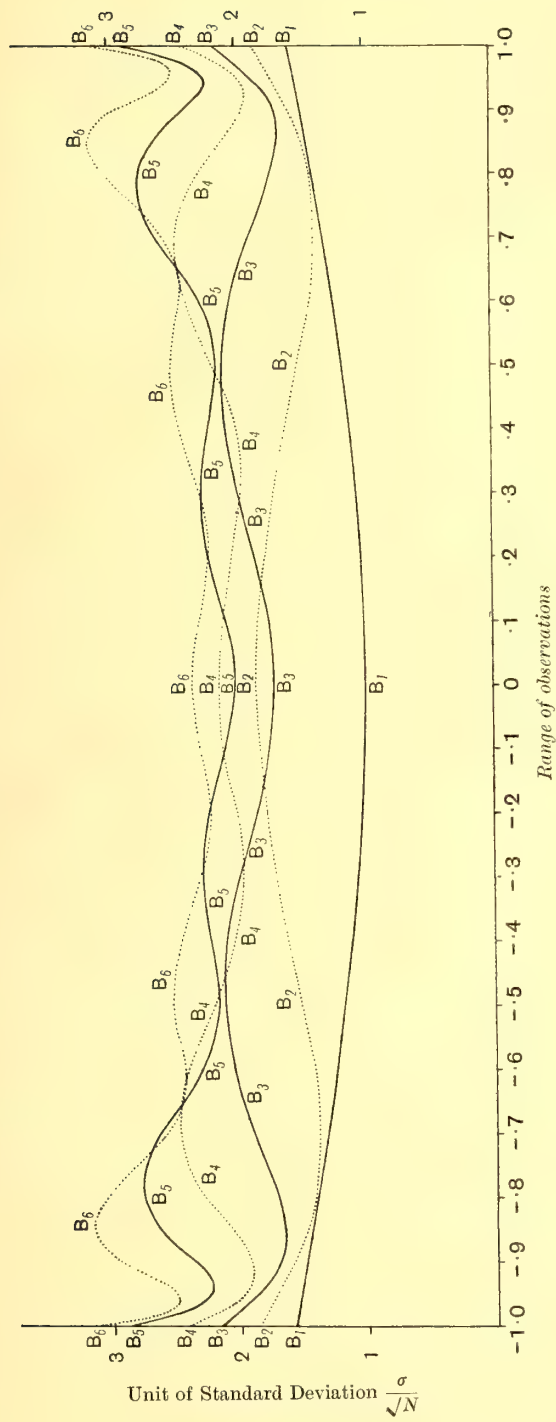


DIAGRAM 8. Function of Sixth Degree.



$B_1 B_1$.	Function of First Degree
$B_2 B_2$.	Second "
$B_3 B_3$.	Third "
$B_4 B_4$.	Fourth "
$B_5 B_5$.	Fifth "
$B_6 B_6$.	Sixth "

DIAGRAM 9. Curves of Standard Deviations. Uniform Distribution with clusters at the ends of range.

where $m_p = \frac{1}{N} \int x^p \frac{\psi(x)}{f(x)} dx$, $\psi(x) dx$ being the number of observations between x and $x + dx$ and the integration being extended over the range of observations.

It is clear that if we have found a suitable curve of squared standard deviation for adjusted y by taking a distribution $\phi(x)$ of observations with constant standard deviations a corresponding curve can be derived for observations with varying standard deviations by using the distribution

$$\psi(x) = k\phi(x) \cdot f(x) \dots\dots\dots(65).$$

As $\int k\phi(x) \cdot f(x) dx = N$ the constant k must be

$$k = \frac{N}{\int \phi(x) \cdot f(x) dx}.$$

Hence we find
$$m_p = \frac{\int x^p \cdot \phi(x) dx}{\int \phi(x) \cdot f(x) dx} = \frac{N\mu_p}{\int \phi(x) \cdot f(x) dx},$$

where μ_p is the p th moment coefficient for the distribution $\phi(x)$, and as

$$\frac{m_p}{\mu_p} = \frac{N}{\int \phi(x) \cdot f(x) dx} = k$$

for any p the determinant may be written

$$\left| \begin{array}{cccccc} \sigma_y^2 \cdot \frac{N}{\sigma^2} k & 1 & x & x^2 & \dots\dots & x^n \\ 1 & 1 & \mu_1 & \mu_2 & \dots\dots & \mu_n \\ x & \mu_1 & \mu_2 & \mu_3 & \dots\dots & \mu_{n+1} \\ x^2 & \mu_2 & \mu_3 & \mu_4 & \dots\dots & \mu_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ x^n & \mu_n & \mu_{n+1} & \mu_{n+2} & \dots\dots & \mu_{2n} \end{array} \right| = 0 \dots\dots\dots(66).$$

We thus find the same determinant as the distribution $\phi(x)$ would give for observations with constant error of observation except that the factor k has come in, that is to say the expression for σ_y^2 has been multiplied by

$$\frac{1}{k} = \frac{1}{N} \int \phi(x) \cdot f(x) dx \dots\dots\dots(67).$$

The goodness of the distribution therefore will partly depend on the value of $\frac{1}{k}$, and because we have found $\phi(x)$ the best distribution for observations with constant standard deviation it does not follow that

$$\psi(x) = k\phi(x) \cdot f(x)$$

is the best distribution for observations with the standard deviation $\sigma\sqrt{f(x)}$. But the deriving of $\psi(x)$ from $\phi(x)$ is nevertheless useful as a means of simplifying the investigations and will be applied in the following special inquiries.

We shall consider two forms of $f(x)$ and try to find the best distributions for functions of the first and of the second degree.

$$(a) \quad f(x) = (1 + \alpha x^2)^2, \text{ where } \alpha > -1,$$

for errors of observation increasing or decreasing in both directions from the middle of the range.

$$(b) \quad f(x) = (1 + \alpha x)^2, \text{ where } 1 > \alpha \geq 0,$$

for error of observations increasing in one direction.

These two forms will roughly cover two distinct and important types of cases, such as occur in practice.

(2) When $f(x) = (1 + \alpha x^2)^2$ we find, according to (67),

$$\frac{1}{k} = 1 + 2\alpha\mu_2 + \alpha^2\mu_4,$$

and as (66) for $n = 1$ gives

$$\sigma_y^2 \cdot \frac{N}{\sigma^2} k = \frac{1}{\mu_2 - \mu_1^2} \{\mu_2 - 2\mu_1 x + x^2\},$$

we have for a function of the first degree

$$\sigma_y^2 = \frac{\sigma^2}{N} \frac{1 + 2\alpha\mu_2 + \alpha^2\mu_4}{\mu_2 - \mu_1^2} \{\mu_2 - 2\mu_1 x + x^2\} \dots\dots\dots(68).$$

This curve has a minimum for $x = \mu_1$ and the maximum in the range is, if $\mu_1 > 0$, at $x = -1$, and if $\mu_1 < 0$, at $x = 1$; it equals in both cases

$$\frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \left\{ 1 + \frac{(1 + [\mu_1])^2}{\mu_2 - \mu_1^2} \right\} \dots\dots\dots(69),$$

$[\mu_1]$ being the numerical value of μ_1 .

Now (69) is a minimum for $\mu_1 = 0$; we therefore ought to choose that value for μ_1 and we then get, from (68),

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \left\{ 1 + \frac{x^2}{\mu_2} \right\} \dots\dots\dots(70),$$

μ_2 and μ_4 may vary between 0 and 1 independently of each other and are only bound by the conditions that

$$\mu_4 \leq \mu_2$$

$$\text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} \leq 1.$$

For any set of values which satisfies these conditions we may determine a distribution consisting of $\frac{\gamma}{2} N$ observations at $x = \pm v$ and $(1 - \gamma) N$ at $x = 0$, since from any two such values we could determine

$$v^2 = \frac{\mu_4}{\mu_2} \leq 1$$

$$\text{and} \quad \gamma = \frac{\mu_2^2}{\mu_4} \leq 1.$$

By introducing v^2 and γ for μ_2 and μ_4 we get two quite independent variables and (70) then takes the form

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + 2\alpha\gamma v^2 + \alpha^2\gamma v^4) \left(1 + \frac{x^2}{\gamma v^2} \right).$$

We now have to determine γ and v^2 so that the maximum value $\sigma_y^{x=1}$ is as small as possible. We find

$$\left[\frac{d\sigma_y^2}{d\gamma} \right]_{x=1} = \frac{\sigma^2}{N} \left(2av^2 + a^2v^4 - \frac{1}{\gamma^2v^2} \right) \dots\dots\dots(71)$$

and $\left[\frac{d\sigma_y^2}{dv^2} \right]_{x=1} = \frac{\sigma^2}{N} \left(2a\gamma + 2a^2\gamma v^2 + a^2 - \frac{1}{\gamma v^4} \right) \dots\dots\dots(72).$

Clearly $\left[\frac{d\sigma_y^2}{d\gamma} \right]_{x=1} = 0$ leads to $\gamma^2 = \frac{1}{av^4(2+av^2)} \dots\dots\dots(73).$

Introducing this value into (72) we obtain

$$\left[\frac{d\sigma_y^2}{dv^2} \right]_{x=1} = \frac{\sigma^2}{N} \cdot a^2 \left(1 + \frac{1}{\sqrt{a(2+av^2)}} \right),$$

which is > 0 .

Hence the minimum for constant v^2 determined by

$$\left[\frac{d\sigma_y^2}{d\gamma} \right]_{x=1} = 0$$

decreases with v^2 .

But when v^2 decreases, γ^2 , as given by (73), increases and the lowest value of v^2 , for which it is real, is that determined by

$$\gamma^2 = \frac{1}{av^4(2+av^2)} = 1.$$

For v^2 smaller than this (73) gives $\gamma^2 > 1$, and as long as $\gamma^2 \geq 1$ we therefore have

$$\left[\frac{d\sigma_y^2}{d\gamma} \right]_{x=1} \leq 0.$$

Hence the minimum of $\sigma_y^{x=1}$ is to be found for $\gamma^2 = 1$.

For this value (72) may be written as

$$\left[\frac{d\sigma_y^2}{dv^2} \right]_{x=1} = \frac{\sigma^2}{N} \frac{1}{v^4} (1 + av^2) (2av^4 + av^2 - 1) \dots\dots\dots(74),$$

which is zero for $v^2 = -\frac{1}{4} + \sqrt{\frac{1 + \frac{8}{a}}{16}} \dots\dots\dots(75)$

and > 0 for v^2 greater than this value.

When the v^2 found lies between 0 and 1, that is when $a > \frac{1}{3}$, we have thus found the minimum sought. When $a \leq \frac{1}{3}$, then $\left[\frac{d\sigma_y^2}{dv^2} \right]_{x=1}$ as given by (74) is < 0 and the minimum of $\sigma_y^{x=1}$ is found by giving v^2 its maximum value, that is 1.

Returning to the variates μ_2 and μ_4 we see that in all cases

$$\gamma^2 = \frac{\mu_2^2}{\mu_4} = \frac{1}{\beta_2} = 1,$$

from which it follows that no distribution of observations other than those arrived at consisting of two equally big groups can give μ_1 , μ_2 and μ_4 the values required.

We accordingly reach the result that: *when observing a function of the first degree for which the standard deviation of the observations is $\sigma(1 + \alpha x^2)$, symmetrical about the middle of the range, we get the best function for σ_y^2 by taking two equally big groups of observations, at the ends of the range if $\alpha \leq \frac{1}{3}$ and at $v = \pm \frac{1}{2} \sqrt{\sqrt{1 + \frac{8}{\alpha}} - 1}$ if $\alpha > \frac{1}{3}$.*

(3) According to (70) the maximum of σ_y^2 for this distribution is

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + \alpha v^2)^2 \left(1 + \frac{1}{v^2}\right),$$

v being equal to 1 for $\alpha \leq \frac{1}{3}$ and v being determined by (75) for $\alpha > \frac{1}{3}$.

We shall next consider the distributions (i) for which $\phi(x)$ is constant from -1 to 1 and (ii) for which $\phi(x)$ consists of $\frac{N}{2}$ observations uniformly distributed from -1 to 1 and $\frac{N}{2}$ into two clusters.

(i) For a uniform distribution from -1 to 1 we have $\mu_2 = \frac{1}{3}$, $\mu_4 = \frac{1}{5}$ and, according to (67),

$$\frac{1}{k} = 1 + \frac{2}{3}\alpha + \frac{1}{5}\alpha^2,$$

the actual distribution is hence, as $\phi(x) = \frac{N}{2}$,

$$\psi(x) = \frac{N}{2} \frac{(1 + \alpha x^2)^2}{1 + \frac{2}{3}\alpha + \frac{1}{5}\alpha^2},$$

and the maximum σ_y^2 as given by (70) for $x = \pm 1$,

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + \frac{2}{3}\alpha + \frac{1}{5}\alpha^2) \cdot 4.$$

(ii) When $\phi(x) = \frac{N}{4}$ with the additional clusters $\frac{N}{4}$ at $\pm u$ we have

$$\mu_2 = \frac{1}{6} + \frac{1}{2}u^2 \quad \text{and} \quad \mu_4 = \frac{1}{10} + \frac{1}{2}u^4.$$

According to (70) the maximum σ_y^2 is then

$$\sigma_y^2 = \frac{\sigma^2}{N} [1 + \alpha(\frac{1}{3} + u^2) + \alpha^2(\frac{1}{10} + \frac{1}{2}u^2)] \left(1 + \frac{6}{3u^2 + 1}\right).$$

We shall now determine u so as to make this a minimum. We find that

$$\frac{d\sigma_y^2}{du^2} = \frac{\sigma^2}{N} \cdot \frac{1}{5} \left[5\alpha \{1 + \alpha(1 + u^2)\} - \frac{2}{(3u^2 + 1)^2} (45 + 7\alpha^2) \right] = 0$$

requires

$$45\alpha^2 u^6 + 15\alpha(3 + 5\alpha)u^4 + 5\alpha(6 + 7\alpha)u^2 - (90 - 5\alpha + 9\alpha^2) = 0 \quad \dots(76),$$

the root u^2 of which is > 1 for $\alpha < .5576$.

For $\alpha \leq .5576$ we hence get the minimum σ_y^2 by taking the clusters at $u = \pm 1$ and for $\alpha > .5576$ at the places $\pm u$ determined by (76).

Table III contains for a series of values of α the values of v , $(1 + \alpha v^2)$ and u of the two distributions above and the maximum σ_y for the three distributions.

TABLE III.

α	v	$1 + \alpha v^2$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from best distri- bution	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = \frac{N}{2}$	u	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = \frac{N}{4}$ and clusters of $\frac{N}{4}$ at $\pm u$
0	1.0000	1.000	1.414	2.000	1.0000	1.581
$\frac{1}{6}$	1.0000	1.167	1.650	2.113	1.0000	1.760
$\frac{1}{3}$	1.0000	1.333	1.886	2.231	1.0000	1.944
$\frac{1}{2}$	.8836	1.390	2.100	2.352	1.0000	2.131
$\frac{2}{3}$	.8071	1.434	2.284	2.477	.9289	2.316
$\frac{3}{4}$	.7510	1.470	2.448	2.603	.8502	2.483
1	.7071	1.500	2.598	2.733	.7797	2.637
2	.5559	1.618	3.330	3.540	.5762	3.438
3	.4782	1.686	3.908	4.382	.4925	4.173
4	.4278	1.732	4.404	5.241	.4612	4.899

The difference between the maxima from the two first distributions taken as a proportion of the maximum of the first decreases from 41 per cent. at $\alpha = 0$ to the minimum 5 per cent. at $\alpha = 1$, and then again increases to 19 per cent. at $\alpha = 4$. For small α , that is in practice $\alpha = 0$, and again for $\alpha > 3$, for which the difference is greater than 12 per cent., the third distribution may therefore be useful as giving a much smaller maximum value than the purely continuous distribution and at the same time offering some justification for the form of the function.

(4) We shall next, still assuming that $f(x) = (1 + \alpha x^2)^2$, consider the choice of observations for a *function of the second degree*.

According to (66) and (67) we find

$$\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{1}{k} \times \left\{ \frac{\mu_2\mu_4 - \mu_3^2 + 2(\mu_2\mu_3 - \mu_1\mu_4)x + (\mu_4 - 3\mu_2^2 + 2\mu_1\mu_3)x^2 + 2(\mu_1\mu_2 - \mu_3)x^3 + (\mu_2 - \mu_1^2)x^4}{\mu_2\mu_4 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4 - \mu_3^2} \right\} \dots\dots\dots(77),$$

$$\text{and} \quad \frac{1}{k} = 1 + 2\alpha\mu_2 + \alpha^2\mu_4,$$

where the μ 's are the moment coefficients about $x = 0$ of the distribution $\phi(x)$ which is connected with the actual distribution $\psi(x)$ by the relation

$$\psi(x) = k\phi(x) \cdot f(x).$$

From any distribution $\phi(x)$ which has μ_1 and $\mu_3 \geq 0$ we can form a symmetrical $\frac{1}{2}\{\phi(x) + \phi(-x)\}$ which has the same μ_2 and μ_4 as $\phi(x)$. We shall prove that the maximum σ_y^2 obtained from the symmetrical distribution is always lower than that obtained from the skew.

Let the factor in curled brackets in (77) be F_s for a skew distribution $\phi(x)$ and F_0 for the corresponding symmetrical distribution.

We then have

$$F_0 = \frac{\mu_2\mu_4 + (\mu_4 - 3\mu_2^2)x^2 + \mu_2x^4}{\mu_2(\mu_4 - \mu_2^2)}.$$

The condition for a maximum or minimum other than that at $x = 0$ is

$$3\mu_2^2 - \mu_4 > 0,$$

or

$$\beta_2 < 3,$$

and as the denominator is positive we have in that case the maximum at $x = 0$. It is thus clear that the maxima of F_0 between -1 and 1 must be either at $x = 0$ or at $x = \pm 1$.

We shall show that

$$[F_s]_{x=0} > [F_0]_{x=0},$$

and that either $[F_s]_{x=1}$ or $[F_s]_{x=-1} > [F_0]_{x=\pm 1}$.

According to what has been proved in Section I (4) the coefficient of x^4 in (77) is positive, the denominator of (77) is therefore positive and we have

$$[F_s - F_0]_{x=0} = \frac{(\mu_2\mu_3 - \mu_1\mu_4)^2}{(\mu_4 - \mu_2^2)(\mu_2\mu_4 - \mu_2^3 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4 - \mu_3^2)} > 0.$$

We shall next compare F_s and F_0 for $x = \pm 1$.

Putting
$$[F_0]_{x=1} = \frac{N}{D}$$

we have
$$[F_s]_{x=1} = \frac{N - \delta}{D - \epsilon},$$

where
$$\delta = \mu_3^2 - 2\mu_1\mu_3 + \mu_1^2 \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}$$

and
$$\epsilon = \mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4.$$

For $\frac{\delta}{\epsilon}$ we find

$$\frac{\delta}{\epsilon} = \frac{(\mu_3 - \mu_1)^2 \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}}{(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)}.$$

Looking first at the case $\frac{\mu_1}{\mu_3} \geq 0$, we have

$$(\mu_3 - \mu_1)^2 < (\mu_3 - \mu_1\mu_2)^2,$$

and if we choose the value for which the other term of the numerator is < 0 ,

$$\frac{\delta}{\epsilon} < 1.$$

When $\frac{\mu_1}{\mu_3} < 0$ we see, from considering the form

$$\frac{\delta}{\epsilon} = 1 - \frac{\mu_1^2(1 - \mu_4) - 2\mu_1\mu_3(1 - \mu_2) \pm 2\{\mu_3(1 - \mu_2) - \mu_1(\mu_2 - \mu_4)\}}{(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)},$$

that for either $x = 1$ or $x = -1$

$$\frac{\delta}{\epsilon} < 1.$$

As $\epsilon > 0$ we have hence for any μ_1 and μ_3 , remembering that $\frac{N}{D}$ being a squared standard deviation multiplied by the number of observations is ≥ 1 ,

$$\frac{N - \delta}{D - \epsilon} > \frac{N}{D} > \frac{\delta}{\epsilon},$$

that is, for either $x = 1$ or -1 ,

$$F_s > F_0.$$

We have thus proved that the maxima of F_0 are below those of F_s .

(5) Our problem is hence reduced to finding the best curve among those represented by

$$\sigma_y^2 = \frac{\sigma^2 (1 + 2a\mu_2 + a^2\mu_4)}{\mu_2 (\mu_4 - \mu_2^2)} \{ \mu_2 \mu_4 + (\mu_4 - 3\mu_2^2) x^2 + \mu_2 x^4 \} \dots\dots\dots (78).$$

As was stated in (2) of this section we get all sets of possible values for μ_2 and μ_4 from three groups of observations symmetrical about $x = 0$, and we may therefore limit our search of the best distribution to these.

Let the observations be $\frac{\gamma}{2}N$ at $x = \pm v$, at $(1 - \gamma)N$ at $x = 0$. The interpolation formula of Lagrange gives, when $\bar{y}_p$ represents the mean of the observations at $x = p$,

$$y = \frac{x^2 - v^2}{-v^2} \bar{y}_0 + \frac{x(x - v)}{2v^2} \bar{y}_{-v} + \frac{x(x + v)}{2v^2} \bar{y}_v,$$

from which we find

$$\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{1}{v^4} \left\{ \frac{(x^2 - v^2)^2}{1 - \gamma} + \frac{x^2 (x^2 + v^2) (1 + av^2)^2}{\gamma} \right\} \dots\dots\dots (79).$$

It is obvious that if for a certain distribution we have

$$\begin{matrix} x=0 & x^2=1 \\ \sigma_y^2 & > \sigma_y^2 \end{matrix}$$

we can get a better distribution by taking more observations at 0. If on the other hand

$$\begin{matrix} x=0 & x^2=1 \\ \sigma_y^2 & < \sigma_y^2 \end{matrix},$$

the curve cannot be the best unless σ_y^2 is a minimum for the present values of v and γ . From (79) we find

$$\left[\frac{d\sigma_y^2}{d\gamma} \right]_{x=1} = \frac{\sigma^2}{N} \cdot \frac{1}{v^4} \left\{ \frac{(1 - v^2)^2}{(1 - \gamma)^2} - \frac{(1 + v^2)(1 + av^2)^2}{\gamma^2} \right\} \dots\dots\dots (80)$$

$$\text{and} \quad \left[\frac{d\sigma_y^2}{dv^2} \right]_{x^2=1} = \frac{\sigma^2}{N} \cdot \frac{1}{v^6} \left\{ -\frac{2(1 - v^2)}{1 - \gamma} - \frac{(2 + v^2 - av^4)(1 + av^2)}{\gamma} \right\},$$

from which we obtain the conditions for maximum or minimum

$$v^2 = \frac{1 \pm 2\sqrt{a}}{a}$$

and

$$\frac{\gamma}{1 - \gamma} = \frac{2\sqrt{a}(1 \pm \sqrt{a})^2}{a \mp 2\sqrt{a} - 1}.$$

The lower sign requires $3 - 2\sqrt{2} \geq a \geq \frac{1}{4}$ and the upper sign $a > 3 + 2\sqrt{2}$ to make $0 \geq v^2 \geq 1$. The case $a < \frac{1}{4}$ has no interest, as we have seen that when $a < \frac{1}{3}$ extrapolation is not even for a linear function advantageous. We have therefore seen that for $a < 3 + 2\sqrt{2}$ * σ_y^2 has no minimum and we have thus proved that the best distribution requires $\sigma_y^2 = \sigma_y^2$, that is

$$\frac{2v^2 - 1}{1 - \gamma} = \frac{(1 + v^2)(1 + av^2)^2}{\gamma},$$

or

$$\frac{1}{1 - \gamma} = 1 + \frac{(1 + v^2)(1 + av^2)^2}{2v^2 - 1} \dots\dots\dots(81).$$

The maximum of the curve is

$$\sigma_y^2 = \frac{\sigma^2}{N} \cdot \frac{1}{1 - \gamma}.$$

To find the minimum of this value we differentiate (81) and get

$$\left[\frac{d\sigma_y^2}{dv^2} \right]_{x=0} = \frac{1 + av^2}{(2v^2 - 1)^2} \{4av^4 - av^2 - 2a - 3\},$$

which is zero for

$$v^2 = \frac{1}{8} \left(1 + \sqrt{33 + \frac{48}{a}} \right) \dots\dots\dots(82)$$

and positive for greater v^2 , so that we have found a minimum.

For $a = 3$ we find from (82) $v^2 = 1$, hence for $a \geq 3$ we have to choose $v^2 = 1$, from which, according to (81), follows

$$\frac{1}{1 - \gamma} = 1 + 2(1 + a)^2 \text{ or } \gamma = \frac{2(1 + a)^2}{1 + 2(1 + a)^2}.$$

When $3 + 2\sqrt{2} > a > 3$, $v^2 = \frac{1}{8} \left(1 + \sqrt{33 + \frac{48}{a}} \right)$ is < 1 , and for the corresponding γ we have

$$\frac{\frac{1}{2}\gamma}{(1 + av^2)^2} = \frac{1 + v^2}{2(2v^2 - 1)} = \frac{5a + 4 + \sqrt{a(33a + 48)}}{8(a + 2)} \dots\dots\dots(83).$$

Returning to the $\phi(x)$ distribution, which is found from this distribution by dividing the frequencies by $k(1 + ax^2)^2$, we therefore find, when $\frac{\epsilon}{2}N$ is the number of observations at $x = \pm v$ and $(1 - \epsilon)N$, that, at $x = 0$,

$$\frac{\frac{\epsilon}{2}}{1 - \epsilon} = \frac{1 + v^2}{2(2v^2 - 1)}$$

* A further examination shows that for $a > 3 + 2\sqrt{2}$ σ_y^2 has a minimum but this is smaller than σ_y^2 when $a < 6.7$. Up to this value we therefore have $\sigma_y^2 = \sigma_y^2$ for the best curves. For $a > 6.7$ the minimum of σ_y^2 determines the best distribution.

or

$$\epsilon = \frac{1 + v^2}{3v^2}.$$

Hence

$$\left. \begin{aligned} \mu_2 &= \frac{1}{3}(1 + v^2) \\ \mu_4 &= \frac{v^2}{3}(1 + v^2) \end{aligned} \right\} \dots\dots\dots(84).$$

and

For $\alpha \equiv 3$ we have found $v^2 = 1$ which according to (84) involves $\mu_2 = \mu_4$, so that only the distribution above consisting of three groups can realise the requisite conditions.

When $\alpha > 3$ we have $v < 1$ and therefore $\mu_4 < \mu_2$, so that it must be possible to satisfy the equation (84) by a continuous distribution of observations. However v^2 is decreasing so slowly for increasing α that practically the distribution determined by (84) cannot differ much from three groups of observations.

Our results are accordingly that for a function of the second degree, of which the standard deviation of the observations is $\sigma(1 + \alpha x^2)$, we get the best function for σ_y^2 when $\alpha \equiv 3$ by taking three groups of observations at the middle and the ends of the range, each group proportional to the squared standard deviation at the place, and when $3 + 2\sqrt{2} > \alpha > 3$ by taking three groups of observations determined by (82) and (83).

(6) From (78) we find

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \frac{\mu_4}{\mu_4 - \mu_2^2},$$

which, when μ_2 and μ_4 are found in accordance with (82) and (84), determines the maximum σ_y arrived at from our special three groups of observations. Besides the numerical evaluation of this standard deviation, we give in Table IV below the maximum of σ_y obtained from a distribution for which $\phi(x)$ is constant from -1 to 1 , that is, since, according to (67),

$$\frac{1}{k} = \left(1 + \frac{2}{3}\alpha + \frac{\alpha^2}{5}\right),$$

the distribution

$$\psi(x) = \frac{(1 + \alpha x^2)^2}{1 + \frac{2}{3}\alpha + \frac{\alpha^2}{5}} \cdot \frac{N}{2}.$$

That maximum is determined by

$$\sigma_y^2 = \frac{\sigma^2}{N} \left(1 + \frac{2}{3}\alpha + \frac{\alpha^2}{5}\right) \cdot 9,$$

$\frac{\sigma^2}{N} \cdot 9$ being the maximum σ_y^2 obtained from a rectangular distribution of observations with the standard deviation σ .

The last column of the same table gives the maximum σ_y arrived at when $\phi(x)$ is the rectangular distribution with clusters at -1 and 1 for which $\sigma_y^2 = \sigma_y^2 \frac{x=0}{x^2=1}$. For this distribution consisting of $\cdot 22026 N$ observations at $+1$ and at -1 and $\cdot 5595 N = 2cN$ uniformly distributed from -1 to 1 , we have found as given in

Table II (p. 50) the maximum $\frac{\sigma}{\sqrt{N}} \cdot 1.862$. Hence when μ_2 and μ_4 are the moment coefficient of this $\phi(x)$ the maximum is found from

$$\sigma_y^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \cdot 1.862.$$

We find $\mu_2 = .6270$, $\mu_4 = .5524$ and $\frac{1}{k} = 1 + 1.2540a + .5524a^2$.

The actual distribution is hence

$$\psi(x) = \frac{.27975(1 + ax^2)^2}{1 + 1.2540a + .5524a^2} \cdot N,$$

together with the clusters

$$\frac{.22026(1 + a)^2}{1 + 1.2540a + .5524a^2} \cdot N,$$

at -1 and 1 .

TABLE IV.

a	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for the best distribution	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = \frac{N}{2}$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = c$ and clusters at ± 1
0	1.732	3.000	1.862
1	3.000	4.099	3.120
2	4.359	5.310	4.453
3	5.745	6.573	5.810
4	7.135	7.861	7.178
5	8.522	9.165	8.551

The difference between the first and second maxima taken as a proportion of the first varies from 79 per cent. at $a = 0$ to 8 per cent. at $a = 5$, while the difference between the first and the third maxima varies from 8 per cent. at $a = 0$ to 0.4 per cent. at $a = 5$. The continuous distribution with clusters is therefore especially useful for smaller a .

For $a = 4$ we find from (82) $v = .9816$ and for $a = 5$, $v = .9700$, both of these values of v are so close to 1 that if instead of using them we take the observations at 1 and -1 and let the numbers of the three groups of observations be proportional to the squared standard deviations we get the maxima 7.141 and 8.544 which only differ quite insignificantly from the corresponding values of Table IV.

(7) For a function of the first degree, of which the standard deviation of the observations is $\sigma(1 + ax)$, where $0 \leq a < 1$, we have, according to (66) and (67),

$$\sigma_y^2 = \frac{\sigma^2}{N} \frac{1 + 2a\mu_1 + a^2\mu_2}{\mu_2 - \mu_1^2} \{\mu_2 - 2\mu_1x + x^2\} \dots\dots\dots(85).$$

For $\mu_1 = -c^2$ the maximum of this function is at $x = 1$, and for $\mu_1 = c^2$ at -1 . As the maximum of $(\mu_2 - 2\mu_1x + x^2)$ has the same value in both cases it is clear

that the negative μ_1 gives the lower maximum for σ_y^2 . We therefore only have to find the conditions for $[\sigma_y^2]_{x=1}$ being a minimum when $\mu_1 < 0$.

We have

$$\sigma_y^2 = \frac{\sigma^2}{N} \frac{[(1 + \alpha\mu_1)^2 + \alpha^2(\mu_2 - \mu_1^2)]}{\mu_2 - \mu_1^2} \{\mu_2 - \mu_1^2 + (1 - \mu_1)^2\} \dots\dots\dots(86),$$

and differentiating with regard to μ_2 ,

$$\left[\frac{d\sigma_y^2}{d\mu_2} \right]_{x=1} = \frac{\sigma^2}{N} \cdot \frac{\alpha^2(\mu_2 - \mu_1^2)^2 - (1 - \mu_1)^2(1 + \alpha\mu_1)^2}{(\mu_2 - \mu_1^2)^2}.$$

As $\alpha < 1$, we have $(1 - \mu_1)(1 + \alpha\mu_1) > 0$ and

$$\alpha(\mu_2 - \mu_1^2) - (1 - \mu_1)(1 + \alpha\mu_1) = (\alpha\mu_2 - 1) + \mu_1(1 - \alpha) < 0,$$

from which it follows that

$$\left[\frac{d\sigma_y^2}{d\mu_2} \right]_{x=1} < 0$$

for any $\mu_1 \geq 0$.

The greatest value μ_2 can take for our range -1 to $+1$ is 1 , the minimum of σ_y^2 must therefore be found for $\mu_2 = 1$, for which value (86) passes into

$$[\sigma_y^2]_{x=1} = \frac{\sigma^2}{N} 2 \left\{ 2\alpha + \frac{(1 - \alpha)^2}{1 + \mu_1} \right\},$$

which, since $\mu_1 \geq 0$, is a minimum and equals $\frac{\sigma^2}{N} \cdot 2(1 + \alpha^2)$ when $\mu_1 = 0$.

The $\phi(x)$ distribution ought accordingly to consist of two equally big groups at the ends of the range and the *actual distribution to be chosen for a function of the first degree, the standard deviation of which is a linear function of the variable, should be two groups at the ends of the working range with numbers proportional to the squared standard deviations at these places.*

(8) For a continuous distribution from -1 to 1 with frequencies proportional to the squared standard deviations we have

$$\mu_1 = 0 \text{ and } \mu_2 = \frac{1}{3},$$

and the maximum

$$\sigma_y^2 = \frac{\sigma^2}{N} \left(1 + \frac{\alpha^2}{3} \right) 4,$$

the actual distribution is

$$\psi(x) = \frac{(1 + \alpha x)^2}{1 + \frac{\alpha^2}{3}} \cdot \frac{N}{2}.$$

Table V contains besides the maxima of σ_y from these two distributions those obtained from a distribution for which $\phi(x)$ is constant with two additional clusters at -1 and 1 each consisting of $\frac{N}{4}$ of the observations.

The actual distribution is, since

$$\mu_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

$$\psi(x) = \frac{(1 + \alpha x)^2}{1 + \frac{2}{3}\alpha^2} \cdot \frac{N}{4},$$

with $\frac{(1-\alpha)^2}{1+\frac{2}{3}\alpha^2} \cdot \frac{N}{4}$ observations at -1
 and $\frac{(1+\alpha)^2}{1+\frac{2}{3}\alpha^2} \cdot \frac{N}{4}$ at $+1$ in addition.

The maximum of σ_y^2 is $\frac{\sigma^2}{N} (1 + \frac{2}{3}\alpha^2)^{\frac{5}{2}}$.

TABLE V.

a	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for best distribution	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = \frac{N}{2}$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ for distribution with $\phi(x) = \frac{N}{4}$ and clusters at ± 1
·0	1·414	2·000	1·581
·1	1·421	2·003	1·587
·2	1·442	2·013	1·602
·3	1·477	2·030	1·628
·4	1·523	2·053	1·663
·5	1·581	2·082	1·708
·6	1·649	2·117	1·761
·7	1·726	2·157	1·821
·8	1·811	2·203	1·889
·9	1·903	2·254	1·962

(9) For a *function of the second degree* we found in (5) that when the standard deviation of the observations was $s_y = \sigma(1 + \alpha x^2)$ and $\alpha \geq 3$ it was advantageous to use the whole working range of observations, much more must this be the case when $s_y = \sigma(1 + \alpha x)$ and $0 \leq \alpha < 1$. We shall therefore try to *find the three best groups of observations* taken at -1 , v , and 1 , supposing v unknown. We do not venture to assert that another form of distribution might not lead to a curve of standard deviation with lower maximum, but the solution of the general problem would involve a more elaborate investigation into the possible variations of μ_1 , μ_2 , μ_3 and μ_4 for distributions with limited range than seems desirable in this connection. We shall further limit our problem by assuming that the best distribution will be found among those which make $\sigma_y^2 \stackrel{x=1}{=} \sigma_y^2 \stackrel{x=-1}{=}$ and both also equal to a maximum situated between $x = -1$ and $x = 1$. This would obviously be right if the maximum were found at $x = v$; this in fact is not the case, but still the maximum value is likely to be chiefly determined by the number of observations at $x = v$ and there is therefore every reason to believe that our assumption is justifiable.

Let there be $N\delta$ observations at -1 , $N \cdot \gamma$ at 1 and $N(1 - \delta - \gamma)$ at v . The interpolation formula of Lagrange then gives

$$y = \frac{(x-v)(x-1)}{(1+v) \cdot 2} \bar{y}_{-1} + \frac{(x-v)(x+1)}{(1-v) \cdot 2} \bar{y}_1 + \frac{x^2-1}{v^2-1} y_v,$$

from which we find

$$\sigma_y^2 = \frac{\sigma^2}{N} \left\{ \frac{(x-v)^2 (x-1)^2}{4(1+v)^2} \cdot \frac{(1-a)^2}{\delta} + \frac{(x-v)^2 (x+1)^2}{4(1-v)^2} \cdot \frac{(1+a)^2}{\gamma} + \frac{(x^2-1)^2}{(v^2-1)^2} \cdot \frac{(1+av)^2}{1-\delta-\gamma} \right\}.$$

The condition for

$$\sigma_y^2 = \sigma_y^{x=1}$$

is

$$\frac{(1+a)^2}{\gamma} = \frac{(1-a)^2}{\delta}.$$

Eliminating δ we obtain for $\sigma_y^2 - \sigma_y^{x=1}$ the value

$$\sigma_y^2 - \sigma_y^{x=1} = \frac{\sigma^2}{N} \cdot \frac{(1+a)^2 (x^2-1)}{(v^2-1)^2} \left\{ \frac{(1+av)^2 (x^2-1)}{(1+a)^2 - 2\gamma(1+a^2)} + \frac{1}{2\gamma} [(1+v^2)x^2 + 2v(1-v^2)x + 2 - 5v^2 + v^4] \right\}$$

or

$$\sigma_y^2 - \sigma_y^{x=1} = \frac{\sigma^2}{N} \cdot \frac{(1+a)^2 (x^2-1)}{(v^2-1)^2 [(1+a)^2 - 2\gamma(1+a^2)]} \{ [(1+a)^2 (1+v^2) - 2\gamma(a-v)^2] x^2 + 2v(1-v^2) [(1+a)^2 - 2\gamma(1+a^2)] x + (1+a)^2 (2 - 5v^2 + v^4) - 2\gamma [(1+a^2) (2 - 5v^2 + v^4) + (1+av)^2] \} \dots\dots\dots (87).$$

Our assumption that the maximum σ_y^2 shall be equal to $\sigma_y^{x=1}$ requires that the expression in curled brackets shall be a perfect square for which the condition is

$$2 \left[\frac{\gamma}{(1+a)^2} \right]^2 \{ a^2(1+a^2)v^6 + 2a(1+a^2)v^5 + (3-a^2-3a^4)v^4 - 4a(3+2a^2)v^3 + (-2+9a^2+5a^4)v^2 + 2a(3+a^2)v - a^2(3+2a^2) \} + \frac{\gamma}{(1+a)^2} \{ -a^2v^6 - 2av^5 + (-5+2a^2)v^4 + 12av^3 - (2+9a^2)v^2 - 2av + 3 + 4a^2 \} + v^4 + 2v^2 - 1 = 0 \dots\dots\dots (88).$$

Now $\sigma_y^{x=1} = \frac{\sigma^2}{N} \cdot \frac{(1+a)^2}{\gamma}$ is the maximum which we want to make as low as possible, hence we have for a certain a to find the v for which $\frac{\gamma}{(1+a)^2}$ as given by (88) is a maximum.

We shall examine the cases $a = .5$ and $a = .9$.

(10) For $a = .5$ (88) takes the form

$$\left[\frac{\gamma}{(1+a)^2} \right]^2 \{ .625v^6 + 2.5v^5 + 5.125v^4 - 14v^3 + 1.125v^2 + 6.5v - 1.75 \} + \frac{\gamma}{(1+a)^2} \{ -.25v^6 - v^5 - 4.5v^4 + 6v^3 - 4.25v^2 - v + 4 \} + v^4 + 2v^2 - 1 = 0,$$

which differentiated with regard to v gives

$$\left[\frac{\gamma}{(1+a)^2} \right]^2 \{ 3.75v^6 + 12.5v^4 + 20.5v^3 - 42v^2 + 2.25v + 6.5 \} + \frac{\gamma}{(1+a)^2} \{ -1.5v^5 - 5v^4 - 18v^3 + 18v^2 - 8.5v - 1 \} + 4v(v^2 + 1) = 0.$$

We find that these two equations have for $v = -.190$ the root $\frac{\gamma}{(1+a)^2} = .2936$ in common which represents a maximum.

The maximum of the curve is hence $\frac{\sigma^2}{N} \cdot 3.405$, which value occurs for $x = \pm 1$ and for $x = .064$ determined by (87).

The distribution of observations is

$$\begin{aligned} &.6607 N \text{ at } 1, \\ &.0734 N \text{ at } -1, \\ \text{and} \quad &.2659 N \text{ at } -.190. \end{aligned}$$

For comparison we shall consider what would result from taking for the $\phi(x)$ distribution three equally big groups of observations at $-1, 0$ and 1 . This would for observations with the constant error σ make the maximum of the curve equal to $\frac{\sigma^2}{N} \cdot 3$ and that multiplied by

$$1 + 2a\mu_1 + a^2\mu_2 = \frac{3.5}{3}$$

gives

$$\frac{\sigma^2}{N} \cdot 3.5.$$

The actual distribution $\psi(x)$ would be

$$\begin{aligned} &.6429 N \text{ at } 1, \\ &.0714 N \text{ at } -1, \\ \text{and} \quad &.2857 N \text{ at } 0. \end{aligned}$$

This last distribution only makes the maximum σ'' about 3 per cent. greater than the value which we obtained by our special distribution and it will therefore for most practical cases be as useful.

(11) When $a = .9$ we find for (87),

$$\begin{aligned} &\left[\frac{\gamma}{(1+a)^2} \right]^2 \{2.9322v^6 + 6.516v^5 + .4434v^4 - 33.264v^3 + 17.141v^2 + 13.716v - 7.4844\} \\ &+ \frac{\gamma}{(1+a)^2} \{-.81v^6 - 1.8v^5 - 3.38v^4 + 10.8v^3 - 9.29v^2 - 1.8v + 6.24\} \\ &+ v^4 + 2v^2 - 1 = 0, \end{aligned}$$

which differentiated with regard to v gives

$$\begin{aligned} &\left[\frac{\gamma}{(1+a)^2} \right]^2 \{17.5932v^5 + 32.58v^4 + 1.7736v^3 - 99.792v^2 + 34.282v + 13.716\} \\ &+ \frac{\gamma}{(1+a)^2} \{-4.86v^5 - 9v^4 - 13.52v^3 + 32.4v^2 - 18.58v - 1.8\} + 4v(v^2 + 1) = 0. \end{aligned}$$

For $v = -.354$ these two equations have the root $\frac{\gamma}{(1+a)^2} = .23214$ in common which is therefore the maximum of $\frac{\gamma}{(1+a)^2}$.

The maximum of the corresponding σ_y^2 is hence

$$\frac{\sigma^2}{N} \cdot \frac{(1 + \alpha)^2}{\gamma} = \frac{\sigma^2}{N} \cdot 4.308.$$

From (87) we find that it occurs at $x = .125$ as well as at $x = \pm 1$. The distribution of observations is then

$$.8380 N \text{ at } 1,$$

$$.0023 N \text{ at } -1,$$

and

$$.1597 N \text{ at } -.354.$$

Comparing again with a distribution consisting of three groups of observations at $-1, 0$ and 1 with frequencies proportional to the squared standard deviations at these places we find that the distribution would be

$$.7814 N \text{ at } 1,$$

$$.0022 N \text{ at } -1,$$

and

$$.2164 N \text{ at } 0,$$

and the maximum of σ_y^2 would be

$$\frac{\sigma^2}{N} \cdot 3(1 + 2\alpha\mu_1 + \alpha^2\mu_2) = \frac{\sigma^2}{N} \cdot 4.62.$$

We thus find that by our special distribution the maximum of σ_y^2 was 7 per cent. lower, the choice of that distribution would thus permit us to reduce the total number of observations at the same rate without raising the maximum of σ_y^2 .

(12) The result of these investigations is that *the maximum σ_y obtained from the best three groups of observations differs so little from that obtained from three groups at $-1, 0$ and 1 that the first grouping only in quite exceptional practice would be preferred.*

We shall therefore in Table VI give the maximum σ_y arrived at from the following three distributions: (1) three groups of observations at $-1, 0$ and 1 in numbers proportional to the squared standard deviations at these places, (2) a distribution for which $\phi(x) = \frac{N}{2}$, and (3) a distribution for which $\phi(x) = .2797 N$ with additional clusters $.2203 N$ at ± 1 (see Table II, p. 50).

Both in Table V and in Table VI the difference between the two first maxima as a proportion of the first decreases with increasing α so that the distribution with uniform $\phi(x)$ is more profitable for $\alpha > 0$ than for observations with constant errors.

VIII. *Best distribution of observations for determining a single constant of the function.*

(1) Our choice of observations has hitherto aimed at giving within the working range of observations a determination of the function as accurate and uniform as possible. We shall now consider what is the best choice of observations for

TABLE VI.

a	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from three groups at 0 and ± 1	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = \frac{N}{2}$	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from distribution for which $\phi(x) = .2797N$ and clusters at ± 1	Maximum of $\sigma_y \frac{\sqrt{N}}{\sigma}$ from best three groups
.0	1.732	3.000	1.862	—
.1	1.738	3.005	1.868	—
.2	1.755	3.020	1.886	—
.3	1.783	3.045	1.914	—
.4	1.822	3.079	1.954	—
.5	1.871	3.122	2.003	1.845
.6	1.929	3.175	2.062	—
.7	1.995	3.236	2.129	—
.8	2.069	3.304	2.205	—
.9	2.149	3.381	2.287	2.076

determining a single constant of the function. The investigations will be carried out for functions of the first and of the second degree for which the standard deviations of the observations are

$$s_y = \sigma(1 + ax^2), \quad a > -1$$

or

$$s_y = \sigma(1 + ax), \quad 1 > a \geq 0.$$

We have in (3) of Section I given the formula (8) for $\sigma_{a_p}^2$ and shall here give only the form to which it is transferred by putting

$$\psi(x) = k\phi(x)f(x),$$

$$\frac{1}{k} = \frac{1}{N} \int \phi(x) \cdot f(x) dx.$$

The formula analogous to that given for σ_y^2 (66) is

$$\left| \begin{array}{cccccccc} \sigma_{a_p}^2 \cdot \frac{N}{\sigma^2} k & 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & 1 & \mu_1 & \mu_2 & \dots & \mu_p & \dots & \mu_n \\ 0 & \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{p+1} & \dots & \mu_{n+1} \\ 0 & \mu_2 & \mu_3 & \mu_4 & \dots & \mu_{p+2} & \dots & \mu_{n+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & \mu_p & \mu_{p+1} & \mu_{p+2} & \dots & \mu_{2p} & \dots & \mu_{n+p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \mu_n & \mu_{n+1} & \mu_{n+2} & \dots & \mu_{p+n} & \dots & \mu_{2n} \end{array} \right| = 0 \dots (89).$$

(2) For a function of the first degree

$$y = a_0 + a_1x,$$

for which the standard deviation of an observation is

$$s_y = \sigma(1 + ax^2), \quad a > -1,$$

and therefore

$$\frac{1}{k} = 1 + 2a\mu_2 + a^2\mu_4,$$

we find, according to (89),

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \left(1 + \frac{\mu_1^2}{\mu_2 - \mu_1^2}\right) \dots\dots\dots (90),$$

and

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \frac{1}{\mu_2 - \mu_1^2} \dots\dots\dots (91).$$

As for any skew distribution of observations we can find a corresponding symmetrical distribution with the same μ_2 and μ_4 , both these expressions are a minimum for $\mu_1 = 0$.

We have already shown in (2) of Section VII that any possible values of μ_2 and μ_4 can be produced by three symmetrical groups of observations, so that by introducing the variables v and γ determined by

$$\mu_2 = v^2\gamma,$$

and

$$\mu_4 = v^4\gamma,$$

and limited by

$$v^2 \leq 1,$$

$$0 \leq \gamma \leq 1,$$

we do not leave out any possibilities.

From (90) we then get

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2a\gamma v^2 + a^2\gamma v^4),$$

which for $a > 0$ is a minimum when $\gamma = v^2 = 0$, and for $a = 0$ is $\frac{\sigma^2}{N}$ for any γ and v^2 .

For $a < 0$ we find, since

$$\frac{d\sigma_{a_0}^2}{dv^2} = \frac{\sigma^2}{N} 2a\gamma (1 + av^2) \text{ and } v^2 < -\frac{1}{a},$$

that for a constant γ , $\sigma_{a_0}^2$ has the least value when v^2 is as great as possible, that is for $v^2 = 1$.

The minimum of $\sigma_{a_0}^2$ is then

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} \{1 + (2 + a)\alpha\gamma\},$$

which, since $a(2 + a) < 0$, is a minimum when γ takes its greatest possible value 1.

The minimum is thus

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + a)^2.$$

Hence we conclude that:

when $a > 0$, $\sigma_{a_0}^2$ is a minimum and equal to $\frac{\sigma^2}{N}$ for N observations at $x = 0$,

when $a = 0$, $\sigma_{a_0}^2$ is a minimum and equal to $\frac{\sigma^2}{N}$ for any distribution for which $\mu_1 = 0$,

and

when $a < 0$, $\sigma_{a_0}^2$ is a minimum and equal to $\frac{\sigma^2}{N} (1 + a)^2$ for two equally big groups of observations at ± 1 .

(3) When we introduce $\mu_1 = 0$, $\mu_2 = \gamma v^2$ and $\mu_4 = \gamma v^4$ in (91) we get

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\gamma v^2 + \alpha^2\gamma v^4) \frac{1}{\gamma v^2}.$$

This for constant v^2 is a minimum when $\gamma = 1$ and then equal to

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2\alpha v^2 + \alpha^2 v^4) \frac{1}{v^2} \dots\dots\dots(92).$$

As

$$\frac{d\sigma_{a_1}^2}{dv^2} = \frac{\sigma^2}{N} \left(\alpha^2 - \frac{1}{v^4} \right),$$

$v^2 = \pm \frac{1}{\alpha}$ when possible, that is for $\alpha \geq 1$ determines a minimum, while for $\alpha < 1$, $\sigma_{a_1}^2$ reaches its lowest value for $v^2 = 1$. From (92) we find for $\alpha \geq 1$ the minimum

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} \cdot 4\alpha,$$

and for $\alpha < 1$ the minimum

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + \alpha)^2,$$

both formulae giving $\sigma_{a_1}^2 = \frac{\sigma^2}{N} \cdot 4$ for $\alpha = 1$.

Our results are accordingly:

when $\alpha > 1$, $\sigma_{a_1}^2$ is a minimum and equal to $\frac{\sigma^2}{N} \cdot 4\alpha$ for two equally big groups of

observations at $x = \pm \frac{1}{\alpha}$ or for any distribution with the same μ_2 and μ_4 ,

and when $\alpha \leq 1$, $\sigma_{a_1}^2$ is a minimum and equal to $\frac{\sigma^2}{N} (1 + \alpha)^2$ for two equally big groups of observations at $x = \pm 1$.

We see that for $\alpha \geq 0$ two equally big groups of observations at ± 1 make both $\sigma_{a_0}^2$ and $\sigma_{a_1}^2$ minima and these groups in addition form the distribution for which $\sigma_{a_1}^2$ has the lowest maximum within the possible range of observations.

(4) For a function of the second degree

$$y = a_0 + a_1 x + a_2 x^2,$$

with the standard deviations of observations

$$s_y = \sigma (1 + \alpha x^2), \quad \alpha > -1,$$

and therefore

$$\frac{1}{k} = 1 + 2\alpha\mu_2 + \alpha^2\mu_4,$$

we find, from (89),

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \cdot \frac{\mu_2\mu_4 - \mu_3^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots\dots\dots(93),$$

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \cdot \frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots\dots\dots(94),$$

and

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_2 + \alpha^2\mu_4) \cdot \frac{\mu_2 - \mu_1^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots\dots\dots(95).$$

We shall prove that the last factor of each of these formulae is a minimum for

$$\mu_1 = \mu_3 = 0.$$

To prove this for (93) we consider the difference

$$\frac{\mu_2\mu_4}{\mu_2\mu_4 - \mu_2^3} - \frac{\mu_3^2}{\mu_3^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4} = \frac{\mu_2(\mu_2\mu_3 - \mu_1\mu_4)^2}{\mu_2(\mu_4 - \mu_2^2)[(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)]} > 0,$$

from which follows

$$\frac{\mu_2\mu_4 - \mu_2^3}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} > \frac{\mu_2\mu_4}{\mu_2\mu_4 - \mu_2^3} > \frac{\mu_3^2}{\mu_3^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4}.$$

For (94) it is at once clear that

$$\frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3 - [(\mu_3 - \mu_1\mu_2)^2 + \mu_1^2(\mu_4 - \mu_2^2)]} > \frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3}.$$

For the case of (95) we compare

$$\frac{\mu_2}{\mu_2\mu_4 - \mu_2^3} \text{ and } \frac{\mu_1^2}{\mu_3^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4},$$

and we find the difference

$$\frac{1}{\mu_4 - \mu_2^2} - \frac{1}{\mu_4 - \mu_2^2 + \left(\frac{\mu_3}{\mu_1} - \mu_2\right)^2} > 0,$$

and hence

$$\frac{\mu_2 - \mu_1^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} > \frac{\mu_2}{\mu_2\mu_4 - \mu_2^3} > \frac{\mu_1^2}{\mu_3^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4}.$$

It is thus proved for the three formulae that a distribution of observations for which $\mu_1 = \mu_3 = 0$ gives lower values than any distribution with the same μ_2 and μ_4 as the former and with $\mu_1 \leq 0$, $\mu_3 \geq 0$.

Hence our problem is reduced to finding the μ_2 and μ_4 which make the following expressions minima:

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \frac{\mu_4}{\mu_4 - \mu_2^2} \dots\dots\dots(96),$$

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \frac{1}{\mu_2} \dots\dots\dots(97),$$

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (1 + 2a\mu_2 + a^2\mu_4) \frac{1}{\mu_4 - \mu_2^2} \dots\dots\dots(98).$$

(5) Introducing $\mu_2 = \gamma v^2$ and $\mu_4 = \gamma v^4$ in (96) we get

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} \left(1 + \frac{\gamma}{1 - \gamma} (1 + av^2)^2 \right),$$

which is seen to be $> \frac{\sigma^2}{N}$ except when $\gamma = 0$.

Hence the minimum value of $\sigma_{a_0}^2 = \frac{\sigma^2}{N}$ can only be obtained by taking all the observations at $x = 0$.

(97) is identical with (91) for $\mu_1 = 0$. The conditions for a minimum of $\sigma_{a_1}^2$ are

therefore the same for a function of the second degree as for a function of the first degree. That is, when $\alpha > 1$, $\sigma_{a_1}^2$ is a minimum and equal to $\frac{\sigma^2}{N} \cdot 4\alpha$ for two equally big groups of observations at $x = \pm \frac{1}{\alpha}$, or for any distribution with the same μ_2 and μ_4 , and when $\alpha \leq 1$, $\sigma_{a_1}^2$ is a minimum and equal to $\frac{\sigma^2}{N} (1 + \alpha)^2$ for two equally big groups of observations at $x = \pm 1$.

With the variates γ and v (98) takes the form

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\gamma v^2 + \alpha^2\gamma v^4) \frac{1}{v^4\gamma(1-\gamma)}.$$

By differentiating with regard to v^2 we get

$$\frac{d\sigma_{a_2}^2}{dv^2} = \frac{\sigma^2}{N} \cdot \frac{2}{\gamma(1-\gamma)v^6} (-1 - \alpha\gamma v^2),$$

which is negative for any α , v and γ within our limits.

For constant γ , $\sigma_{a_2}^2$ is therefore least when $v^2 = 1$ and the minimum value is

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} \left(\frac{1}{\gamma} + 2\alpha + \alpha^2 \right) \frac{1}{1-\gamma} \dots\dots\dots (99).$$

This is again a minimum when

$$\frac{d\sigma_{a_2}^2}{d\gamma} = \frac{\sigma^2}{N} \cdot \frac{1}{\gamma^2(1-\gamma)^2} \{ \alpha(2+\alpha)\gamma^2 + 2\gamma - 1 \} = 0,$$

that is for $\gamma = \frac{1}{2+\alpha}$ which gives a minimum both for positive and negative α .

Thus the distribution that makes $\sigma_{a_2}^2$ a minimum has a $\phi(x)$ -distribution consisting of $\frac{N}{2(2+\alpha)}$ observations at -1 and 1 and $\frac{1+\alpha}{2+\alpha}N$ observations at 0 .

We have

$$\mu_2 = \mu_4 = \frac{1}{2+\alpha}$$

and

$$\frac{1}{k} = (1+\alpha).$$

The relation

$$\psi(x) = k\phi(x)f(x)$$

then gives us

$$\psi(0) = \frac{N}{2+\alpha}$$

and

$$\psi(\pm 1) = \frac{1+\alpha}{2(2+\alpha)} \cdot N.$$

From (99) we find the minimum value

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (2+\alpha)^2.$$

Our result is thus that $\sigma_{a_2}^2$ is a minimum and equal to $\frac{\sigma^2}{N} (2+\alpha)^2$ for a distribution consisting of $\frac{N}{2+\alpha}$ observations at 0 and $\frac{1+\alpha}{2(2+\alpha)}N$ at ± 1 .

(6) When the standard deviation of an observation is

$$s_y = \sigma(1 + \alpha x) \quad \text{and} \quad 0 \leq \alpha < 1,$$

we have

$$\frac{1}{k} = 1 + 2\alpha\mu_1 + \alpha^2\mu_2,$$

and according to (89) we find for a function of the first degree

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_1 + \alpha^2\mu_2) \frac{\mu_2}{\mu_2 - \mu_1^2} \dots\dots\dots(100)$$

and

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_1 + \alpha^2\mu_2) \frac{1}{\mu_2 - \mu_1^2} \dots\dots\dots(101).$$

By differentiating (100) we find

$$\frac{d\sigma_{a_0}^2}{d\mu_1} = \frac{\sigma^2}{N} \frac{2\mu_2(1 + \alpha\mu_1)(\mu_1 + \alpha\mu_2)}{(\mu_2 - \mu_1^2)^2}$$

and

$$\frac{d\sigma_{a_0}^2}{d\mu_2} = \frac{\sigma^2}{N} \frac{(\alpha\mu_2 - 2\alpha\mu_1^2 - \mu_1)(\mu_1 + \alpha\mu_2)}{(\mu_2 - \mu_1^2)^2}.$$

Both of these can only be zero when

$$\mu_1 + \alpha\mu_2 = 0 \quad \dots\dots\dots(102),$$

which is seen to determine a minimum of $\sigma_{a_0}^2$ the value of which is $\frac{\sigma^2}{N}$. The

condition $\mu_2 = -\frac{\mu_1}{\alpha}$ can be fulfilled by an infinity of different distributions. From

$$0 \leq \mu_2 \leq 1$$

follows the condition

$$0 \geq \mu_1 \geq -\alpha.$$

We shall confine our attention to those distributions which consist of two groups of observations. Let there be $N\gamma$ observations at v_1 and $(1 - \gamma)N$ at v_2 , we then have

$$\mu_1 = v_2 + \gamma(v_1 - v_2),$$

$$\mu_2 = v_2^2 + \gamma(v_1^2 - v_2^2),$$

from which by means of (102) is found

$$-\frac{\gamma}{v_2(1 + \alpha v_2)} = \frac{1 - \gamma}{v_1(1 + \alpha v_1)} = \frac{1}{(v_1 - v_2)\{1 + \alpha(v_1 + v_2)\}}$$

and

$$\frac{1}{k} = 1 + \alpha\mu_1 = \frac{(1 + \alpha v_1)(1 + \alpha v_2)}{1 + \alpha(v_1 + v_2)}.$$

Thus we find that the $\phi(x)$ -distribution consists of

$$\frac{-v_2(1 + \alpha v_2)}{(v_1 - v_2)\{1 + \alpha(v_1 + v_2)\}} N \text{ at } v_1$$

and

$$\frac{v_1(1 + \alpha v_1)}{(v_1 - v_2)\{1 + \alpha(v_1 + v_2)\}} N \text{ at } v_2,$$

while the actual distribution

$$\psi(x) = \frac{1 + \alpha(v_1 + v_2)}{(1 + \alpha v_1)(1 + \alpha v_2)} (1 + \alpha x)^2 \phi(x)$$

consists of
$$\left. \begin{aligned} & -v_2 \frac{(1 + \alpha v_1)}{v_1 - v_2} N \text{ at } v_1 \\ & v_1 \frac{(1 + \alpha v_2)}{v_1 - v_2} N \text{ at } v_2 \end{aligned} \right\} \dots\dots\dots(103).$$

and

We thus see that for any two points v_1 and v_2 of which one is negative and the other positive we can choose the numbers of observations so as to make $\sigma_{a_0}^2 = \frac{\sigma^2}{N}$ as it of course would be by taking a single group of observations at $x = 0$.

(7) By differentiating (101) we get

$$\frac{d\sigma_{a_1}^2}{d\mu_1} = \frac{\sigma^2}{N} \frac{2}{(\mu_2 - \mu_1)^2} (1 + \alpha\mu_1)(\mu_1 + \alpha\mu_2) \dots\dots\dots(104)$$

and

$$\frac{d\sigma_{a_1}^2}{d\mu_2} = -\frac{\sigma^2}{N} \frac{(1 + \alpha\mu_1)^2}{(\mu_2 - \mu_1)^2}.$$

As the latter is always negative $\sigma_{a_1}^2$ is for constant μ_1 least when μ_2 has its greatest value, that is 1.

Introducing this in (104) we get as condition for a minimum,

$$\mu_1 + \alpha = 0.$$

There is only one distribution for which $\mu_2 = 1$ and $\mu_1 = -\alpha$, and it is that consisting of two groups of observations at -1 and 1 included in the distributions examined in (6).

From (103) we find that the actual distribution consists of $\frac{1-\alpha}{2}N$ observations at -1 and $\frac{1+\alpha}{2}N$ at 1 . The minimum of $\sigma_{a_1}^2$ is from (101) found to be $\frac{\sigma^2}{N}$.

The minimum $\frac{\sigma^2}{N}$ of $\sigma_{a_1}^2$ can thus only be obtained by taking two groups of observations at the limits of the range with numbers proportional to the standard deviation of observations at these places. This distribution makes also $\sigma_{a_0}^2$ a minimum, but it is not, except when $\alpha = 0$, the distribution which gives σ_y^2 the lowest maximum value within the possible range of observations.

(8) For a function of the second degree,

$$y = a_0 + a_1x + a_2x^2$$

with the standard deviation

$$\sigma_y = \sigma(1 + \alpha x),$$

where

$$0 \leq \alpha < 1$$

we have

$$\frac{1}{k} = 1 + 2\alpha\mu_1 + \alpha^2\mu_2,$$

and from (89),

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_1 + \alpha^2\mu_2) \frac{\mu_2\mu_4 - \mu_3^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots(105),$$

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_1 + \alpha^2\mu_2) \frac{\mu_4 - \mu_2^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots(106),$$

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (1 + 2\alpha\mu_1 + \alpha^2\mu_2) \frac{\mu_2 - \mu_1^2}{\mu_2\mu_4 - \mu_2^3 - \mu_3^2 + 2\mu_1\mu_2\mu_3 - \mu_1^2\mu_4} \dots(107).$$

(105) may be brought into the form

$$\sigma_{a_0}^2 = \frac{\sigma^2}{N} \left\{ 1 + \frac{\mu_2 \cdot (\mu_2 \mu_4 - \mu_3^2) \left(a + \frac{\mu_1}{\mu_2} \right)^2 + \frac{1}{\mu_2} (\mu_2^2 - \mu_1 \mu_3)^2}{\mu_2 \mu_4 - \mu_3^2 - \mu_2^2 + 2\mu_1 \mu_2 \mu_3 - \mu_1^2 \mu_4} \right\},$$

where the denominator and $\mu_2 \mu_4 - \mu_3^2$ are always positive. Hence the condition for $\sigma_{a_0}^2$, taking its minimum value $\frac{\sigma^2}{N}$, is

$$\mu_1 + a\mu_2 = 0 \quad \text{and} \quad \mu_2^2 - \mu_1 \mu_3 = 0$$

$$\text{or} \quad \frac{\mu_3}{\mu_2} = \frac{\mu_2}{\mu_1} = -\frac{1}{a} \dots \dots \dots (108).$$

We shall examine the possible distributions consisting of three groups of observations with the frequencies γ_1 , γ_2 and γ_3 at v_1 , v_2 and v_3 . The conditions (108) require

$$\frac{\gamma_1 v_1^2 + \gamma_2 v_2^2 + \gamma_3 v_3^2}{\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3} = \frac{\gamma_1 v_1^3 + \gamma_2 v_2^3 + \gamma_3 v_3^3}{\gamma_1 v_1^2 + \gamma_2 v_2^2 + \gamma_3 v_3^2} = \frac{\gamma_2 v_2^2 (v_2 - v_1) + \gamma_3 v_3^2 (v_3 - v_1)}{\gamma_2 v_2 (v_2 - v_1) + \gamma_3 v_3 (v_3 - v_1)} = -\frac{1}{a}$$

$$\text{or} \quad \frac{\gamma_1 v_1 (1 + av_1)}{v_2 - v_3} = \frac{\gamma_2 v_2 (1 + av_2)}{v_3 - v_1} = \frac{\gamma_3 v_3 (1 + av_3)}{v_1 - v_2} \dots \dots \dots (109).$$

Now $\frac{v_1}{v_2 - v_3}$, $\frac{v_2}{v_3 - v_1}$ and $\frac{v_3}{v_1 - v_2}$ can never all have the same sign and $(1 + av)$ is for any $v \geq -1$ positive, from which it follows that (109) leads to negative frequencies. Nor can (109) be satisfied by two groups of observations as $\gamma_2 = 0$ requires $v_1 = v_3 = 0$, that is one group of observations at $x = 0$ which of course gives $\sigma_{a_0}^2 = \frac{\sigma^2}{N}$.

(9) We may write (106)

$$\sigma_{a_1}^2 = \frac{\sigma^2}{N} \left(\frac{1}{\mu_2} + \frac{1}{\mu_2} \cdot \frac{(\mu_4 - \mu_2^2) (\mu_1 + a\mu_2)^2 + (\mu_3 - \mu_1 \mu_2)^2}{(\mu_4 - \mu_2^2) (\mu_2 - \mu_1^2) - (\mu_3 - \mu_1 \mu_2)^2} \right),$$

where the last ratio is seen to be positive unless

$$\mu_1 + a\mu_2 = 0 \quad \text{and} \quad \mu_3 - \mu_1 \mu_2 = 0 \dots \dots \dots (110).$$

If therefore any distribution of observations can give $\frac{1}{\mu_2}$ its minimum value 1 and at the same time fulfil those conditions it will make $\sigma_{a_1}^2$ a minimum and equal to $\frac{\sigma^2}{N}$. But $\mu_2 = 1$ together with (110) lead to

$$\mu_3 = \mu_1 = -a,$$

which require $\frac{1+a}{2}N$ observations at -1 and $\frac{1-a}{2}N$ at 1 , whereas the actual distribution must consist of $\frac{1-a}{2}N$ observations at -1 and $\frac{1+a}{2}N$ at 1 .

Thus the only distribution which makes $\sigma_{a_1}^2$ a minimum and equal to $\frac{\sigma^2}{N}$ is that consisting of $\frac{1-a}{2}N$ observations at -1 and $\frac{1+a}{2}N$ at 1 .

(10) The general minimum conditions for σ_{a_2} cannot be found without more elaborate investigations into the possible variations of the moment coefficients than are at present available and we shall limit our research to the case of three groups of observations.

Let us suppose $\gamma_1 N$, $\gamma_2 N$ and $(1 - \gamma_1 - \gamma_2) N$ observations taken at x_1 , x_2 and x_3 , and let the corresponding means be $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$.

We then find, when

$$\Delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1),$$

$$a_2 = \frac{1}{\Delta} \{ \bar{y}_1(x_3 - x_2) + \bar{y}_2(x_1 - x_3) + \bar{y}_3(x_2 - x_1) \},$$

and

$$\sigma_{a_2}^2 = \frac{\sigma^2}{\Delta^2 \cdot N} \left\{ \frac{(x_3 - x_2)^2 (1 + ax_1)^2}{\gamma_1} + \frac{(x_1 - x_3)^2 (1 + ax_2)^2}{\gamma_2} + \frac{(x_2 - x_1)^2 (1 + ax_3)^2}{1 - \gamma_1 - \gamma_2} \right\} \dots\dots\dots (111).$$

Differentiations first with regard to γ_1 and then with regard to γ_2 give the minimum conditions

$$\frac{\gamma_1^2}{(x_3 - x_2)^2 (1 + ax_1)^2} = \frac{\gamma_2^2}{(x_1 - x_3)^2 (1 + ax_2)^2} = \frac{(1 - \gamma_1 - \gamma_2)^2}{(x_2 - x_1)^2 (1 + ax_3)^2},$$

or, when we suppose $x_1 < x_2 < x_3$,

$$\frac{\gamma_1}{(x_3 - x_2)(1 + ax_1)} = \frac{-\gamma_2}{(x_1 - x_3)(1 + ax_2)} = \frac{1 - \gamma_1 - \gamma_2}{(x_2 - x_1)(1 + ax_3)} = \frac{1}{2(x_3 - x_1)(1 + ax_2)} \dots\dots\dots (112).$$

With these values for γ_1 and γ_2 we get from (111)

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} \left\{ \frac{2(x_3 - x_1)(1 + ax_2)}{\Delta} \right\}^2 = \frac{\sigma^2}{N} \left\{ \frac{2(1 + ax_2)}{(x_2 - x_1)(x_3 - x_2)} \right\}^2.$$

This for constant x_2 is obviously a minimum for $x_1 = -1$ and $x_3 = 1$ and is then equal to

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} \left\{ \frac{2(1 + ax_2)}{1 - x_2^2} \right\}^2.$$

From this we find

$$\frac{d\sigma_{a_2}}{dx_2} = \frac{\sigma}{\sqrt{N}} \frac{2(ax_2^2 + 2x_2 + a)}{(1 - x_2^2)^2},$$

which shows that

$$x_2 = \sqrt{\frac{1}{a^2} - 1} - \frac{1}{a}$$

determines a minimum.

The minimum value is

$$\sigma_{a_2}^2 = \frac{\sigma^2}{N} (1 + \sqrt{1 - a^2})^2,$$

and the frequencies found from (112) are

$$\frac{1}{4a} \sqrt{1 - a} (\sqrt{1 + a} - \sqrt{1 - a}) \cdot N \text{ at } -1,$$

$$\frac{1}{4\alpha} \sqrt{1+\alpha} (\sqrt{1+\alpha} - \sqrt{1-\alpha}) \cdot N \text{ at } 1,$$

and
$$\frac{1}{2} N \text{ at } -\frac{1}{\alpha} (1 - \sqrt{1-\alpha^2}).$$

IX. *Adjustment with regard to both of two variates connected by a linear relation.*

(1) The case often occurs when both of the variates observed have errors of observations of the same order so that adjustment only of one of them is unsatisfactory. We shall therefore in this section consider adjustment with regard to both of the variates and give the adjusted relation between them and the standard deviations of the constants.

Let x' be observed with the standard deviation $\sqrt{\alpha}\sigma$ and y' with the standard deviation $\sqrt{\gamma}\sigma$, we shall then for the sake of greater perspicuity exchange the variates for $x = \frac{x'}{\sqrt{\alpha}}$ and $y = \frac{y'}{\sqrt{\gamma}}$ so that both of our variates have the same standard deviation σ . Let $\frac{1}{N} \sum \{x^r y^s\}$ taken over the N pairs of observations be denoted by $\mu_{r,s}$, we then find, by adjusting only the y 's according to (3),

$$\begin{vmatrix} y & 1 & x \\ \mu_{01} & 1 & \mu_{10} \\ \mu_{11} & \mu_{10} & \mu_{20} \end{vmatrix} = 0,$$

or
$$y - \mu_{01} = \frac{\mu_{11} - \mu_{01}\mu_{10}}{\mu_{20} - \mu_{10}^2} (x - \mu_{10}) \dots\dots\dots(113).$$

By adjusting only the x 's we get

$$y - \mu_{01} = \frac{\mu_{02} - \mu_{01}^2}{\mu_{11} - \mu_{01}\mu_{10}} (x - \mu_{10}) \dots\dots\dots(114),$$

which only coincide with (113) when

$$(\mu_{20} - \mu_{10}^2)(\mu_{02} - \mu_{01}^2) = (\mu_{11} - \mu_{01}\mu_{10})^2,$$

that is when there is perfect correlation between x and y and no casual errors of observation.

(2) Adjusting at the same time with regard to x and y may be transformed to the problem of finding the straight line for which the sum of the squared distances of the observed points (x, y) is a minimum.

Let the line sought be

$$x \cos v + y \sin v + p = 0.$$

The sum which we want to make a minimum is then

$$S = \mu_{20} \cos^2 v + \mu_{02} \sin^2 v + 2\mu_{11} \cos v \sin v + 2p\mu_{10} \cos v + 2p\mu_{01} \sin v + p^2,$$

$$\frac{dS}{dp} = 0 \text{ requires } p = -\mu_{10} \cos v - \mu_{01} \sin v,$$

indicating that the line passes through the mean (μ_{10}, μ_{01}) ; this determines a minimum for constant v .

The corresponding S is

$$S = (\mu_{20} - \mu_{10}^2) \cos^2 v + (\mu_{02} - \mu_{01}^2) \sin^2 v + 2(\mu_{11} - \mu_{01}\mu_{10}) \cos v \sin v \dots (115),$$

which differentiated with regard to v gives

$$\frac{dS}{dv} = -\{\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)\} \sin 2v + 2(\mu_{11} - \mu_{01}\mu_{10}) \cos 2v.$$

It thus follows that

$$\tan 2v = \frac{2(\mu_{11} - \mu_{01}\mu_{10})}{\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)} = \frac{2 \tan v}{1 - \tan^2 v},$$

or

$$\tan v = \frac{\frac{1}{2}\{\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2) \pm \sqrt{[\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2)]^2 + 4[\mu_{11} - \mu_{01}\mu_{10}]^2}\}}{\mu_{11} - \mu_{01}\mu_{10}} \dots (116)$$

determine a maximum and a minimum of S .

Substituting in (115) we find

$$S = \frac{1}{2}\{\mu_{20} - \mu_{10}^2 + \mu_{02} - \mu_{01}^2 \pm \sqrt{[\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)]^2 + 4[\mu_{11} - \mu_{01}\mu_{10}]^2}\},$$

so that the minimum corresponds to the negative sign of the root in (116).

The adjusted function connecting x and y is hence a line through the general mean forming an angle u with the x -axis which is determined by

$$\tan u = -\cot v = \frac{\mu_{02} - \mu_{01}^2 - (\mu_{20} - \mu_{10}^2) + \sqrt{[\mu_{20} - \mu_{10}^2 - (\mu_{02} - \mu_{01}^2)]^2 + 4[\mu_{11} - \mu_{01}\mu_{10}]^2}}{2(\mu_{11} - \mu_{01}\mu_{10})} \dots (117).$$

For the variates x' and y' there must to this value of the tangent be added the factor $\sqrt{\frac{\gamma}{\alpha}}$, expressed by the moment coefficients of x' and y' we therefore find

$$\tan u = \frac{\alpha(\mu'_{02} - \mu_{01}'^2) - \gamma(\mu'_{20} - \mu_{10}'^2) + \sqrt{[\gamma(\mu'_{20} - \mu_{10}'^2) - \alpha(\mu'_{02} - \mu_{01}'^2)]^2 + 4\alpha\gamma[\mu'_{11} - \mu_{01}'\mu_{10}']^2}}{2\alpha(\mu'_{11} - \mu_{01}'\mu_{10}')} \dots (118).$$

(3) We shall prove that the line is situated between the two regression curves (113) and (114).

Making (μ_{10}, μ_{01}) the zero point of the coordinates, the three tangents to be compared are

$$\frac{\mu_{11}}{\mu_{20}}, \quad \frac{\mu_{02}}{\mu_{11}} \quad \text{and} \quad \frac{1}{2\mu_{11}}\{\mu_{02} - \mu_{20} + \sqrt{(\mu_{02} - \mu_{20})^2 + 4\mu_{11}^2}\} = \tan u,$$

where the μ 's now are the moment coefficients about the mean.

According to $\mu_{11} \geq 0$ we have

$$\frac{\mu_{11}}{\mu_{20}} \leq \frac{\mu_{02}}{\mu_{11}},$$

since

$$\mu_{11}^2 < \mu_{20} \cdot \mu_{02}.$$

As

$$\sqrt{(\mu_{02} - \mu_{20})^2 + 4\mu_{11}^2} < \mu_{02} + \mu_{20},$$

we have

$$\tan u \leq \frac{\mu_{02}}{\mu_{11}}.$$

It rests to compare $\tan u$ and $\frac{\mu_{11}}{\mu_{20}}$, we find

$$\tan u - \frac{\mu_{11}}{\mu_{20}} = \frac{1}{2\mu_{11}} \left\{ \mu_{02} - \mu_{20} - \frac{2\mu_{11}^2}{\mu_{20}} + \sqrt{\left[\mu_{02} - \mu_{20} - \frac{2\mu_{11}^2}{\mu_{20}} \right]^2 + \frac{4\mu_{11}^2}{\mu_{20}^2} (\mu_{02}\mu_{20} - \mu_{11}^2)} \right\}.$$

The factor in curled brackets is hence positive and we have $\tan u >$ or $< \frac{\mu_{11}}{\mu_{20}}$ according as

$$\mu_{11} > \text{ or } < 0,$$

we have thus proved that

$$\frac{\mu_{11}}{\mu_{20}} \leq \tan u \leq \frac{\mu_{02}}{\mu_{11}}.$$

(4) In order to find the standard deviations of the constants of the line we shall express the observations, the standard deviations of which are $\sqrt{a}\sigma$ and $\sqrt{\gamma}\sigma$, by a parameter r to get an equation for each observation.

Suppose

$$x_i = a + r_i \cos u,$$

$$y_i = b + r_i \sin u,$$

and suppose we have a good approximation for $a, b, u, r_1, r_2, \dots, r_N$ from which is calculated x and y corresponding to the observations. The differences between observed and calculated x and y can then be expressed by

$$\begin{aligned} \Delta x_i &= \Delta a - r_i \sin u \cdot \Delta u + \cos u \cdot \Delta r_i \\ \Delta y_i &= \Delta b + r_i \cos u \cdot \Delta u + \sin u \cdot \Delta r_i \end{aligned} \dots\dots\dots(119),$$

and we can carry out an adjustment, $\Delta a, \Delta b, \Delta u, \Delta r_1, \Delta r_2, \dots, \Delta r_N$ being the elements.

The normal equations are:

$$\begin{aligned} \frac{1}{a} \sum \{x_i\} &= \frac{N}{a} \Delta a + 0 \cdot \Delta b - \sum \{r_i\} \frac{\sin u}{a} \Delta u + \frac{\cos u}{a} \Delta r_1 + \dots + \frac{\cos u}{a} \Delta r_N, \\ \frac{1}{\gamma} \sum \{y_i\} &= 0 \cdot \Delta a + \frac{N}{\gamma} \Delta b + \sum \{r_i\} \frac{\cos u}{\gamma} \Delta u + \frac{\sin u}{\gamma} \Delta r_1 + \dots + \frac{\sin u}{\gamma} \Delta r_N, \\ \sum \left\{ r_i \left[-\frac{\sin u}{a} \Delta x_i + \frac{\cos u}{\gamma} \Delta y_i \right] \right\} &= -\sum \{r_i\} \frac{\sin u}{a} \Delta a + \sum \{r_i\} \frac{\cos u}{\gamma} \Delta b + \sum \{r_i\}^2 \left[\frac{\sin^2 u}{a} + \frac{\cos^2 u}{\gamma} \right] \Delta u + r_1 \left(\frac{1}{\gamma} - \frac{1}{a} \right) \cos u \sin u \Delta r_1 + \dots \\ &\quad + r_N \left(\frac{1}{\gamma} - \frac{1}{a} \right) \cos u \sin u \Delta r_N, \\ \frac{\cos u}{a} \Delta x_1 + \frac{\sin u}{\gamma} \Delta y_1 &= \frac{\cos u}{a} \Delta a + \frac{\sin u}{\gamma} \Delta b + r_1 \left(\frac{1}{\gamma} - \frac{1}{a} \right) \cos u \sin u \Delta u + \left(\frac{\cos^2 u}{a} + \frac{\sin^2 u}{\gamma} \right) \Delta r_1 + \dots + 0 \cdot \Delta r_N, \\ &\dots\dots\dots \\ \frac{\cos u}{a} \Delta x_N + \frac{\sin u}{\gamma} \Delta y_N &= \frac{\cos u}{a} \Delta a + \frac{\sin u}{\gamma} \Delta b + r_N \left(\frac{1}{\gamma} - \frac{1}{a} \right) \cos u \sin u \Delta u + 0 \cdot \Delta r_1 + \dots + \left(\frac{\cos^2 u}{a} + \frac{\sin^2 u}{\gamma} \right) \Delta r_N \end{aligned}$$

Eliminating $r_1, r_2, \dots, r_N$ from the first and the third of these equations by means of the last N equations, we obtain

$$\sum \{ \sin u \Delta x_i - \cos u \Delta y_i \} = N \sin u \Delta a - N \cos u \Delta b - \sum \{ r_i \} \Delta u \dots\dots(120),$$

and

$$\sum \{ r_i [\sin u \Delta x_i - \cos u \Delta y_i] \} = \sum \{ r_i \} \sin u \Delta a - \sum \{ r_i \} \cos u \Delta b - \sum \{ r_i^2 \} \Delta u \dots(121).$$

By eliminating the r 's from the second of the normal equations we get an equation identical with (120), which shows that we have one more element than we can determine.

From (120) and (121) we are however able to find

$$(\sin u \Delta a - \cos u \Delta b) \text{ and } \Delta u; \text{ we get}$$

$$\sin u \Delta a - \cos u \Delta b = \frac{1}{N (m_2 - m_1^2)} \Sigma \{ (m_2 - m_1 r_i) (\sin u \Delta x_i - \cos u \Delta y_i) \}$$

and

$$\Delta u = \frac{1}{N (m_2 - m_1^2)} \Sigma \{ (m_1 - r_i) (\sin u \Delta x_i - \cos u \Delta y_i) \},$$

where

$$m_1 = \frac{1}{N} \Sigma \{ r_i \} \text{ and } m_2 = \frac{1}{N} \Sigma (r_i^2).$$

For a point of the adjusted line corresponding to r_q we find, according to (119),

$$p_q = \sin u \Delta x_q - \cos u \Delta y_q = \sin u \Delta a - \cos u \Delta b - r_p \Delta u.$$

The standard deviation of p_q is seen to be the standard deviation of the position of the adjusted point (x_p, y_p) in the direction at right angles to the line.

We find

$$p_q = \frac{1}{N (m_2 - m_1^2)} \Sigma \{ [m_2 - m_1 r_i - r_p (m_1 - r_i)] (\sin u \Delta x_i - \cos u \Delta y_i) \}$$

and

$$\sigma_{p_q}^2 = \frac{\sigma^2}{N} (\alpha \sin^2 u + \gamma \cos^2 u) \left\{ 1 + \frac{(r_p - m_1)^2}{m_2 - m_1^2} \right\}.$$

This standard deviation is quite analogous to that obtained for an adjusted ordinate when the abscissa is errorless and gives the same indications for the distribution of the observations.

For σ_u we find

$$\sigma_u^2 = \frac{\sigma^2}{N} \frac{(\alpha \sin^2 u + \gamma \cos^2 u)}{(m_2 - m_1^2)},$$

again emphasising that the standard deviation of the r 's ought to be a maximum to give the best determination of the line.

In conclusion I should like to express my thanks to Miss H. Gertrude Jones for the care she has devoted to the preparation of the diagrams in this paper.

ON THE PRODUCT-MOMENTS OF VARIOUS ORDERS OF THE NORMAL CORRELATION SURFACE OF TWO VARIATES.

BY K. PEARSON AND A. W. YOUNG.

(1) In several recent investigations we have found it desirable to have the values of product-moment coefficients about the mean of the normal correlation surface. The present paper deals with the case of two variates. If the correlation surface be

$$z = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)} \dots\dots\dots(i),$$

where σ_x and σ_y are the standard deviations of the two variates x and y and r their correlation, then we define the s th- t th product-moment coefficient to be

$$q_{s,t} = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^s y^t z dx dy \dots\dots\dots(ii).$$

Further we write

$$p_{s,t} = q_{s,t}/(\sigma_x^s \sigma_y^t) \dots\dots\dots(iii),$$

so that $p_{s,t}$ is a purely numerical quantity and a function of the variable r only.

Clearly from the symmetry of the surface

$$p_{2s, 2t+1} = p_{2s+1, 2t} = 0.$$

We are accordingly only concerned with cases in which $s+t$ is *even*.

We propose therefore first to give the general algebraical expressions for the lower values of $p_{s,t}$, and secondly to provide tables for the numerical values of these product moments proceeding by increments of $\cdot 05$ in r .

Since $s+t$ must be even if $p_{s,t}$ be not zero, it follows that s and t must either both be even or both be odd. In the former case $p_{s,t}$ does not alter when r changes sign; in the latter case $p_{s,t}$ for negative r is simply $p_{s,t}$ for positive r with the sign changed. It is accordingly only needful to table $p_{s,t}$ for positive values of r .

For the purpose of testing computations the following formulae are of value:

$$p_{s,t} = (s+t-1)rp_{s-1,t-1} + (s-1)(t-1)(1-r^2)p_{s-2,t-2} \dots\dots\dots(iv),$$

$$p_{s,t} = (t-1)p_{s,t-2} + srp_{s-1,t-1} = (s-1)p_{s-2,t} + trp_{s-1,t-1} \dots\dots\dots(v).$$

Or, again we may write

$$p_{s,t} = (sr)^s \pi_{s,t} \dots\dots\dots(vi),$$

and we have

$$\pi_{s,t} = (t-1)\pi_{s,t-2} + \pi_{s-1,t-1} \dots\dots\dots(vii),$$

which is capable of numerical evaluation in a single machine operation.

The general values for any normal product-moment coefficient are

$$p_{2s, 2t} = \frac{2s! 2t!}{2^{s+t}} \sum_{u=0}^{u=t} \left\{ \frac{(2r)^{2u}}{(s-u)!(t-u)!2u!} \right\} \dots\dots\dots(viii),$$

$$p_{2s+1, 2t+1} = \frac{r(2s+1)!(2t+1)!}{2^{s+t}} \sum_{u=0}^{u=t} \left\{ \frac{(2r)^{2u}}{(s-u)!(t-u)!(2u+1)!} \right\} \dots\dots\dots(ix).$$

(2) We are now in a position to set down the algebraical values of the product-moment coefficients.

$$(a) \quad s \text{ or } t = 0. \quad p_{0,2t} = p_{2t,0} = 1 \cdot 3 \cdot 5 \dots (2t-1), \\ p_{0,0} = 1, \quad p_{2,0} = 1, \quad p_{4,0} = 3, \quad p_{6,0} = 15, \quad p_{8,0} = 105, \quad p_{10,0} = 945.$$

These are of course the simple moment coefficients of the normal curve when the unit of abscissal length is the standard deviation.

$$(b) \quad s \text{ or } t = 1. \quad p_{2t+1,1} = p_{1,2t+1} = 1 \cdot 3 \cdot 5 \dots (2t+1)r, \\ p_{11} = r, \quad p_{3,1} = 3r, \quad p_{5,1} = 15r, \quad p_{7,1} = 105r, \quad p_{9,1} = 945r.$$

Generally $p_{2t-1,1}/p_{2t,0} = r$ and provides a means of finding r and testing how far the correlation of two variates is normal.

$$(c) \quad s \text{ or } t = 2. \\ p_{2,2t} = p_{2t,2} = 1 \cdot 3 \cdot 5 \dots (2t-1) \{1 + 2tr^2\}, \\ p_{2,2} = 1 + 2r^2, \quad p_{2,4} = 3(1 + 4r^2), \quad p_{2,6} = 15(1 + 6r^2), \\ p_{2,8} = 105(1 + 8r^2), \quad p_{2,10} = 945(1 + 10r^2).$$

$$(d) \quad s \text{ or } t = 3. \\ p_{3,2t+1} = p_{2t+1,3} = 1 \cdot 3 \cdot 5 \dots (2t+1)r \{3 + 2tr^2\}, \\ p_{3,3} = 3r(3 + 2r^2), \quad p_{3,5} = 15r(3 + 4r^2), \quad p_{3,7} = 105r(3 + 6r^2), \quad p_{3,9} = 945r(3 + 8r^2).$$

$$(e) \quad s \text{ or } t = 4. \\ p_{4,2t} = p_{2t,4} = 1 \cdot 3 \cdot 5 \dots (2t-1) \{3 + 12tr^2 + 4t(t-1)r^4\}, \\ p_{4,4} = 3(3 + 24r^2 + 8r^4), \quad p_{4,6} = 15(3 + 36r^2 + 24r^4), \\ p_{4,8} = 105(3 + 48r^2 + 48r^4), \quad p_{4,10} = 945(3 + 60r^2 + 80r^4).$$

$$(f) \quad s \text{ or } t = 5. \\ p_{5,2t+1} = p_{2t+1,5} = 1 \cdot 3 \cdot 5 \dots (2t+1)r \{15 + 20tr^2 + 4t(t-1)r^4\}, \\ p_{5,5} = 15r(15 + 40r^2 + 8r^4), \quad p_{5,7} = 105r(15 + 60r^2 + 24r^4), \\ p_{5,9} = 945r(15 + 80r^2 + 48r^4).$$

$$(g) \quad s \text{ or } t = 6. \\ p_{6,2t} = p_{2t,6} = 1 \cdot 3 \cdot 5 \dots (2t-1) \{15 + 90tr^2 + 60t(t-1)r^4 + 8t(t-1)(t-2)r^6\} \\ p_{6,6} = 15(15 + 270r^2 + 360r^4 + 48r^6), \quad p_{6,8} = 105(15 + 360r^2 + 720r^4 + 192r^6), \\ p_{6,10} = 945(15 + 450r^2 + 1200r^4 + 480r^6).$$

$$(h) \quad s \text{ or } t = 7. \\ p_{7,2t+1} = p_{2t+1,7} = 1 \cdot 3 \cdot 5 \dots (2t+1)r \\ \{105 + 210tr^2 + 84t(t-1)r^4 + 8t(t-1)(t-2)r^6\}, \\ p_{7,7} = 105r(105 + 630r^2 + 504r^4 + 48r^6), \\ p_{7,9} = 945r(105 + 840r^2 + 1008r^4 + 192r^6).$$

$$(i) \quad s \text{ or } t = 8. \\ p_{8,2t} = p_{2t,8} = 1 \cdot 3 \cdot 5 \dots (2t-1) \{105 + 840r^2 + 840t(t-1)r^4 \\ + 224t(t-1)(t-2)r^6 + 16t(t-1)(t-2)(t-3)r^8\}, \\ p_{8,8} = 105(105 + 3360r^2 + 10080r^4 + 5376r^6 + 384r^8), \\ p_{8,10} = 945(105 + 4200r^2 + 16800r^4 + 13440r^6 + 1920r^8).$$

(j) s or $t = 9$.

$$p_{9, 2t+1} = p_{2t+1, 9} = 1 \cdot 3 \cdot 5 \dots (2t+1) r \{945 + 2520tr^2 + 1512t(t-1)r^4 \\ + 288t(t-1)(t-2)r^6 + 16t(t-1)(t-2)(t-3)r^8\}, \\ p_{9, 9} = 945r(945 + 10080r^2 + 18144r^4 + 6912r^6 + 384r^8).$$

(k) s or $t = 10$.

$$p_{10, 2t} = p_{2t, 10} = 1 \cdot 3 \cdot 5 \dots (2t-1) \{945 + 9450tr^2 + 12600t(t-1)r^4 \\ + 5040t(t-1)(t-2)r^6 + 720t(t-1)(t-2)(t-3)r^8 \\ + 32t(t-1)(t-2)(t-3)(t-4)r^{10}\}, \\ p_{10, 10} = 945(945 + 47250r^2 + 252000r^4 + 302400r^6 + 864000r^8 + 3840r^{10}).$$

The table on pp. 90-1 gives the numerical values of these coefficients. We proceed to illustrate their use.

Illustration I. In discussing the relation of auricular height (y) with age (x) of a girl's head a sample of 2272 individuals was found to provide the following product-moment coefficients:

$$q_{1, 1} = 3.113,712, \quad q_{3, 1} = 74.447,616, \\ q_{2, 1} = -1.957,022, \quad q_{4, 1} = -108.701,559.$$

Are these incompatible with normal correlation? (See K. Pearson, *On the General Theory of Skew Correlation and non-linear Regression*, Drapers' Company Research Memoirs, Biometric Series II, p. 35.) We have

$$\sigma_x = 3.064,819, \quad \sigma_y = 3.454,125,$$

and

$$r = .294,128,$$

and the leading subscript above corresponds to the x coordinate. We need first the values of $q_{2, 1}$, $q_{3, 1}$ and $q_{4, 1}$ on the hypothesis of normality. Clearly $q_{2, 1}$ and $q_{4, 1}$ will be zero, and using linear interpolation:

$$q_{3, 1} = \sigma_x^3 \sigma_y p'_{3, 1} \\ = 99.437,979 \times .88256 \\ = 87.759,983 = 87.7600, \text{ say.}$$

In the next place we require the probable errors of these q 's. The general expression for the probable error of a product-moment about the mean is given in *Biometrika*, Vol. IX, p. 3. In our present notation it is

$$N\sigma_{qs, t}^2 = q_{2s, 2t} - q_{s, t}^2 + s^2 q_{2, 0} q_{s-1, t}^2 + t^2 q_{0, 2} q_{s, t-1}^2 \\ + 2st q_{1, 1} q_{s-1, t} q_{s, t-1} - 2s q_{s+1, t} q_{s-1, t} - 2t q_{s, t+1} q_{s, t-1}.$$

Now remembering that for a normal distribution q vanishes when $s+t$ is odd, and that $q_{4, 0} = 3\sigma_x^4$, while

$$q_{s, t} = 10^{\frac{s+t}{2}-2} p'_{s, t} \sigma_x^s \sigma_y^t,$$

we have

$$N\sigma_{q_{2, 1}}^2 = q_{4, 2} + 4q_{2, 0} q_{1, 1}^2 + q_{0, 2} q_{2, 0}^2 + 4q_{1, 1}^2 q_{2, 0} - 4q_{3, 1} q_{1, 1} - 2q_{2, 2} q_{2, 0} \\ = \sigma_x^4 \sigma_y^2 \{10p'_{4, 2} + 8r^2 + 1 - 4rp'_{3, 1} - 2p'_{2, 2}\}, \\ \sigma_{q_{2, 1}} = \frac{\sigma_x^2 \sigma_y}{\sqrt{N}} \{10p'_{4, 2} + 8r^2 + 1 - 4rp'_{3, 1} - 2p'_{2, 2}\}^{\frac{1}{2}} \dots \dots \dots (x).$$

$$N\sigma_{q_{3,1}}^2 = q_{6,2} - q_{3,1}^2 = \sigma_x^6 \sigma_y^2 (100p'_{6,2} - p'^2_{3,1}),$$

$$\sigma_{q_{3,1}} = \frac{\sigma_x^3 \sigma_y}{\sqrt{N}} \{100p'_{6,2} - p'^2_{3,1}\}^{\frac{1}{2}} \dots\dots\dots (xi).$$

$$\begin{aligned} N\sigma_{q_{4,1}}^2 &= q_{8,2} - q_{4,1}^2 + 16q_{2,0}q_{3,1}^2 + q_{0,2}q_{4,0}^2 \\ &\quad + 8q_{1,1}q_{3,1}q_{4,0} - 8q_{5,1}q_{3,1} - 2q_{4,2}q_{4,0} \\ &= \sigma_x^8 \sigma_y^2 \{1000p'_{8,2} + 16p'^2_{3,1} + 9 + 24rp'_{3,1} - 80p'_{5,1}p'_{3,1} - 60p'_{4,2}\}, \\ \sigma_{q_{4,1}} &= \frac{\sigma_x^4 \sigma_y}{\sqrt{N}} \{1000p'_{8,2} + 16p'^2_{3,1} + 9 + 24rp'_{3,1} - 80p'_{5,1}p'_{3,1} - 60p'_{4,2}\}^{\frac{1}{2}} \\ &\dots\dots\dots (xii). \end{aligned}$$

We require accordingly to determine the following p 's: $p'_{2,2}$, $p'_{3,1}$, $p'_{4,2}$, $p'_{5,1}$, $p'_{6,2}$ and $p'_{8,2}$ by aid of our table with second differences or direct calculation from the algebraic values in terms of r . We have

$$\begin{aligned} p'_{2,2} &= 1.173,0226, & p'_{3,1} &= .782,3840, & p'_{4,2} &= .403,8135, \\ p'_{5,1} &= .441,1920, & p'_{6,2} &= .227,8602, & p'_{8,2} &= .177,6695. \end{aligned}$$

Also $.6744898/\sqrt{N} = .014,1505.$

Substituting we find the following probable errors:

$$\begin{aligned} \text{P.E. of } r &= .012,926, \\ \text{P.E. of } q_{2,1} &= .720,631, \\ \text{P.E. of } q_{3,1} &= 6.625,903, \\ \text{P.E. of } q_{4,1} &= 51.267,688. \end{aligned}$$

We can now sum up our results for these data:

$$\begin{aligned} r &= .2941 \pm .0129, \\ q_{2,1} &= 0 \pm .7206, \\ q_{3,1} &= 87.7600 \pm 6.6259, \\ q_{4,1} &= 0 \pm 51.2677. \end{aligned}$$

The probable errors would have been to some extent modified had we been able to calculate them on the true and not the observed r . We have

$$\begin{aligned} \Delta q_{2,1}/\text{P.E. of } q_{2,1} &= -2.716, \\ \Delta q_{3,1}/\text{P.E. of } q_{3,1} &= -2.009, \\ \Delta q_{4,1}/\text{P.E. of } q_{4,1} &= -2.120. \end{aligned}$$

Thus none of the deviations are excessive in terms of their probable errors. The system accordingly does not diverge very widely from the normal. At the same time the deviations are all in one sense, i.e. in defect of the normal value, and are all greater than twice the probable error. It appears therefore probable that there is some significant if slight deviation from normal correlation in the growth of the auricular height.

Illustration II. For the correlation of the contemporaneous barometric heights at Laudale and Southampton the following values have been found:

$$\begin{aligned} \text{Southampton } (x) & \quad \sigma_x = 3.250,067 \\ \text{Laudale } (y) & \quad \sigma_y = 3.932,290 \end{aligned} \left. \vphantom{\begin{aligned} \sigma_x &= 3.250,067 \\ \sigma_y &= 3.932,290 \end{aligned}} \right\} r = .780,225, \\ & \quad \quad \quad N = 2922. \end{aligned}$$

Table of the Product-Moment Coefficients of the Normal Surface for two Variates in terms of the Correlation Coefficient r^{**}

r	$p'_{1,1}$	$p'_{1,3}$	$p'_{1,5}$	$p'_{1,7}$	$p'_{1,9}$	$p'_{2,2}$	$p'_{2,4}$	$p'_{2,6}$	$p'_{2,8}$	$p'_{2,10}$	$p'_{3,3}$	$p'_{3,7}$	$p'_{3,9}$	r
-0.0	-0.0	-0.0	-0.0	-0.0,0	-0.00,0	1.000	.300	-150,00	-105,0	-094,5000	-000,000	-000,000,00	-000,00000	-0.0
-0.5	-0.5	-1.5	-0.75	-0.52,5	-0.47,25	1.005	.303	-152,25	-107,1	-096,8625	-045,075	-015,828,75	-014,2695	-0.5
-1.0	-1.0	-3.0	-1.50	-1.05,0	-0.94,50	1.020	.312	-159,00	-113,4	-103,9500	-090,600	-032,130,00	-029,1060	-1.0
-1.5	-1.5	-4.5	-2.25	-1.57,5	-1.41,75	1.045	.327	-170,25	-123,9	-115,7625	-137,025	-049,376,25	-045,0765	-1.5
-2.0	-2.0	-6.0	-3.00	-2.10,0	-1.89,00	1.080	.348	-186,00	-138,6	-132,3000	-184,800	-068,040,00	-062,7480	-2.0
-2.5	-2.5	-7.5	-3.75	-2.62,5	-2.36,25	1.125	.375	-206,25	-157,5	-153,3625	-234,375	-088,593,75	-082,6875	-2.5
-3.0	-3.0	-9.0	-4.50	-3.15,0	-2.83,50	1.180	.408	-231,00	-180,6	-179,5500	-286,200	-111,510,00	-105,4620	-3.0
-3.5	-3.5	-1.05	-5.25	-3.67,5	-3.30,75	1.245	.447	-260,25	-207,9	-210,2625	-340,725	-137,261,25	-131,6385	-3.5
-4.0	-4.0	-1.20	-6.00	-4.20,0	-3.78,00	1.320	.492	-294,00	-239,4	-245,7000	-398,400	-166,320,00	-161,7840	-4.0
-4.5	-4.5	-1.35	-6.75	-4.72,5	-4.25,25	1.405	.543	-332,25	-275,1	-285,8625	-459,675	-199,158,75	-196,4655	-4.5
-5.0	-5.0	-1.50	-7.50	-5.25,0	-4.72,50	1.500	.600	-375,00	-315,0	-330,7500	-525,000	-236,250,00	-236,2500	-5.0
-5.5	-5.5	-1.65	-8.25	-5.77,5	-5.19,75	1.605	.663	-422,25	-359,1	-380,3625	-594,825	-278,066,25	-281,7045	-5.5
-6.0	-6.0	-1.80	-9.00	-6.30,0	-5.67,00	1.720	.732	-474,00	-407,4	-434,7000	-669,600	-325,080,00	-333,3960	-6.0
-6.5	-6.5	-1.95	-9.75	-6.82,5	-6.14,25	1.845	.807	-530,25	-459,9	-493,7625	-749,775	-377,763,75	-391,8915	-6.5
-7.0	-7.0	-2.10	-1.050	-7.35,0	-6.61,50	1.980	.888	-591,00	-516,6	-557,5500	-835,800	-436,590,00	-457,7580	-7.0
-7.5	-7.5	-2.25	-1.125	-7.87,5	-7.08,75	2.125	.975	-656,25	-577,5	-626,0625	-928,125	-500,625	-531,5625	-7.5
-8.0	-8.0	-2.40	-1.200	-8.40,0	-7.56,00	2.280	1.068	-726,00	-642,6	-699,3000	-1,027,200	-574,560,00	-613,8720	-8.0
-8.5	-8.5	-2.55	-1.275	-8.92,5	-8.03,25	2.445	1.167	-800,25	-711,9	-777,2625	-1,133,475	-654,648,75	-705,2535	-8.5
-9.0	-9.0	-2.70	-1.350	-9.45,0	-8.50,50	2.620	1.272	-879,00	-785,4	-859,9500	-1,247,400	-742,770,00	-806,2740	-9.0
-9.5	-9.5	-2.85	-1.425	-9.97,5	-8.97,75	2.805	1.383	-962,25	-863,1	-947,3625	-1,369,425	-839,396,25	-917,5005	-9.5
-1.00	-1.00	-3.00	-1.500	-1.050,0	-9.45,00	3.000	1.500	-1,050,00	-945,0	-1,039,5000	-1,500,000	-945,000,00	-1,039,5000	-1.00

r	$p'_{5,4}$	$p'_{4,6}$	$p'_{5,8}$	$p'_{5,10}$	$p'_{5,12}$	$p'_{5,14}$	$p'_{5,16}$	$p'_{5,18}$	$p'_{5,20}$	$p'_{5,22}$	$p'_{5,24}$	$p'_{5,26}$	$p'_{5,28}$	$p'_{5,30}$	r
-0.0	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.00,0000	-0.0
-0.5	-0.91,8015	-0.46,352,25	-0.32,763,15	-0.29,772,225	-0.28,350,000	-0.27,928,800	-0.27,516,600	-0.27,104,400	-0.26,692,200	-0.26,280,000	-0.25,867,800	-0.25,455,600	-0.25,043,400	-0.24,631,200	-0.5
-1.0	-0.97,2240	-0.50,436,00	-0.36,590,40	-0.34,095,600	-0.31,490,225	-0.28,885,800	-0.26,281,400	-0.23,677,000	-0.21,072,600	-0.18,468,200	-0.15,863,800	-0.13,259,400	-0.10,655,000	-0.08,050,600	-1.0
-1.5	-1.06,3215	-0.57,332,25	-0.43,095,15	-0.41,490,225	-0.38,885,800	-0.36,281,400	-0.33,677,000	-0.31,072,600	-0.28,468,200	-0.25,863,800	-0.23,259,400	-0.20,655,000	-0.18,050,600	-0.15,446,200	-1.5
-2.0	-1.19,1840	-0.67,176,00	-0.52,466,40	-0.52,239,600	-0.49,635,200	-0.47,030,800	-0.44,426,400	-0.41,822,000	-0.39,217,600	-0.36,613,200	-0.34,008,800	-0.31,404,400	-0.28,800,000	-0.26,195,600	-2.0
-2.5	-1.35,9375	-0.80,156,25	-0.64,968,75	-0.66,740,625	-0.64,516,000	-0.62,291,375	-0.60,066,750	-0.57,842,125	-0.55,617,500	-0.53,392,875	-0.51,168,250	-0.48,943,625	-0.46,719,000	-0.44,494,375	-2.5
-3.0	-1.56,7440	-0.96,516,00	-0.80,942,40	-0.85,303,600	-0.83,078,975	-0.80,854,350	-0.78,629,725	-0.76,405,100	-0.74,180,475	-0.71,955,850	-0.69,731,225	-0.67,506,600	-0.65,281,975	-0.63,057,350	-3.0
-3.5	-1.81,8015	-1.16,552,25	-1.00,803,15	-1.09,152,225	-1.07,927,600	-1.06,703,000	-1.05,478,400	-1.04,253,800	-1.03,029,200	-1.01,804,600	-1.00,580,000	-99,355,400	-98,130,800	-96,906,200	-3.5
-4.0	-2.11,3440	-1.40,616,00	-1.25,042,40	-1.38,423,600	-1.37,200,000	-1.35,976,400	-1.34,751,800	-1.33,527,200	-1.32,302,600	-1.31,078,000	-1.29,853,400	-1.28,628,800	-1.27,404,200	-1.26,179,600	-4.0
-4.5	-2.45,6415	-1.69,112,25	-1.54,227,15	-1.74,168,225	-1.73,000,000	-1.71,831,800	-1.70,663,600	-1.69,495,400	-1.68,327,200	-1.67,159,000	-1.65,990,800	-1.64,822,600	-1.63,654,400	-1.62,486,200	-4.5
-5.0	-2.85,0000	-2.02,500,00	-1.89,000,00	-2.17,350,000	-2.16,181,800	-2.15,013,600	-2.13,845,400	-2.12,677,200	-2.11,509,000	-2.10,340,800	-2.09,172,600	-2.08,004,400	-2.06,836,200	-2.05,668,000	-5.0
-6.0	-3.80,3040	-2.86,056,00	-2.78,258,40	-3.30,447,600	-3.29,279,200	-3.28,110,800	-3.26,942,400	-3.25,774,000	-3.24,605,600	-3.23,437,200	-3.22,268,800	-3.21,100,400	-3.19,932,000	-3.18,763,600	-6.0
-6.5	-4.37,0415	-3.37,412,25	-3.34,407,15	-4.02,858,225	-4.01,689,800	-4.00,521,400	-3.99,353,000	-3.98,184,600	-3.97,016,200	-3.95,847,800	-3.94,679,400	-3.93,511,000	-3.92,342,600	-3.91,174,200	-6.5
-7.0	-5.00,4240	-3.96,036,00	-3.99,470,40	-4.87,995,600	-4.86,827,200	-4.85,658,800	-4.84,490,400	-4.83,322,000	-4.82,153,600	-4.80,985,200	-4.79,816,800	-4.78,648,400	-4.77,480,000	-4.76,311,600	-7.0
-7.5	-5.70,9375	-4.62,056,25	-4.74,468,75	-5.80,490,625	-5.79,322,200	-5.78,153,800	-5.76,985,400	-5.75,817,000	-5.74,648,600	-5.73,480,200	-5.72,311,800	-5.71,143,400	-5.70,000,000	-5.68,831,600	-7.5
-8.0	-6.49,1040	-5.38,056,00	-5.60,498,40	-7.00,887,600	-7.00,000,000	-6.99,112,400	-6.98,224,800	-6.97,337,200	-6.96,449,600	-6.95,562,000	-6.94,674,400	-6.93,786,800	-6.92,899,200	-6.92,011,600	-8.0
-8.5	-7.35,4815	-6.23,072,25	-6.58,731,15	-8.32,644,225	-8.31,475,800	-8.30,307,400	-8.29,139,000	-8.27,970,600	-8.26,802,200	-8.25,633,800	-8.24,465,400	-8.23,297,000	-8.22,128,600	-8.20,960,200	-8.5
-9.0	-8.30,6640	-7.18,596,00	-7.70,414,40	-9.83,631,600	-9.82,463,200	-9.81,294,800	-9.80,126,400	-9.78,958,000	-9.77,789,600	-9.76,621,200	-9.75,452,800	-9.74,284,400	-9.73,116,000	-9.71,947,600	-9.0
-9.5	-9.35,2815	-8.25,572,25	-8.96,871,15	-1,155,834,225	-1,154,665,800	-1,153,497,400	-1,152,329,000	-1,151,160,600	-1,150,000,000	-1,148,831,600	-1,147,663,200	-1,146,494,800	-1,145,326,400	-1,144,158,000	-9.5
-1.00	-1.050,0000	-9.45,000,00	-1.039,500,00	-1,351,350,000	-1,350,181,600	-1,349,013,200	-1,347,844,800	-1,346,676,400	-1,345,508,000	-1,344,339,600	-1,343,171,200	-1,342,002,800	-1,340,834,400	-1,339,666,000	-1.00

Table of the Product-Moment Coefficients—(cont.)

r	$p'_{6,8}$	$p'_{6,10}$	$p'_{7,7}$	$p'_{7,9}$	$p'_{8,8}$	$p'_{8,10}$	$p'_{9,9}$	$p'_{10,10}$	r
-00	-015,750,0000*	-014,175,0000*	-000,000,0000*	-000,000,0000*	-011,025,0000*	-009,922,5000*	-000,000,0000*	-008,930,2500	-00
-05	-016,699,7281§	-015,245,2196¶	-005,595,3529	-005,060,7728	-011,913,6238	-010,924,6923	-004,584,7313	-010,061,4596	-05
-10	-019,605,8016*	-018,541,3536*	-011,691,7970	-010,725,8437	-014,659,4049	-014,051,5319	-009,900,0214	-013,636,3809	-10
-15	-024,640,0213§	-024,322,3793¶	-018,810,3347	-017,635,4700	-019,505,2551	-019,670,9860	-016,741,5854	-020,215,1252	-15
-20	-032,092,5024*	-033,028,4304*	-027,511,9891	-026,502,5416	-026,866,6699	-028,420,4096	-026,038,0339	-030,785,9754	-20
-25	-042,377,3437§	-045,293,5547¶	-038,418,3105 +	-038,150,6836	-037,347,8027	-041,243,1392	-038,823,8025 -	-046,849,7939	-25
-30	-056,040,5664*	-061,963,5744*	-052,232,4785 -	-053,554,5017	-051,764,1913	-059,440,8526	-056,819,9330	-070,542,7472	-30
-35	-073,770,3193§	-084,119,0498¶	-069,761,1971	-073,882,6839	-071,172,3587	-084,742,2749	-081,525,4417	-104,801,9520	-35
-40	-096,409,3536*	-113,103,3456*	-091,937,5834	-100,544,6837	-096,906,5742	-119,390,2162	-115,322,1153	-153,580,0407	-40
-45	-124,969,7641§	-150,555,8006¶	-119,845,2453	-135,241,6719	-130,623,1232	-166,247,8128	-161,095,7024	-222,116,0976	-45
-50	-160,650,0000*	-198,450,0000*	-154,743,7500*	-180,022,5000*	-174,352,5000*	-228,926,2500*	-222,476,6250*	-317,271,9375*	-50
-55	-204,854,1421§	-259,137,1511¶	-198,095,6790	-237,345,3879	-230,559,9982	-311,935,9691	-304,003,5094	-447,944,3024	-55
-60	-259,213,4496*	-335,394,5616*	-251,595,4094	-310,146,0244	-302,215,2401	-420,863,8078	-411,313,0492	-625,565,2565 +	-60
-65	-325,610,1733§	-430,479,2213¶	-317,200,2402	-401,912,8210	-392,871,2462	-562,578,7885 +	-551,359,9358	-864,704,8679	-65
-70	-406,203,6384*	-548,186,4864*	-397,162,8007	-516,770,0234	-506,753,7153	-745,469,5569	-732,670,8594	-1183,792,2028	-70
-75	-503,458,5937§	-692,913,8672¶	-494,067,0410	-659,569,3945 -	-648,861,2402	-979,716,7529	-965,636,8527	-1605,972,7171	-75
-80	-620,175,8304*	-869,729,9184*	-610,865,9021	-835,991,1867	-825,077,2586	-1277,603,8922	-1262,848,5756	-2160,122,3635 -	-80
-85	-759,525,0673§	-1084,448,2327¶	-750,922,1241	-1052,655,1143	-1042,294,5915 +	-1653,870,6101	-1639,479,4539	-2882,041,0849	-85
-90	-925,080,1056*	-1343,706,5376*	-918,051,9718	-1317,242,0439	-1308,553,4936	-2126,112,4158	-2113,721,9649	-3815,850,9426	-90
-95	-1120,856,2501§	-1655,050,8941¶	-1116,572,1342	-1638,627,1137	-1633,194,1971	-2715,231,3838	-2707,282,7295	-5015,626,8384	-95
-100	-1351,350,0000*	-2027,025,0000*	-1351,350,0000*	-2027,025,0000*	-2027,025,0000	-3445,942,5000*	-3445,942,5000*	-6547,290,7500*	-100

** In this table $p'_{s,t} = p_{s,t}/10^2$.

* Exact values.

† Final figure 1 = 1.25, 2 = 1.75, 3 = 2.75, 4 = 3.75, 5 = 5.25, 6 = 6.25, 9 = 8.75 for exact values.

‡ Final figure 0 = -0.25, 2 = 2.25, 3 = 3.25, 6 = 5.75, 9 = 9.25.

|| Add .125 to obtain the final figures of the exact value.

§ Add .5 to obtain the final figure of the exact value.

¶ For exact value subtract beyond final figure .125.

The β 's of the marginal distributions show a markedly skew and non-normal system. The regression is, however, closely linear. Discuss the values of the product-moment coefficients:

$$\begin{aligned}q_{2,1} &= 11.919,404, \\ q_{1,2} &= 15.598,613, \\ q_{2,2} &= 401.523,496.\end{aligned}$$

For a normal system with the above correlation coefficient we should have:

$$q_{2,1} = q_{1,2} = 0 \text{ and } q_{2,2} = \sigma_x^2 \sigma_y^2 p'_{2,2} = 362.192,761.$$

Thus

$$\begin{aligned}\Delta q_{2,1} &= 11.919,404, \\ \Delta q_{1,2} &= 15.598,613, \\ \Delta q_{2,2} &= 39.330,735.\end{aligned}$$

We require to consider the probable errors of the q 's, which are given by .6744898 times the following standard deviations:

$$\begin{aligned}\sigma_{q_{2,1}} &= \frac{\sigma_x^2 \sigma_y}{\sqrt{N}} \{10p'_{4,2} + 8r^2 + 1 - 4rp'_{3,1} - 2p'_{2,2}\}^{\frac{1}{2}}, \\ \sigma_{q_{1,2}} &= \frac{\sigma_x \sigma_y^2}{\sqrt{N}} \{10p'_{2,4} + 8r^2 + 1 - 4rp'_{1,3} - 2p'_{2,2}\}^{\frac{1}{2}}, \\ \sigma_{q_{2,2}} &= \frac{\sigma_x^2 \sigma_y^2}{\sqrt{N}} \{100p'_{4,4} - p'^2_{2,2}\}^{\frac{1}{2}} \dots\dots\dots(\text{xiii}).\end{aligned}$$

We determine for the above value of r :

$$\begin{aligned}p'_{2,2} &= 2.217,5021, & p'_{3,1} &= p'_{1,3} = 2.340,6750, \\ p'_{2,4} &= p'_{4,2} = 1.030,50246, & p'_{4,4} &= .617,239437.\end{aligned}$$

Our results for a normal distribution are:

$$\begin{aligned}\text{P.E. of } r &= .004,882, \\ \text{P.E. of } q_{2,1} &= 1.091,473, \\ \text{P.E. of } q_{1,2} &= 1.320,585, \\ \text{P.E. of } q_{2,2} &= 15.360,681.\end{aligned}$$

Hence

$$\begin{aligned}\Delta q_{2,1}/\text{P.E. of } q_{2,1} &= 10.920, \\ \Delta q_{1,2}/\text{P.E. of } q_{1,2} &= 11.812, \\ \Delta q_{2,2}/\text{P.E. of } q_{2,2} &= 2.560.\end{aligned}$$

The deviations in the higher moment coefficients are at once seen to be markedly significant. But it will be noted that $q_{2,2}$ as in the previous case does not differ so markedly in value from the normal as the odd moment coefficients. It seems therefore likely, when a distribution is markedly skew, but the regression linear, that the even-even product-moment coefficients will not differ widely from the normal values, but that the even-odd ones will do so. It is possible that this is related to the fact that in distributions (such as 3×3 tables) which can be reduced in various ways to a tetrachoric table, correlation calculated from regression line diagonal cells is usually far more accurate than correlation calculated from non-regression line diagonal cells.

Equations (x) to (xiii) are of value beyond the present illustrations. Further uses of the above formulae and tables are provided in a memoir on "Generalised Tchebycheff Theorems" which will shortly be published. We have to thank Dr Kirstine Smith for much help in the preparation of this paper.

THE CORRELATION COEFFICIENT OF A POLYCHORIC TABLE.

By A. RITCHIE-SCOTT, B.Sc.

§ 1. INTRODUCTION.

We have at our disposal a considerable number of methods for finding the coefficient of correlation between two characters from a table of frequencies. These methods may be summarily named and classified as follows:

1. Product Moment.
2. Tetrachoric r .
3. Marginal centroids.
4. Biserial r .
5. Three Row η .
6. Variate difference methods including the correlation of grades and ranks.
7. Equiprobable tetrachoric r .
8. Mean contingency.
9. Mean square contingency.

Each of these methods has its own specially appropriate field of usefulness, but there still remains one class of table for which no entirely satisfactory methods have been devised, namely those which contain more than 2×2 cells and fewer than 4×4 , to which the tetrachoric and mean square contingency methods respectively may be applied.

It was with a view to investigating satisfactory methods for such tables that the following work was undertaken. Such tables arise under many circumstances, particularly when we can, as in many psychological investigations which depend upon the instinctive judgment of some character, definitely assign individuals with pronounced characters to either end of a scale, but are compelled to relegate doubtful cases to an intermediate but somewhat indefinite category. We have, in a word, good, indifferent, bad; present, doubtful, absent—classifications resulting in a frequency table with three categories for one or both characters.

In the present memoir a normal distribution has been assumed as it has been found to be not infrequently applicable and its assumption has given fairly satisfactory results even with distributions which are not strictly normal.

§ 2. NOTATION.

Let the normal surface (when standard deviations are used as units)

$$Nz(x, y) = \frac{N}{2\pi\sqrt{1-r^2}} e^{-\frac{x^2+y^2-2rxy}{2(1-r^2)}}$$

be divided as in the diagram by planes drawn parallel to the yz plane at the points $x = h_1, x = h_2 \dots$ and by planes parallel to the xz plane at the points $y = k_1, y = k_2 \dots$. Let the planes intersect in lines whose projections are the points $(h_1, k_1), (h_2, k_2), \text{etc.}$, contracted to 11, 12, etc., where the first figure is the suffix of h and the second figure is the suffix of k .

	h_1	h_2	h_3	h_4		h_{p-1}	h_p	
	n_{11}	n_{21}	n_{31}				n_{p1}	$n_{.1}$
k_1	11	21	31	41				$\left. \begin{array}{l} n_{.1} \\ n_{p1} \end{array} \right\} m_{.1}$
	n_{12}	n_{22}	n_{32}				n_{p2}	$n_{.2}$
k_2	12	22	32	42				$\left. \begin{array}{l} n_{.2} \\ n_{p2} \end{array} \right\} m_{.2}$
	n_{13}	n_{23}	n_{33}					$n_{.3}$
k_3	13	23	33	43				$\left. \begin{array}{l} n_{.3} \\ n_{p3} \end{array} \right\} m_{.3}$
	14	24	...	...				
k_{q-1}	n_{1q}	n_{2q}	...				n_{pq}	$n_{.}$
k_q	$n_{1.}$	$n_{2.}$	$n_{3.}$	...			$n_{p.}$	$N.$
	$m_{1.}$							
	$m_{2.}$							
	$m_{3.}$							

The frequencies in the cells and the marginal totals are indicated by $n_{11}, n_{12}, \text{etc.}$, and $n_{1.}, n_{2.} \dots n_{.1}, n_{.2}, \text{etc.}$, as shown in the diagram.

The surface may also be regarded as divided at each point 11, 12, ..., into four quadrants. One of these quadrants is shown by the dotted lines in the diagram. The quadrant in the position shown, viz. the left upper or (--) quadrant will be

regarded in what follows as the leading quadrant and its frequency denoted by m . Thus the quadrant shown will have the frequency m_{43} .

In the ordinary scheme for a tetrachoric table, the quadrants are denoted by a, b, c, d , and when necessary these letters will be used with the appropriate suffix. Thus the division at the point $s . t$ would be represented as in the diagram,

		h_s	
	(m_{st})		
	a_{st}	b_{st}	$m_{.t}$
k_t	st		
	c_{st}	d_{st}	$m_{.t'}$
	$m_{s.}$	$m_{s' .}$	N

The marginal totals corresponding to the leading quadrant are denoted by $m_{s.}, m_{.t}$ and the complementary totals by $m_{s' .}, m_{.t'}$.

Clearly any cell frequency may be expressed in terms of quadrant frequencies since

$$n_{st} = m_{st} - m_{s-1, t} - m_{s, t-1} + m_{s-1, t-1}.$$

§ 3. ENNEACHORIC METHOD.

In order to determine r , since we assume the distribution to be normal and we know the marginal totals, only one more datum is required. This for example may be a frequency block (or the total frequency on a continuous system of cells). As special cases we have the "briquette" or frequency on a rectangle of cells or again the quadrant frequency. The block may be the frequency contents of a group of corner, marginal or internal cells*. Consider (for future use) the general case of a quadrant frequency.

Let
$$\frac{m_{s.}}{N} = {}_s\tau_0, \quad \frac{m_{.t}}{N} = {}_t\theta_0,$$

where ${}_s\tau$ and ${}_t\theta$ are the tetrachoric coefficients. Then

$$\begin{aligned} \frac{m_{st}}{N} &= {}_s\tau_0 {}_t\theta_0 + {}_s\tau_1 {}_t\theta_1 r + {}_s\tau_2 {}_t\theta_2 r^2 + \dots \\ &= {}_s\tau_0 {}_t\theta_0 + \mathfrak{T}({}_s\tau, {}_t\theta, r) \dots\dots\dots(1). \end{aligned}$$

In using Everitt's Tables of the tetrachoric functions in which τ_0 and θ_0 must be less than $\frac{1}{2}$ we must either rearrange the table or adjust the above formula for the position of the mean with regard to the quadrants. It is more convenient to adjust the formula as follows, dropping the suffixes as we are dealing with any point of division.

* A "cell" is the least element for which the frequency is provided in the original data. Cells grouped together for any working purposes are collectively termed a "block."

Let $1 - \tau_0 = \tau_0', \quad 1 - \theta_0 = \theta_0'.$

Mean in a ,

$$\begin{aligned} \frac{m}{N} &= \tau_0 + \theta_0 - 1 + \frac{d}{N} \\ &= \tau_0 + \theta_0 - 1 + \tau_0' \theta_0' + \tau_1' \theta_1' r + \tau_2' \theta_2' r^2 + \dots \\ &= \tau_0 + \theta_0 - 1 - (1 - \tau_0) (1 - \theta_0) + \mathfrak{T}(\tau', \theta', r) \\ &= \tau_0 \theta_0 + \mathfrak{T}(\tau', \theta', r) \dots \dots \dots (2). \end{aligned}$$

Mean in b ,

$$\begin{aligned} \frac{m}{N} &= \tau_0 - \frac{c}{N} \\ &= \tau_0 - (\tau_0 \theta_0' + \tau_1 \theta_1' (-r) - \tau_2 \theta_2' (-r)^2 + \dots) \\ &= \tau_0 - \tau_0 (1 - \theta_0) - \mathfrak{T}(\tau, \theta', -r) \\ &= \tau_0 \theta_0 - \mathfrak{T}(\tau, \theta', -r) \dots \dots \dots (3). \end{aligned}$$

Mean in c ,

$$\begin{aligned} \frac{m}{N} &= \theta_0 - \frac{b}{N} \\ &= \theta_0 - (\tau_0' \theta_0 + \tau_1' \theta_1 (-r) + \tau_2' \theta_2' (-r)^2 + \dots) \\ &= \theta_0 - (1 - \tau_0) \theta_0 - \mathfrak{T}(\tau', \theta, -r) \\ &= \tau_0 \theta_0 - \mathfrak{T}(\tau', \theta, -r) \dots \dots \dots (4). \end{aligned}$$

$$\text{Mean in } d, \quad \frac{m}{N} = \tau_0 \theta_0 + \mathfrak{T}(\tau, \theta, r) \dots \dots \dots (5).$$

In place of taking a quadrant we may take a marginal or internal block. I shall only consider the latter as the case of a marginal block may be deduced from that of an internal block by removing one of the bounding planes to an infinite distance.

In discussing the central block we in effect reduce any table to a 3×3 (ennea-choric) table as we consider it to be constituted of a central block (or group of cells) and 4 marginal and 4 corner blocks. I may therefore use the nomenclature for a 3×3 table without any loss of generality.

$$\begin{aligned} n_{22} &= m_{22} - m_{12} - m_{21} + m_{11}, \\ \therefore \frac{n_{22}}{N} &= {}_2\tau_0 {}_2\theta_0 + \mathfrak{T}({}_2\tau, {}_2\theta, r) - {}_1\tau_0 {}_2\theta_0 - \mathfrak{T}({}_1\tau, {}_2\theta, r) \\ &\quad - {}_2\tau_0 {}_1\theta_0 - \mathfrak{T}({}_2\tau, {}_1\theta, r) + {}_1\tau_0 {}_1\theta_0 + \mathfrak{T}({}_1\tau, {}_1\theta, r) \\ &= ({}_2\tau_0 - {}_1\tau_0) ({}_2\theta_0 - {}_1\theta_0) + ({}_2\tau_1 - {}_1\tau_1) ({}_2\theta_1 - {}_1\theta_1) r \\ &\quad + ({}_2\tau_2 - {}_1\tau_2) ({}_2\theta_2 - {}_1\theta_2) r^2 + \text{etc.} \\ &= \frac{n_{2.} n_{.2}}{N} + ({}_2\tau_1 - {}_1\tau_1) ({}_2\theta_1 - {}_1\theta_1) r + ({}_2\tau_2 - {}_1\tau_2) ({}_2\theta_2 - {}_1\theta_2) r^2 + \text{etc.} \\ &\dots \dots \dots (6). \end{aligned}$$

As an example of the rearrangement of the formula for computation consider the case when the mean is in n_{22} :

$$(\text{mean in } a) \quad \frac{m_{22}}{N} = {}_2\tau_0 {}_2\theta_0 + {}_2\tau_1' {}_2\theta_1' r + {}_2\tau_2' {}_2\theta_2' r^2 + {}_2\tau_3' {}_2\theta_3' r^3 + \dots,$$

$$(\text{mean in } b) \quad \frac{m_{12}}{N} = {}_1\tau_0 {}_2\theta_0 + {}_1\tau_1 {}_2\theta_1' r - {}_1\tau_2 {}_2\theta_2' r^2 + {}_1\tau_3 {}_2\theta_3' r^3 - \dots$$

$$(\text{mean in } c) \quad \frac{m_{21}}{N} = {}_2\tau_0 {}_1\theta_0 + {}_2\tau_1' {}_1\theta_1 r - {}_2\tau_2' {}_1\theta_2 r^2 + {}_2\tau_3' {}_1\theta_3 r^3 - \dots$$

$$(\text{mean in } d) \quad \frac{m_{11}}{N} = {}_1\tau_0 {}_1\theta_0 + {}_1\tau_1 {}_1\theta_1 r + {}_1\tau_2 {}_1\theta_2 r^2 + {}_1\tau_3 {}_1\theta_3 r^3 + \dots$$

$$\begin{aligned} \therefore \frac{n_{22}}{N} &= \frac{1}{N} (m_{22} - m_{12} - m_{21} + m_{11}) \\ &= \frac{n_{2.} n_{.2}}{N} + ({}_2\tau_1' - {}_1\tau_1) ({}_2\theta_1' - {}_1\theta_1) r + ({}_2\tau_2' + {}_1\tau_2) ({}_2\theta_2' + {}_1\theta_2) r^2 \\ &\quad + ({}_2\tau_3' - {}_1\tau_3) ({}_2\theta_3' - {}_1\theta_3) r^3 + \text{etc.} \dots\dots\dots(7). \end{aligned}$$

It will be noted that when one set of categories is symmetrical about the mean, i.e. when say ${}_2\tau_1' = {}_1\tau_1$ all the terms of odd degree in r vanish. This corresponds to the fact that symmetrical categories may be reversed without altering the numerical value of the marginal totals and their relation to the central frequency; but such reversal will change the sign of r .

§ 4. STANDARD DEVIATION OF r BY ENNEACHORIC METHOD.

We have now to determine the probable error of r found in this manner.

Throughout what follows differentials will be used to indicate random sample variations, i.e. it is always supposed that the variations are small as compared to any quantity varied so that all the dn 's are small, or all the n 's are large quantities.

$$m_{st} = N \int_{-\infty}^{h_s} \int_{-\infty}^{k_t} z(x, y, r) dx \cdot dy = f(h_s, k_t, r) \dots\dots\dots(8).$$

Since the variations of the means and the standard deviations are, in this form of m_{st} , involved in the variations of h_s and k_s , we have

$$dm_{st} = \frac{\partial f}{\partial h_s} dh_s + \frac{\partial f}{\partial k_t} dk_t + \frac{\partial f}{\partial r} dr \dots\dots\dots(9).$$

Evaluating the differential coefficients,

$$\frac{\partial f}{\partial h_s} = N \int_{-\infty}^{k_t} z(h_s, y, r) dy = N \frac{e^{-h_s^2/2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k_t - rh_s} e^{-y^2/2} dy \dots\dots\dots(10).$$

This is the area of that portion of the dichotomic plane $x = h_s$ which bounds the quadrant m_{st} . But the area of the whole dichotomic plane is

$$N \int_{-\infty}^{+\infty} z(h_s, y, r) dy = \frac{N}{\sqrt{2\pi}} e^{-h_s^2/2} = NH_s \dots\dots\dots(11),$$

so that if we write

$$\frac{\partial f}{\partial h_s} = NH_s A_{st} \dots\dots\dots(12),$$

where

$$A_{st} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k_t - rh_s} e^{-y^2/2} dy \dots\dots\dots(13),$$

the factor A_{st} will be that fraction of the whole dichotomic plane section, which bounds the quadrant m_{st} and will have no dimensions. The value of A_{st} may be taken directly from Sheppard's tables of the probability integral entering with

the argument $\frac{k_t - rh_s}{\sqrt{1 - r^2}}$. It will be convenient to refer to this tabled integral as $\mathfrak{E}$, so that

$$A_{st} = \mathfrak{E} \left(\frac{k_t - rh_s}{\sqrt{1 - r^2}} \right) \dots\dots\dots (14)$$

It is convenient to note that with this notation

$$\mathfrak{E}(-x) = 1 - \mathfrak{E}(x).$$

Further since

$$m_s = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{h_s} e^{-\frac{1}{2}x^2} dx,$$

$$\frac{dm_s}{dh_s} = \frac{N}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2} = NH_s,$$

and

$$dh_s = \frac{dm_s}{NH_s}.$$

Hence

$$\frac{\partial f}{\partial h_s} dh_s = NA_{st}H_s \cdot \frac{dm_s}{NH_s} = A_{st}dm_s \dots\dots\dots (15).$$

Similarly

$$\frac{\partial f}{\partial k_t} dk_t = B_{st}dm_t \dots\dots\dots (16),$$

where

$$B_{st} = \mathfrak{E} \left(\frac{h_s - rk_t}{\sqrt{1 - r^2}} \right) \dots\dots\dots (17).$$

Lastly

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial N}{\partial r} \int_{-\infty}^{h_s} \int_{-\infty}^{k_t} z(x, y, r) dx dy \\ &= N \int_{-\infty}^{h_s} \int_{-\infty}^{k_t} \frac{d}{dr} z(x, y, r) dx dy \\ &= N \int_{-\infty}^{h_s} \int_{-\infty}^{k_t} \frac{d^2}{dx dy} z(x, y, r) dx dy \\ &= Nz(h_s, k_t, r) \\ &= \chi_{st} \dots\dots\dots (18), \end{aligned}$$

which is the length of the ordinate at the point (s, t) . We may now write

$$dm_{st} = A_{st}dm_s + B_{st}dm_t + \chi_{st}dr \dots\dots\dots (19),$$

and

$$-\chi_{st}dr = A_{st}dm_s + B_{st}dm_t - dm_{st} \dots\dots\dots (20).$$

Considerable use will be made of this formula later and the following abbreviated notation will be used:

$$A_{st}dm_s + B_{st}dm_t - dm_{st} = \delta P_{st} = -\chi_{st}dr \dots\dots\dots (21),$$

and

$$A_{st}m_s + B_{st}m_t - m_{st} = P_{st} \dots\dots\dots (22).$$

The reader must be careful to note that δP_{st} is not dP_{st} but only a part of it, and this symbol is used here as at once a conventional abbreviation and a *memoria technica*.

Since

$$\begin{aligned} n_{22} &= m_{22} - m_{12} - m_{21} + m_{11}, \\ \therefore dn_{22} &= dm_{22} - dm_{12} - dm_{21} + dm_{11} \\ &= A_{22}dm_2 + B_{22}dm_2 + \chi_{22}dr - A_{12}dm_1 - B_{12}dm_2 - \chi_{12}dr \\ &\quad - A_{21}dm_2 - B_{21}dm_1 - \chi_{21}dr + A_{11}dm_1 + B_{11}dm_1 + \chi_{11}dr \dots\dots (23), \end{aligned}$$

$$\therefore -(\chi_{22} - \chi_{12} - \chi_{21} + \chi_{11}) dr = (A_{22} - A_{21}) dm_{2\cdot} - (A_{12} - A_{11}) dm_{1\cdot} \\ + (B_{22} - B_{12}) dm_{\cdot 2} - (B_{21} - B_{11}) dm_{\cdot 1} - dn_{22} \dots\dots\dots(24).$$

Reference to the diagram shows that $A_{22} - A_{21}$ and the other coefficients are the proportions of the areas of the trapezettes bounding the briquette of volume n_{22} . These areas may be systematically named for the whole table thus,

$$\begin{array}{c} | \\ \alpha_{st} \\ \text{---} \beta_{st} \text{---} | \\ st \end{array}$$

that is, the areas of the planes meeting in the line of which the point s, t is the projection, in the direction shown, are named from the point so that

$$A_{s, t} - A_{s, t-1} = \alpha_{st}, \\ B_{s, t} - B_{s-1, t} = \beta_{st}.$$

Hence we may write

$$A_{22} - A_{21} = \alpha_{22}, \\ A_{12} - A_{11} = \alpha_{12}, \\ B_{22} - B_{12} = \beta_{22}, \\ B_{21} - B_{11} = \beta_{21}.$$

If now we notice that since $m_{2\cdot} = N - n_{3\cdot}$ etc., and $m_{1\cdot} = n_{1\cdot}$,

$$\therefore dm_{2\cdot} = -dn_{3\cdot}, \\ dm_{\cdot 2} = -dn_{\cdot 3}, \\ dm_{1\cdot} = dn_{1\cdot}, \\ dm_{\cdot 1} = dn_{\cdot 1},$$

we then have

$$-(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}) dr = -(\alpha_{22} dn_{3\cdot} + \alpha_{12} dn_{1\cdot} + \beta_{22} dn_{\cdot 3} + \beta_{21} dn_{\cdot 1} + dn_{22}) \\ \dots\dots\dots(25).$$

Expanding this in terms of frequency volumes this becomes

$$(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}) dr = (\alpha_{12} + \beta_{21}) dn_{11} + \beta_{21} dn_{21} + (\alpha_{22} + \beta_{21}) dn_{31} \\ + \alpha_{12} dn_{12} + dn_{22} + \alpha_{22} dn_{32} + (\alpha_{12} + \beta_{22}) dn_{13} + \beta_{22} dn_{23} + (\alpha_{22} + \beta_{22}) dn_{33} \\ \dots\dots\dots(26).$$

It has already been shown by Pearson (*Biometrika*, vol. ix, p. 1) that when random samples are taken from a population so large that its composition is not appreciably affected by removing the samples we have the following relations:

$$\sigma_{n_{ss'}} = n_{ss'} \left(1 - \frac{n_{ss'}}{N}\right) \dots\dots\dots(27),$$

$$\text{Mean } (dn_{ss'} dn_{tt'}) = -\frac{n_{ss'} n_{tt'}}{N} \dots\dots\dots(28a),$$

$$\text{Mean } (dn_s \cdot dn_t \cdot) = -\frac{n_s \cdot n_t \cdot}{N} \dots\dots\dots(28b),$$

$$\text{Mean } (dn_s \cdot dn_{\cdot s'}) = n_{ss'} - \frac{n_s \cdot n_{\cdot s'}}{N} \dots\dots\dots(28c),$$

$$\text{Mean } (dn_{ss'} dn_{\cdot t}) = -\frac{n_{ss'} n_{\cdot t}}{N} \dots\dots\dots(28d),$$

$$\text{Mean } (dn_{ss'} dn_{t'}) = - \frac{n_{ss'} n_{t'}}{N} \dots\dots\dots (28e),$$

$$\text{Mean } (dn_{ss'} dn_{s'}) = n_{ss'} \left(1 - \frac{n_{s'}}{N} \right) \dots\dots\dots (28f).$$

Hence squaring both sides, summing for all possible samples and dividing by the number of samples we have,

$$\begin{aligned} (\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})^2 \sigma_r^2 &= (\alpha_{12} + \beta_{21})^2 n_{11} + \beta_{21}^2 n_{21} + (\alpha_{22} + \beta_{21})^2 n_{31} \\ &+ \alpha_{12}^2 n_{12} + n_{22} + \alpha_{22}^2 n_{32} + (\alpha_{12} + \beta_{22})^2 n_{13} + \beta_{22}^2 n_{23} + (\alpha_{22} + \beta_{22})^2 n_{33} \\ &- \frac{1}{N} \left\{ \begin{aligned} &(\alpha_{12} + \beta_{21}) n_{11} + \beta_{21} n_{21} + (\alpha_{22} + \beta_{21}) n_{31} \\ &+ \alpha_{12} n_{12} + n_{22} + \alpha_{22} n_{32} \\ &+ (\alpha_{12} + \beta_{22}) n_{13} + \beta_{21} n_{23} + (\alpha_{22} + \beta_{22}) n_{22} \end{aligned} \right\}^2 \dots\dots\dots (29). \end{aligned}$$

The expression within the large brackets

$$= \alpha_{12} n_{1\cdot} + \alpha_{22} n_{3\cdot} + \beta_{21} n_{\cdot 1} + \beta_{22} n_{\cdot 3} + n_{22}.$$

Calling this Nm we have

$$\begin{aligned} (\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})^2 \sigma_r^2 &= (\alpha_{12} + \beta_{21} - m)^2 n_{11} + (\beta_{21} - m)^2 n_{21} \\ &+ (\alpha_{22} + \beta_{21} - m)^2 n_{31} + (\alpha_{12} - m)^2 n_{12} + (1 - m)^2 n_{22} + (\alpha_{22} - m)^2 n_{32} \\ &+ (\alpha_{12} + \beta_{22} - m)^2 n_{13} + (\beta_{22} - m)^2 n_{23} + (\alpha_{22} + \beta_{22} - m)^2 n_{33} \dots\dots (30). \end{aligned}$$

The following form for m is instructive although giving an apparently less symmetrical form than the above,

$$\begin{aligned} m &= \frac{1}{N} (\alpha_{12} n_{1\cdot} + \alpha_{22} n_{3\cdot} + \beta_{21} n_{\cdot 1} + \beta_{22} n_{\cdot 3} + n_{22}) \\ &= \frac{1}{N} \left\{ (A_{12} - A_{11}) m_{1\cdot} + (B_{22} - B_{12}) (N - m_{\cdot 2}) + (B_{21} - B_{11}) m_{\cdot 1} \right. \\ &\quad \left. + (B_{22} - B_{12}) (N - m_{\cdot 2}) + m_{22} - m_{21} - m_{12} + m_{11} \right\} \\ &= \frac{1}{N} \left\{ A_{12} m_{1\cdot} + B_{12} m_{\cdot 2} - m_{12} - A_{11} m_{1\cdot} - B_{11} m_{\cdot 1} + m_{11} - A_{22} m_{3\cdot} \right. \\ &\quad \left. - B_{22} m_{\cdot 2} + m_{22} + A_{21} m_{2\cdot} + B_{21} m_{\cdot 1} - m_{21} + N (A_{22} - A_{21} + B_{22} - B_{12}) \right\} \\ &= \alpha_{22} + \beta_{32} - \frac{P_{11} - P_{12} - P_{21} + P_{22}}{N} \\ &= \alpha_{22} + \beta_{22} - \frac{\mathfrak{P}}{N} \dots\dots\dots (31). \end{aligned}$$

We may then write

$$\begin{aligned} (\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})^2 \sigma_r^2 &= \left(\alpha_{12} - \alpha_{22} + \beta_{21} - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{11} + \left(0 - \alpha_{22} + \beta_{21} - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{21} \\ &+ \left(\beta_{21} - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{31} + \left(\alpha_{12} - \alpha_{22} + 0 - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{12} \\ &+ \left(\frac{1}{2} - \alpha_{22} + \frac{1}{2} - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{22} + \left(0 - \beta_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{32} \\ &+ \left(\alpha_{12} - \alpha_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{13} + \left(0 - \alpha_{22} + \frac{\mathfrak{P}}{N} \right)^2 n_{23} + \left(\frac{\mathfrak{P}}{N} \right)^2 n_{33} \dots\dots (32). \end{aligned}$$

It will be shown in a later paper that the coefficients of the cell frequencies are functions of moments of the frequencies about the mean.

If any relation between h_1, h_2, k_1, k_2 makes

$$\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22} = 0,$$

while the right side of the σ_r^2 equation remains finite the value of the expression for σ_r^2 will become infinite. $\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}$ obviously vanishes (1) when $h_1 = h_2$ or $k_1 = k_2$, i.e. when either of the central categories becomes vanishingly small, and (2) when $h_1 = -h_2, k_1 = -k_2$ and $r = 0$, i.e. when both sets of the extreme categories have equal frequencies and the correlation is very small.

(1) When $h_1 = h_2$. Then $\chi_{11} = \chi_{12}, \chi_{21} = \chi_{22}, a_{12} = a_{22} = a$ say, $\beta_{21} = \beta_{22} = 0, n_{21} = n_{22} = n_{23} = 0, n_{2\cdot} = 0$.

Then
$$m = \frac{\alpha (n_{1\cdot} + n_{3\cdot})}{N} = \alpha,$$

and the right side of the σ_r^2 equation reduces to zero giving the indeterminate form $\frac{0}{0}$ for σ_r^2 .

(2) When $h_1 = -h_2, k_1 = -k_2$ and $r = 0$. This case will be discussed in the next section.

§ 5. STANDARD DEVIATION OF ENNEACHORIC r IN SPECIAL CASES.

Two particular cases are of interest, (1) when $r = 0$, (2) when the table degenerates into a 2×2 table.

$$\begin{aligned} (1) \quad \text{When } r = 0 \quad A_{st} &= \mathfrak{E}(k_t) = \frac{m_{\cdot t}}{N}, \\ B_{st} &= \mathfrak{E}(h_s) = \frac{m_{s\cdot}}{N}, \\ P_{st} &= A_{st}m_{s\cdot} + B_{st}m_{\cdot t} - m_{st} \\ &= \frac{m_{\cdot t}m_{s\cdot}}{N} + \frac{m_{s\cdot}m_{\cdot t}}{N} - m_{st} \\ &= m_{st}. \end{aligned}$$

Hence
$$a_{22} = A_{22} - A_{21} = \frac{m_{\cdot 2} - m_{\cdot 1}}{N} = \frac{n_{\cdot 2}}{N},$$

$$a_{12} = A_{12} - A_{11} = \frac{m_{\cdot 2} - m_{\cdot 1}}{N} = \frac{n_{\cdot 2}}{N}.$$

Similarly
$$\beta_{22} = \beta_{21} = \frac{n_{2\cdot}}{N},$$

$$\begin{aligned} m &= a_{22} + \beta_{22} - \frac{P_{11} - P_{12} - P_{21} + P_{22}}{N} \\ &= \frac{n_{\cdot 2}}{N} + \frac{n_{2\cdot}}{N} - \frac{1}{N} (m_{11} - m_{12} - m_{21} + m_{22}) \\ &= \frac{n_{\cdot 2}}{N} + \frac{n_{2\cdot}}{N} - \frac{n_{22}}{N}. \end{aligned}$$

$$\begin{aligned}\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22} &= N (H_1 K_1 - H_1 K_2 - H_2 K_1 - H_2 K_2) \\ &= N (H_1 - H_2) (K_1 - K_2),\end{aligned}$$

and the right-hand side of the equation reduces to

$$\begin{aligned}&\left(\frac{n_{22}}{N}\right)^2 (n_{11} + n_{13} + n_{31} + n_{33}) + \left(\frac{n_{22} - n_{2\cdot}}{N}\right)^2 (n_{21} + n_{23}) \\ &+ \left(\frac{n_{22} - n_{2\cdot}}{N}\right)^2 (n_{12} + n_{32}) + \left(1 - \frac{n_{2\cdot}}{N} - \frac{n_{2\cdot}}{N} + \frac{n_{22}}{N}\right)^2 n_{22}.\end{aligned}$$

Remembering that $\frac{n_{2\cdot} n_{\cdot 2}}{N} = n_{22}$ this resolves into

$$\begin{aligned}&\left(\frac{n_{2\cdot} n_{\cdot 2}}{N}\right)^2 \frac{(N - n_{2\cdot})(N - n_{\cdot 2})}{N} + \left(\frac{N - n_{2\cdot}}{N}\right)^2 \left(\frac{n_{\cdot 2}}{N}\right)^2 \frac{n_{2\cdot}}{N} (N - n_{\cdot 2}) \\ &+ \left(\frac{n_{2\cdot}}{N}\right)^2 \left(\frac{N - n_{\cdot 2}}{N}\right)^2 \frac{n_{\cdot 2}}{N} (N - n_{2\cdot}) + \left(\frac{N - n_{2\cdot}}{N}\right)^2 \left(\frac{N - n_{\cdot 2}}{N}\right)^2 \frac{n_{2\cdot} n_{\cdot 2}}{N} \\ &= \frac{(N - n_{2\cdot})(N - n_{\cdot 2}) n_{2\cdot} n_{\cdot 2}}{N^5} \{n_{2\cdot} n_{\cdot 2} + (N - n_{2\cdot}) n_{\cdot 2} + (N - n_{\cdot 2}) n_{2\cdot} \\ &\quad + (N - n_{2\cdot})(N - n_{\cdot 2})\} \\ &= \frac{n_{2\cdot} (N - n_{2\cdot}) n_{\cdot 2} (N - n_{\cdot 2})}{N^3} \dots\dots\dots(33).\end{aligned}$$

Hence

$$N^2 (H_1 - H_2)^2 (K_1 - K_2)^2 \sigma_r^2 = n_{2\cdot} \frac{(N - n_{2\cdot})}{N} \cdot \frac{n_{\cdot 2} (N - n_{\cdot 2})}{N} \cdot \frac{1}{N} = \frac{\sigma_{n_{2\cdot}}^2 \sigma_{n_{\cdot 2}}^2}{N}$$

and

$$\sigma_r = \frac{1}{\sqrt{N}} \cdot \frac{\sigma_{n_{2\cdot}} \sigma_{n_{\cdot 2}}}{N (H_1 - H_2) (K_1 - K_2)} \dots\dots\dots(34).$$

This may also be expressed as follows:

$$\begin{aligned}\sigma_{n_{2\cdot}}^2 \sigma_{n_{\cdot 2}}^2 &= \frac{n_{2\cdot} (N - n_{2\cdot})}{N} \frac{n_{\cdot 2} (N - n_{\cdot 2})}{N} \\ &= \frac{n_{2\cdot} n_{\cdot 2}}{N} \cdot \frac{n_{1\cdot} + n_{3\cdot}}{N} \cdot \frac{n_{\cdot 1} + n_{\cdot 3}}{N} \\ &= n_{22} (n_{11} + n_{13} + n_{31} + n_{33}) \dots\dots\dots(35)\end{aligned}$$

and

$$\sigma_r = \frac{\sqrt{\frac{1}{N} n_{22} (n_{11} + n_{13} + n_{31} + n_{33})}}{N (H_1 - H_2) (K_1 - K_2)} \dots\dots\dots(36).$$

When the extreme categories are nearly equal and r is zero $H_1 = H_2$ and $K_1 = K_2$ and the value for σ_r^2 becomes infinite. It is necessary then to keep second powers of differences to determine σ_r^2 . Keeping second order terms we have:

$$\begin{aligned}dm &= \frac{\partial f}{\partial h} dh + \frac{\partial f}{\partial k} dk + \frac{\partial f}{\partial r} dr + \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial h^2} (dh)^2 + \frac{\partial^2 f}{\partial k^2} (dk)^2 + \frac{\partial^2 f}{\partial r^2} (dr)^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 f}{\partial h \partial k} (dh dk) + 2 \frac{\partial^2 f}{\partial h \partial r} (dh dr) + \frac{\partial^2 f}{\partial k \partial r} (dk dr) \right\} \dots\dots\dots(37).\end{aligned}$$

To find $\frac{\partial^2 f}{\partial h^2}$ etc. we may proceed as follows.

By straightforward differentiation we find that

$$\begin{aligned}\frac{\partial^2 f}{\partial h^2} &= \frac{\partial}{\partial h} \left\{ \frac{N}{2\pi\sqrt{1-r^2}} \int_{-\infty}^k e^{-\frac{y^2+h^2-2rhy}{2(1-r^2)}} dy \right\} \\ &= -\frac{hN}{2\pi\sqrt{1-r^2}} \int_{-\infty}^k e^{-\frac{y^2+h^2-2rhy}{2(1-r^2)}} dy + \frac{rN}{2\pi\sqrt{1-r^2}} e^{-\frac{h^2+k^2-2rhk}{2(1-r^2)}} \\ &= N \left(-hHA - \frac{r\chi}{N} \right) \dots\dots\dots(38),\end{aligned}$$

$$\frac{\partial^2 f}{\partial k^2} = N \left(-kKB - \frac{r\chi}{N} \right) \dots\dots\dots(39),$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial \chi}{\partial r} = \frac{N}{1-r^2} \left\{ r + \frac{(h-kr)(k-rh)}{1-r^2} \right\} \chi \dots\dots\dots(40),$$

$$\frac{\partial^2 f}{\partial h \partial r} = \frac{\partial \chi}{\partial h} = -\frac{h-rk}{1-r^2} \chi \dots\dots\dots(41),$$

$$\frac{\partial^2 f}{\partial k \partial r} = \frac{\partial \chi}{\partial k} = -\frac{k-rh}{1-r^2} \chi \dots\dots\dots(42),$$

$$\frac{\partial^2 f}{\partial h \partial k} = \frac{\partial f}{\partial r} = \chi \dots\dots\dots(43).$$

Hence summing for all samples and dividing by the number of samples and denoting this operation by $\mathfrak{S}$, we have, since quantities of the first order disappear, an equation involving $\mathfrak{S}(dh)^2$, $\mathfrak{S}(dhdk)$, etc. We may determine these values thus:

$$\begin{aligned}dh_s &= \frac{dm_{s\cdot}}{NH_s}, \\ \therefore \mathfrak{S}(dh_s)^2 &= \mathfrak{S} \left(\frac{dm_{s\cdot}}{NH_s} \right)^2 = \frac{m_{s\cdot} \left(1 - \frac{m_{s\cdot}}{N} \right)}{N^2 H_s^2} \dots\dots\dots(44).\end{aligned}$$

$$\text{Similarly} \quad \mathfrak{S}(dk_t)^2 = \frac{m_{\cdot t} \left(1 - \frac{m_{\cdot t}}{N} \right)}{N^2 K_t^2} \dots\dots\dots(45),$$

$$\mathfrak{S}(dh_s dk_t) = \mathfrak{S} \left(\frac{dm_{s\cdot} dm_{\cdot t}}{N^2 HK} \right) = \frac{m_{st} - \frac{m_{s\cdot} m_{\cdot t}}{N}}{N^2 HK} \dots\dots\dots(46).$$

To find $(dhdr)$ we have

$$\begin{aligned}-\chi dr &= A_{st} dm_{s\cdot} + B_{st} dm_{\cdot t} - dm_{st}, \\ \therefore -\chi dr dh_s &= -\frac{\chi dr dm_{s\cdot}}{NH_s}, \\ \therefore -\chi NH_s \mathfrak{S}(dr dh_s) &= A_{st} \mathfrak{S}(dm_{s\cdot})^2 + B_{st} \mathfrak{S}(dm_{s\cdot} dm_{\cdot t}) - \mathfrak{S}(dm_{st} dm_{\cdot t}) \\ &= A_{st} m_{s\cdot} \left(1 - \frac{m_{s\cdot}}{N} \right) + B_{st} \left(m_{st} - \frac{m_{s\cdot} m_{st}}{N} \right) - m_{st} \left(1 - \frac{m_{s\cdot}}{N} \right) \dots\dots\dots(47).\end{aligned}$$

After simplification this becomes

$$\left(1 - \frac{m_{s\cdot}}{N} \right) P_{st} - B_{st} (m_{\cdot t} - m_{st}),$$

$$\therefore \mathfrak{S}(drdh) = -\frac{1}{NH\chi} \left\{ \left(1 - \frac{m_{s\cdot}}{N}\right) P_{st} - B(m_{\cdot t} - m_{st}) \right\} \dots\dots\dots(48).$$

Similarly $\mathfrak{S}(drdk) = -\frac{1}{NK\chi} \left\{ \left(1 - \frac{m_{\cdot t}}{N}\right) P_{st} - A(m_{s\cdot} - m_{st}) \right\} \dots\dots\dots(49).$

Equation (37) then becomes

$$\begin{aligned} 0 = & N \left(-hHA - \frac{r\chi}{N} \right) \mathfrak{S}(dh)^2 + N \left(-kKB - \frac{r\chi}{N} \right) \mathfrak{S}(dk)^2 \\ & + \frac{\chi}{1-r^2} \left\{ r + \frac{(h-kr)(k-rh)}{1-r^2} \right\} \mathfrak{S}(dr)^2 - 2 \frac{h-rk}{1-r^2} \chi \cdot \mathfrak{S}(dhdr) \\ & - 2 \frac{k-rh}{1-r^2} \mathfrak{S}(dkdr) + 2\chi \mathfrak{S}(dhdk) \dots\dots\dots(50). \end{aligned}$$

$\therefore$ Substituting the values of the $\mathfrak{S}$'s, and transposing σ_r^2 we find

$$\begin{aligned} \frac{\chi}{1-r^2} \left\{ r + \frac{(h-kr)(k-rh)}{1-r^2} \right\} \sigma_r^2 = & -N \left(hHA + r \frac{\chi}{N} \right) \frac{m_{s\cdot} \left(1 - \frac{m_{s\cdot}}{N}\right)}{N^2 H_s^2} \\ & - N \left(kKB + r \frac{\chi}{N} \right) \frac{m_{\cdot t} \left(1 - \frac{m_{\cdot t}}{N}\right)}{N^2 K_t^2} \\ & + 2 \frac{h-rk}{1-r^2} \chi \cdot \frac{1}{NH\chi} \left\{ \left(1 - \frac{m_{s\cdot}}{N}\right) P_{st} - B_{st}(m_{\cdot t} - m_{st}) \right\} \\ & + 2 \frac{k-rh}{1-r^2} \chi \cdot \frac{1}{NH\chi} \left\{ \left(1 - \frac{m_{\cdot t}}{N}\right) P_{st} - A_{st}(m_{s\cdot} - m_{st}) \right\} \\ & + 2\chi \cdot \frac{m_{st} - \frac{m_{s\cdot} m_{\cdot t}}{N}}{N^2 HK} \dots\dots\dots(51). \end{aligned}$$

When $r=0$ and $h_1 = -h_2$, $k_1 = -k_2$, χ reduces to NHK , etc., as above and the equation reduces to

$$-NHKhk\sigma_r^2 = -\frac{hm_{st}}{NH} \left(1 - \frac{m_{s\cdot}}{N}\right) - \frac{km_{st}}{NK} \left(1 - \frac{m_{\cdot t}}{N}\right) \dots\dots\dots(52),$$

If we take this equation for each of the four points 11, 12, 21, 22 and combine them according to the scheme $m_{22} = m_{11} - m_{12} - m_{21} + m_{22}$ as before, the left member of the equation becomes

$$\begin{aligned} -NHK(h_1k_1 - h_1k_2 - h_2k_1 + h_2k_2)\sigma_r^2 = & -NHK(h_1k_1 + h_1k_1 + h_1k_1 + h_1k_1)\sigma_r^2 \\ = & -4NHKh_1k_1\sigma_r^2 \dots\dots\dots(53). \end{aligned}$$

In a similar manner the right side reduces to

$$\frac{h_1}{NH_1} \left(n_{12} + \frac{m_{1\cdot}}{N} n_{22} \right) + \frac{k_1}{NK_1} \left(n_{21} + \frac{m_{\cdot 1}}{N} n_{22} \right) \dots\dots\dots(54),$$

which may be further simplified as follows

$$\begin{aligned} n_{12} + \frac{m_{1\cdot}}{N} n_{22} = & n_{12} + \frac{m_{1\cdot}}{N} (m_{\cdot 2} - 2n_{12}) \\ = & 2n_{12} \left(1 - \frac{m_{1\cdot}}{N}\right) \end{aligned}$$

$$\begin{aligned}
&= 2 \frac{m_{1\cdot} m_{\cdot 2}}{N} \left(1 - \frac{N - m_{2\cdot}}{N} \right) \\
&= 2 \frac{m_{1\cdot} m_{\cdot 2} m_{2\cdot}}{N \cdot N} \dots\dots\dots(55).
\end{aligned}$$

Similarly
$$n_{21} + \frac{m_{\cdot 1}}{N} n_{22} = 2 \frac{m_{\cdot 1} m_{2\cdot} m_{\cdot 2}}{N N} \dots\dots\dots(56).$$

Hence substituting we have

$$-4NHKh_1 k_1 \sigma_r^2 = 2 \frac{n_{22}}{N^2} \left\{ \frac{h_1}{H_1} m_{1\cdot} + \frac{k_1}{K_1} m_{\cdot 1} \right\} \dots\dots\dots(57),$$

and

$$\sigma_r^2 = - \frac{n_{22}}{2N^3 H K h_1 k_1} \left\{ \frac{h_1}{H_1} m_{1\cdot} + \frac{k_1}{K_1} m_{\cdot 1} \right\} \dots\dots\dots(58)$$

(h_1 and k_1 are of course negative numbers) which remains finite when the extreme categories are equal to $r = 0$.

(2) When we put $h_2 = k_2 = \infty$ the enneachoric table degenerates into a tetrachoric table and we have

$$n_{13} = n_{23} = n_{33} = 0,$$

$$n_{31} = n_{32} = n_{33} = 0,$$

$$n_{3\cdot} = n_{\cdot 3} = 0,$$

$$\chi_{12} = \chi_{21} = \chi_{22} = 0,$$

and

$$m = \frac{\alpha_{12} n_{1\cdot} + \beta_{21} n_{\cdot 1} + n_{22}}{N}.$$

But

$$\begin{aligned}
\alpha_{12} &= A_{12} - A_{11} \\
&= \mathfrak{E} \left(\frac{\infty - r h_1}{\sqrt{1 - r^2}} \right) - \mathfrak{E} \left(\frac{k_1 - r h_1}{\sqrt{1 - r^2}} \right) = 1 - A_{11}.
\end{aligned}$$

Similarly

$$\beta_{21} = 1 - B_{11}.$$

Hence
$$m = \frac{(1 - A_{11}) n_{1\cdot} + (1 - B_{11}) n_{\cdot 1} + N - n_{1\cdot} - n_{\cdot 1} + n_{11}}{N}$$

$$= \frac{N - (A_{11} n_{1\cdot} + B_{11} n_{\cdot 1} - n_{11})}{N}$$

$$= 1 - \frac{P_{11}}{N}.$$

$$\begin{aligned}
\therefore \chi_{11}^2 \sigma_r^2 &= \left(\frac{P_{11}}{N} - A_{11} - B_{11} + 1 \right)^2 n_{11} + \left(\frac{P_{11}}{N} - B_{11} \right)^2 n_{21} \\
&\quad + \left(\frac{P_{11}}{N} - A_{11} \right)^2 n_{12} + \left(\frac{P_{11}}{N} \right)^2 n_{22} \dots\dots\dots(59).
\end{aligned}$$

This form of the standard deviation of a tetrachoric correlation will be referred to again later, and will then be reduced to a still more symmetrical form.

From the above it will be seen that we can determine the correlation coefficient from any frequency block, assuming a normal distribution, but the accuracy of this determination varies with the position and size of the frequency block. The probable error ($\cdot 67449\sigma$) varies from cell to cell, and an unlucky choice of the working cell may lead to a correlation coefficient with large probable error. A correction

of this latitude might in some cases be obtained by using another cell. But as the r from this cell would probably differ from that previously found, and as neither of them would be identical with that of the normal surface from which they are supposed to be sampled, we must find some means of approximating to the "best" r . The most general method of doing this, following on the above, would seem to be to weight each of the frequency blocks and determine the weights so that the resulting probable error of the weighted r is a minimum. In doing this we must have regard to the fact that the variates we are dealing with are not independent but correlated. We must consider this method.

Let the polychoric table have p rows and q columns so that the indices of the last row and column are $1q, 2q, \dots pq$ and $p1, p2, \dots pq$ respectively, and let each of the frequency volumes be weighted by an arbitrary weight $w_{11}, w_{12}, \dots$ indicated by the same suffix as their respective cells.

$$\begin{aligned} & \text{Then} \quad w_{11}n_{11} + w_{12}n_{12} + \dots \\ & = w_{11}m_{11} + w_{12}(m_{12} - m_{11}) + w_{21}(m_{21} - m_{11}) + w_{22}(m_{22} - m_{12} - m_{21} + m_{11}) + \dots \\ & \quad + w_{st}(m_{st} - m_{s-1,t} - m_{s,t-1} + m_{s-1,t-1}) + \dots \\ & = (w_{11} - w_{12} - w_{21} + w_{22})m_{11} + \dots + (w_{st} - w_{s+1,t} - w_{s,t+1} + w_{s+1,t+1})m_{st} + \dots \\ & = \omega_{11}m_{11} + \omega_{12}m_{12} + \dots + \omega_{st}m_{st} \dots \dots \dots (60). \end{aligned}$$

$$\begin{aligned} \text{Then} \quad \frac{1}{N}(w_{11}n_{11} + w_{12}n_{12} + \dots) &= \frac{1}{N}(\omega_{11}m_{11} + \omega_{12}m_{12} + \dots) \\ &= \omega_{11}({}_1\tau_{01}\theta_0 + \mathfrak{T}_{11}) + \omega_{12}({}_1\tau_{02}\theta_0 + \mathfrak{T}_{12}) + \dots \dots \dots (61), \end{aligned}$$

which is an equation to find r , using $\mathfrak{T}_{st}$ as an abbreviation for

$${}_s\tau_{1..t}\theta_1r + {}_s\tau_{2..t}\theta_2r^2 + {}_s\tau_{3..t}\theta_3r^3 + \text{etc.}$$

(Compare the usage in equations (2) to (5).)

$m_{1q}, m_{2q}, \dots m_{p1}, m_{p2} \dots$ are complete segments of the normal solid and are independent of r , i.e. $\frac{m_{sq}}{N} = {}_s\tau_{0q}\theta_0$, and these terms disappear from both sides of the equation. The w 's of the cells in the last row and last column of the table may therefore be dropped and we have $(p-1)(q-1)$ w 's to determine. The probable error may be written down in the manner already shown and as there are $(p-1)(q-1)$ independent frequencies there is sufficient data to determine the w 's so that the probable error is a minimum.

There is no essential difficulty in carrying this out except that the coefficients rapidly become very cumbersome as tables increase beyond 3×3 for then the simplification $dm_2 = -dn_3$ is no longer available. It will be found however that the same result may be derived more simply from the method discussed below.

§ 6. POLYCHORIC METHOD.

The frequency surface divided into p columns and q rows is divided at each point $11, 12 \dots$ into four quadrants, and for each of these divisions a value for r , viz., $r_{11}, r_{12} \dots$, may be found by the tetrachoric method. These may be regarded as approximations to the true value of r , and their weighted mean found, the weights

being determined so that the probable error of the mean r so found shall be a minimum.

$$\text{Let } r = \frac{C_{11}r_{11} + C_{12}r_{12} + \dots}{C_{11} + C_{12} + \dots} \dots\dots\dots(62).$$

$$\text{Then } (C_{11} + C_{12} + \dots) dr = C_{11}dr_{11} + C_{12}dr_{12} + \dots \dots\dots(63).$$

Squaring, summing for all possible values and dividing by the number of samples,

$$(\Sigma C_{st})^2 \sigma_r^2 = \Sigma (C_{st} \sigma_{st})^2 + 2 \Sigma (C_{st} C_{s't'} \sigma_{st} \sigma_{s't'} R_{st, s't'}) \dots\dots\dots(64),$$

$$\text{where } \sigma_{st} = \sigma_{r_{st}}, \quad R_{st, s't'} = r_{r_{st}, r_{s't'}}.$$

$$\begin{aligned} \text{Let } S &= \Sigma (C_{st} \sigma_{st})^2 + 2 \Sigma (C_{st} C_{s't'} \sigma_{st} \sigma_{s't'} R_{st, s't'}), \\ C &= \Sigma (C_{st}). \end{aligned}$$

Then for a minimum

$$\begin{aligned} dS &= \frac{\partial S}{\partial C_{11}} dC_{11} + \frac{\partial S}{\partial C_{12}} dC_{12} + \dots = 0, \\ dC &= dC_{11} + dC_{12} + \dots = 0, \\ \therefore \left(\frac{\partial S}{\partial C_{11}} - \lambda \right) dC_{11} &= \left(\frac{\partial S}{\partial C_{12}} - \lambda \right) dC_{12} = \dots = 0. \\ \therefore C_{11} \sigma_{11}^2 + C_{12} \sigma_{11} \sigma_{12} R_{11, 12} + C_{13} \sigma_{11} \sigma_{13} R_{11, 13} + \dots &= \lambda, \\ C_{11} \sigma_{11} \sigma_{12} R_{11, 12} + C_{12} \sigma_{12}^2 + C_{13} \sigma_{12} \sigma_{13} R_{12, 13} + \dots &= \lambda \dots\dots\dots(65). \end{aligned}$$

Let

$$\Delta = \begin{vmatrix} 1 & \frac{1}{\sigma_{11}} & \frac{1}{\sigma_{12}} & \frac{1}{\sigma_{13}} & \dots \\ \frac{1}{\sigma_{11}} & 1 & R_{11, 12} & R_{11, 13} & \dots \\ \frac{1}{\sigma_{12}} & R_{11, 12} & 1 & R_{12, 13} & \dots \\ \frac{1}{\sigma_{13}} & R_{11, 13} & R_{12, 13} & 1 & \dots \end{vmatrix} \dots\dots\dots(66).$$

Then

$$\begin{aligned} C_{11} \sigma_{11} &= - \frac{\lambda \Delta_{0011}}{\Delta_{0000}}, \\ C_{12} \sigma_{12} &= - \frac{\lambda \Delta_{0012}}{\Delta_{0000}} \dots\dots\dots(67). \end{aligned}$$

Since λ is arbitrary we may put $\Sigma (C_{st}) = 1$,

$$\begin{aligned} \Sigma (C_{st}) &= - \frac{\lambda}{\Delta_{0000}} \left\{ \frac{\Delta_{0000}}{1} + \frac{\Delta_{0011}}{\sigma_{11}} + \frac{\Delta_{0012}}{\sigma_{12}} + \dots - \Delta_{0000} \right\} \\ &= - \frac{\lambda (\Delta - \Delta_{0000})}{\Delta_{0000}} = 1 \dots\dots\dots(68), \end{aligned}$$

$$\therefore \lambda = - \frac{\Delta_{0000}}{\Delta - \Delta_{0000}} \dots\dots\dots(69).$$

$$\therefore C_{11} \sigma_{11} = \frac{\Delta_{0011}}{\Delta - \Delta_{0000}} \dots\dots\dots(70).$$

$$\begin{aligned}
 \therefore \Sigma (C_{st} \sigma_{st})^2 + 2 \Sigma (C_{st} C_{s't'} \sigma_{st} \sigma_{s't'} R_{st, s't'}) \\
 = \frac{1}{\Delta - \Delta_{0000}} \Sigma \left\{ C_{st} \sigma_{st} \left(\frac{\Delta_{0000}}{\sigma_{st}} + \Delta_{0011} R_{11st} + \Delta_{0012} R_{12 \cdot st} \right. \right. \\
 \left. \left. + \Delta_{00st} + \dots - \frac{\Delta_{0000}}{\sigma_{st}} \right) \right\} \\
 = \frac{1}{\Delta - \Delta_{0000}} \Sigma \left\{ C_{st} \sigma_{st} \left(- \frac{\Delta_{0000}}{\sigma_{st}} \right) \right\} \\
 = - \frac{\Sigma C_{st} \cdot \Delta_{0000}}{\Delta - \Delta_{0000}} = \frac{\Delta_{0000}}{\Delta_{0000} - \Delta} \dots \dots \dots (71),
 \end{aligned}$$

$$\text{and since } \Sigma C_{st} = 1 \qquad \sigma_r^2 = \frac{\Delta_{0000}}{\Delta_{0000} - \Delta} \dots \dots \dots (72).$$

§ 7. COMPARISON OF POLYCHORIC AND ENNEACHORIC COEFFICIENTS OF CORRELATION.

We may now compare the polychoric coefficient (r_p) with the enneachoric coefficient (r_e) previously found.

$$\begin{aligned}
 \text{Since} \qquad \qquad \qquad \Sigma (C_{st}) = 1, \\
 r_p = C_{11} r_{11} + C_{12} r_{12} + \dots \dots \dots (73),
 \end{aligned}$$

$$\therefore dr_p = C_{11} dr_{11} + C_{12} dr_{12} + \dots = - C_{11} \frac{\delta P_{11}}{\chi_{11}} - C_{12} \frac{\delta P_{12}}{\chi_{12}} \dots \dots \dots (74),$$

$$\begin{aligned}
 \therefore - (C_{11} dr_{11} + C_{12} dr_{12} + \dots) \\
 = \frac{C_{11}}{\chi_{11}} (A_{11} dm_{1 \cdot} + B_{11} dm_{\cdot 1} - dm_{11}) + \frac{C_{12}}{\chi_{12}} (A_{12} dm_{1 \cdot} + B_{12} dm_{\cdot 2} - dm_{12}) + \text{etc.} \\
 \dots \dots \dots (75).
 \end{aligned}$$

Transposing and rearranging we find

$$\begin{aligned}
 \frac{C_{11}}{\chi_{11}} dm_{11} + \frac{C_{12}}{\chi_{12}} dm_{12} + \dots \\
 = \frac{C_{11}}{\chi_{11}} (A_{11} dm_{1 \cdot} + B_{11} dm_{\cdot 1} + \chi_{11} dr_{11}) + \frac{C_{12}}{\chi_{12}} (A_{12} dm_{1 \cdot} + B_{12} dm_{\cdot 2} + \chi_{12} dr_{12}) + \text{etc.} \\
 = \frac{C_{11}}{\chi_{11}} \left(\frac{\partial f_{11}}{\partial h_1} dh_1 + \frac{\partial f_{11}}{\partial k_1} dk_1 + \frac{\partial f_{11}}{\partial r_{11}} dr_{11} \right) + \frac{C_{12}}{\chi_{12}} \left(\frac{\partial f_{12}}{\partial h_1} dh_1 + \frac{\partial f_{12}}{\partial k_2} dk_2 + \frac{\partial f_{12}}{\partial r_{12}} dr_{12} \right) + \dots \\
 \dots \dots \dots (76),
 \end{aligned}$$

$$\text{where} \qquad \qquad \qquad f_{st} = N \int_{-\infty}^{h_s} \int_{-\infty}^{k_t} z(x, y, r) dx dy = m_{st},$$

$$\begin{aligned}
 \frac{C_{11}}{\chi_{11}} m_{11} + \frac{C_{12}}{\chi_{12}} m_{12} + \dots = \frac{C_{11}}{\chi_{11}} ({}_1\tau_0 {}_1\theta_0 + \mathfrak{T}_{10}) + \frac{C_{12}}{\chi_{12}} ({}_1\tau_0 {}_2\theta_0 + \mathfrak{T}_{12}) + \dots \\
 \dots \dots \dots (77),
 \end{aligned}$$

using $\mathfrak{T}_{st}$ as in equation (61), page 106, which is an equation of which r_p is a root. If in this equation we put $C_{11} = \chi_{11}$, $C_{12} = -\chi_{12}$, $C_{21} = -\chi_{21}$, $C_{22} = \chi_{22}$, and the remaining C 's = 0 we have the equation for the enneachoric coefficient and r_e now appears to be the weighted mean of four tetrachoric r 's,

$$r_e = \frac{\chi_{11} r_{11} - \chi_{12} r_{12} - \chi_{21} r_{21} + \chi_{22} r_{22}}{\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}}.$$

Comparing this with the generalized equation (61) for enneachoric r previously found in which all the frequency volumes were weighted, we have

$$\begin{aligned}\omega_{11} &= \frac{C_{11}}{\chi_{11}}, \\ \omega_{12} &= \frac{C_{12}}{\chi_{12}}, \\ &\dots\dots\dots \\ \omega_{st} &= \frac{C_{st}}{\chi_{st}},\end{aligned}$$

and since

$$\omega_{st} = w_{st} - w_{s+1, t} - w_{s, t+1} + w_{s+1, t+1},$$

it can be easily shown* if there are p rows and q columns that

$$\begin{aligned}w_{st} &= \omega_{st} + \omega_{s, t+1} + \omega_{s, t+2} + \dots \omega_{s, q-1} + \omega_{s+1, t} + \omega_{s+1, t+1} + \dots \omega_{s+1, q-1} \\ &\dots\dots\dots \\ &+ \omega_{p-1, t} + \omega_{p-2, t+1} + \dots \omega_{p, q} \dots\dots\dots (78),\end{aligned}$$

that is the sum of all the weights having the same suffixes as the points contained within the d quadrant of which n_{st} occupies the corner.

* Consider a two-fold extension ruled and named in a manner similar to the polychoric scheme on page 94.

Then

$$\begin{aligned}m_{11} &= n_{11} \\ m_{12} &= m_{11} + m_{12} \\ &\dots\dots\dots \\ m_{st} &= n_{11} + n_{12} + n_{13} \dots\dots + m_{1t} \\ &\quad + n_{21} + n_{22} + n_{23} \dots\dots + m_{2t} \\ &\quad \dots\dots\dots \\ &\quad + n_{s1} + n_{s2} + n_{s3} \dots\dots + n_{st} \\ &\quad \dots\dots\dots\end{aligned}$$

Hence $\omega_{11}m_{11} + \omega_{12}m_{12} + \dots\dots$

$$\begin{aligned}&= \omega_{11}n_{11} + \omega_{12}(n_{11} + n_{12}) + \omega_{13}(n_{11} + n_{12} + n_{13}) \\ &\quad + \dots\dots \\ &\quad + \omega_{st}(n_{11} + n_{12} + \dots\dots n_{1t} \\ &\quad + n_{21} + n_{22} + \dots\dots n_{2t} \\ &\quad + \dots\dots \\ &\quad + n_{s1} + n_{s2} + \dots\dots n_{st}).\end{aligned}$$

If we rearrange this in terms of n_{11}, n_{12} we shall have

$$\begin{aligned}&n_{11}(\omega_{11} + \omega_{12} + \dots\dots \omega_{st}) \\ &+ n_{12}(\omega_{12} + \omega_{13} + \dots\dots \omega_{st}) \\ &= w_{11}n_{11} + w_{12}n_{12} + \dots\dots w_{st}n_{st} + \dots\dots w_{pq}n_{pq}.\end{aligned}$$

It is clear that w_{st} will be the sum of the ω 's belonging to all the m 's of which n_{st} is a constituent. that is, from the figure, all the n 's whose boundary lines lie beyond the lines $h = s - 1, k = t - 1$, i.e.

$$w_{st} = \sum_{h=1}^s \sum_{k=1}^t \omega_{hk}.$$

The relation

$$n_{st} = m_{st} - m_{s-1, t} - m_{s, t-1} + m_{s-1, t-1}$$

may be compared to the partial finite difference in two variables

$$\Delta_x \Delta_y U_{x, y} = U_{x+1, y+1} - U_{x+1, y} - U_{x, y+1} + U_{x, y},$$

which may help to make the above relation clearer.

§ 8. COMPUTATION OF R 'S AND σ 'S.

As the computation of r_p even for an enneachoric table is somewhat lengthy, it is necessary to have a definite scheme to work to. In addition to this the values of the R 's when resolved into their constituents present some interesting features.

A new expression for tetrachoric r has already been deduced from the degeneration of an enneachoric table. The following is a derivation of it directly and in a more symmetrical form.

Consider the tetrachoric table

	D		
a		b	$n_{.1}$
F	E	L	
c		d	$n_{.2}$
	G		
$n_{1.}$		$n_{2.}$	N

Let A and B have the same significance as before, i.e. A is the fraction which the area of the plane DE is of the whole dichotomic plane and B the same for FE , and write

$${}_aP = An_{1.} + Bn_{.1} - a \dots\dots\dots(79),$$

where the a suffix is used to indicate that it is the P of the leading or a quadrant.

Then since the fractional area of EG will be $1 - A$ and of EL , $1 - B$, the corresponding P for b quadrant will be

$$\begin{aligned} {}_bP &= An_{2.} + (1 - B) n_{.1} - b \\ &= A(N - n_{1.}) + (1 - B) n_{.1} - (n_{.1} - a) \\ &= AN - (An_{1.} + Bn_{.1} - a) \\ &= AN - {}_aP \dots\dots\dots(80). \end{aligned}$$

Similarly ${}_cP = BN - {}_aP \dots\dots\dots(81),$

$${}_dP = -N(A + B - 1) + {}_aP \dots\dots\dots(82).$$

Hence we have ${}_aP + {}_bP + {}_cP + {}_dP = N \dots\dots\dots(82) \text{ bis.}$

We have already seen (20) that

$$-\chi^2 dr = Adm_{1.} + Bdm_{.1} - dm_{11} = \delta {}_aP_{11} \dots\dots\dots(83).$$

Using the symbol Σ as before to denote the operation "sum for all possible samples and divide by the number of samples"* we have

* It would be useful to have a distinctive name for this operation, verb as well as substantive.

$$\begin{aligned}
\chi^2 \sigma_r^2 &= \mathfrak{S} (\delta_a P)^2 \\
&= \mathfrak{S} (A d m_{1.} + B d m_{.1} - d m_{11})^2 \\
&= A^2 m_{1.} + B^2 m_{.1} + m_{11} + 2 A B m_{11} - 2 A m_{11} - 2 B m_{11} - \frac{a P^2}{N} \\
&= A^2 (a + c) + B^2 (a + b) + a + (2 A B - 2 A - 2 B) a - \frac{a P^2}{N} \\
&= (A + B - 1)^2 a + A^2 c + B^2 b - \frac{a P^2}{N} \dots\dots\dots (84).
\end{aligned}$$

But

$${}_a P = A (a + c) + B (a + b) - a = (A + B - 1) a + A c + B b + 0 \cdot d \dots\dots (85)$$

and

$$a + b + c + d = N,$$

$$\begin{aligned}
\therefore \mathfrak{S} (\delta_a P)^2 &= \left(A + B - 1 - \frac{a P}{N} \right)^2 a + \left(B - \frac{a P}{N} \right)^2 b + \left(A - \frac{a P}{N} \right)^2 c + \left(0 - \frac{a P}{N} \right)^2 d \\
&= \left(- \frac{a P}{N} \right)^2 a + \left(\frac{c P}{N} \right)^2 b + \left(\frac{b P}{N} \right)^2 c + \left(- \frac{a P}{N} \right)^2 d \\
&= \frac{1}{N^2} \left\{ {}_a P^2 a + {}_c P^2 b + {}_b P^2 c + {}_a P^2 d \right\} \dots\dots\dots (86).
\end{aligned}$$

Further

$$\begin{aligned}
&(-{}_a P) a + ({}_c P) b + ({}_b P) c + (-{}_a P) a \\
&= a \{ N (A + B - 1) - {}_a P \} + b (B N - {}_a P) + c (A N - {}_a P) + d (-{}_a P) \\
&= N \{ (A + B - 1) a + B b + A c \} - N {}_a P \\
&= N \{ A (a + c) + B (a + b) - a \} - N {}_a P \\
&= N {}_a P - N {}_a P = 0 \dots\dots\dots (87).
\end{aligned}$$

The above form (86) of the square of the standard deviation (omitting factor) is interesting as involving only the squares of the P 's. Since the P 's are connected by the relation (82) *bis* and (87) their values may be determined from any two of them.

The R 's.

Since

$$-\chi_{st} \delta r_{st} = \delta P_{st}$$

and

$$-\chi_{s't'} \delta r_{s't'} = \delta P_{s't'},$$

$$\chi_{st} \chi_{s't'} \delta r_{st} \delta r_{s't'} = \delta P_{st} \delta P_{s't'}$$

and

$$\chi_{st} \chi_{s't'} \sigma_{st} \sigma_{s't'} R_{st.s't'} = \mathfrak{S} (\delta P_{st} \delta P_{s't'}) = S_{st.s't'} \text{ (say).}$$

In conformity with this notation

$$\chi_{st}^2 \sigma_{st}^2 = S_{st.st}.$$

It is useful to have a verbal rule for writing down such mean products as

$$\mathfrak{S} (\delta P_{st} \delta P_{s't'}).$$

The following will serve.

Multiply the detached coefficients of the differentials in the δP 's as in ordinary multiplication; strike out the products in which the related frequencies have no common frequency and insert the common part of the frequencies after the related coefficients. From the whole subtract the full products of the P 's divided by the total frequency. This may be proved as follows:

Let p and q be any frequencies in a given distribution in which the population N is so large that sampling does not alter its composition; then we have the well-known results (p. 99)

$$\sigma_p^2 = p \left(1 - \frac{p}{N}\right),$$

$$\mathfrak{S}(dpdq) = -\frac{pq}{N}.$$

Now let p and q have a common part c so that

$$p = p' + c,$$

$$q = q' + c.$$

$$\begin{aligned} \text{Then } \mathfrak{S}(dpdq) &= \mathfrak{S}d(p' + c) \cdot d(q' + c) \\ &= \mathfrak{S}(dp'dq' + dp'dc + dq'dc + dc^2) \\ &= -\frac{p'q'}{N} - \frac{p'c}{N} - \frac{q'c}{N} + c \left(1 - \frac{c}{N}\right) \\ &= -\frac{(q' + c)(p' + c)}{N} + c \\ &= c - \frac{pq}{N}. \end{aligned}$$

Now the mean product of any two linear functions of p 's and q 's,

$$S = \mathfrak{S}d(i_1p_1 + i_2p_2 + \dots) \cdot d(k_1q_1 + k_2q_2 + \dots),$$

will consist of the sum of the mean products of terms such as

$$i_s dp_s \cdot k_t dq_t.$$

But

$$\begin{aligned} \mathfrak{S}(i_s dp_s \cdot k_t dq_t) &= i_s k_t \mathfrak{S}(dp_s \cdot dq_t) \\ &= i_s k_t \left(c_{st} - \frac{p_s q_t}{N}\right), \end{aligned}$$

where c is the common part of p_s and q_t .

Therefore

$$\begin{aligned} S &= \Sigma \left\{ i_s k_t \left(c_{st} - \frac{p_s q_t}{N} \right) \right\} \\ &= \Sigma i_s k_t c_{st} - \Sigma i_s k_t \frac{p_s q_t}{N}. \end{aligned}$$

Hence the rule.

As an example consider $S_{11 \cdot 21}$,

$$\begin{aligned} S_{11 \cdot 21} &= \mathfrak{S}(\delta P_{11} \delta P_{21}) \\ &= \mathfrak{S}(A_{11} dm_{1 \cdot} + B_{11} dm_{\cdot 1} - dm_{11})(A_{21} dm_{2 \cdot} + B_{21} dm_{\cdot 1} - dm_{21}) \\ &= A_{11} A_{21} m_{1 \cdot} + A_{11} B_{21} m_{11} - A_{11} m_{11} + B_{11} A_{21} m_{21} + B_{11} B_{21} m_{\cdot 1} - B_{11} m_{21} \\ &\quad - A_{21} m_{11} - B_{21} m_{11} + m_{11} - \frac{P_{11} P_{21}}{N} \\ &= (A_{11} + B_{11} - 1)(A_{21} + B_{21} - 1)n_{11} + B_{11}(A_{21} + B_{21} - 1)n_{21} + B_{11} B_{21} n_{31} \\ &\quad + A_{11} A_{21} n_{12} \qquad \qquad \qquad + 0 \cdot A_{21} \cdot n_{22} \qquad \qquad \qquad + 0 \cdot 0 \cdot n_{32} \\ &\quad + A_{11} A_{21} n_{13} \qquad \qquad \qquad + 0 \cdot A_{21} \cdot n_{23} \qquad \qquad \qquad + 0 \cdot 0 \cdot n_{33} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{N} \left\{ \begin{array}{l} (A_{11} + B_{11} - 1) n_{11} + B_{11} n_{21} + B_{11} n_{31} \\ + \quad A_{11} n_{12} + 0 \cdot n_{22} + 0 \cdot n_{32} \\ + \quad A_{11} n_{13} + 0 \cdot n_{23} + 0 \cdot n_{33} \end{array} \right\} \\
& \times \left\{ \begin{array}{l} (A_{21} + B_{21} - 1) n_{11} + (A_{21} + B_{21} - 1) n_{21} + B_{21} n_{31} \\ + \quad A_{21} n_{12} + A_{21} n_{22} + 0 \cdot n_{32} \\ + \quad A_{21} n_{13} + A_{21} n_{23} + 0 \cdot n_{33} \end{array} \right\} \\
& = \left(A_{11} + B_{11} - 1 - \frac{P_{11}}{N} \right) \left(A_{21} + B_{21} - 1 - \frac{P_{21}}{N} \right) n_{11} \\
& \quad + \left(B_{11} - \frac{P_{11}}{N} \right) \left(A_{21} + B_{21} - 1 - \frac{P_{21}}{N} \right) n_{21} + \left(B_{11} - \frac{P_{11}}{N} \right) \left(B_{21} - \frac{P_{21}}{N} \right) n_{31} \\
& \quad + \left(A_{11} - \frac{P_{11}}{N} \right) \left(A_{21} - \frac{P_{21}}{N} \right) n_{12} + \left(0 - \frac{P_{11}}{N} \right) \left(A_{21} - \frac{P_{21}}{N} \right) n_{22} \\
& \quad + \left(0 - \frac{P_{11}}{N} \right) \left(0 - \frac{P_{21}}{N} \right) n_{32} + \left(A_{11} - \frac{P_{11}}{N} \right) \left(A_{21} - \frac{P_{21}}{N} \right) n_{13} \\
& \quad + \left(0 - \frac{P_{11}}{N} \right) \left(A_{21} - \frac{P_{21}}{N} \right) n_{23} + \left(0 - \frac{P_{11}}{N} \right) \left(0 - \frac{P_{21}}{N} \right) n_{33} \dots\dots\dots(88).
\end{aligned}$$

In the above the P 's are ${}_aP$'s and remembering that

$$A_{11} + B_{11} - 1 - \frac{{}_aP_{11}}{N} = -\frac{{}_aP_{11}}{N}, \text{ etc.,}$$

we have

$$\begin{aligned}
S_{11 \cdot 21} &= \left(-\frac{{}_aP_{11}}{N} \right) \left(-\frac{{}_aP_{21}}{N} \right) n_{11} + \left(\frac{{}_cP_{11}}{N} \right) \left(-\frac{{}_aP_{21}}{N} \right) n_{21} + \left(\frac{{}_cP_{11}}{N} \right) \left(\frac{{}_cP_{21}}{N} \right) n_{31} \\
& \quad + \left(\frac{{}_bP_{11}}{N} \right) \left(\frac{{}_bP_{21}}{N} \right) n_{12} + \left(-\frac{{}_aP_{11}}{N} \right) \left(\frac{{}_bP_{21}}{N} \right) n_{22} + \left(-\frac{{}_aP_{11}}{N} \right) \left(-\frac{{}_aP_{12}}{N} \right) n_{32} \\
& \quad + \left(\frac{{}_bP_{11}}{N} \right) \left(\frac{{}_bP_{21}}{N} \right) n_{13} + \left(-\frac{{}_aP_{11}}{N} \right) \left(\frac{{}_bP_{21}}{N} \right) n_{23} + \left(-\frac{{}_aP_{11}}{N} \right) \left(-\frac{{}_aP_{21}}{N} \right) n_{33} \\
&= \frac{1}{N^2} \{ {}_aP_{11} {}_aP_{21} n_{11} - {}_cP_{11} {}_aP_{21} n_{21} + {}_cP_{11} {}_cP_{21} n_{31} \\
& \quad + {}_bP_{11} {}_bP_{21} (n_{12} + n_{13}) - {}_aP_{11} {}_bP_{21} (n_{22} + n_{23}) + {}_aP_{11} {}_aP_{21} (n_{32} + n_{33}) \} \\
& \quad \dots\dots\dots(89).
\end{aligned}$$

The relation between the coefficients in the above expression is very simple. We have already seen that

$$N^2 \chi^2 \sigma_r^2 = a (-{}_aP)^2 + b ({}_cP)^2 + c ({}_bP)^2 + d (-{}_aP)^2 \dots\dots\dots(90).$$

In the quadrants of a tetrachoric table write the P coefficients. Thus

$$\begin{array}{c|c} -{}_aP & +{}_cP \\ \hline +{}_bP & -{}_aP \end{array}.$$

The a frequency is related to $-{}_aP$, etc.

Consider now the empty scheme of an enneachoric table regarded as a tetrachoric table with the point of division first at, say, 11 and second at 12, and write in the P coefficients as above.

Divided at 11.

$- {}_aP_{11}$	${}_cP_{11}$	${}_cP_{11}$
${}_bP_{11}$	$- {}_aP_{11}$	$- {}_aP_{11}$
${}_bP_{11}$	$- {}_aP_{11}$	$- {}_aP_{11}$

Divided at 21.

$- {}_aP_{21}$	$- {}_aP_{21}$	${}_cP_{21}$
${}_bP_{21}$	${}_bP_{21}$	$- {}_aP_{21}$
${}_bP_{21}$	${}_bP_{21}$	$- {}_aP_{21}$

Now if we superpose these two schemes upon an enneachoric table with a frequency in each cell, each cell will then contain the P coefficient and frequency of each term of the expansion of $S_{11 \cdot 21}$ (with the omission of the factor $\frac{1}{N^2}$) thus

$(- {}_aP_{11}) (- {}_aP_{21}) n_{11}$	$({}_cP_{11}) (- {}_aP_{21}) n_{21}$	$({}_cP_{11}) ({}_cP_{21}) n_{31}$
$({}_bP_{11}) ({}_bP_{21}) n_{12}$	$(- {}_aP_{11}) ({}_bP_{21}) n_{22}$	$(- {}_aP_{11}) (- {}_aP_{21}) n_{32}$
$({}_bP_{11}) ({}_bP_{21}) n_{13}$	$(- {}_aP_{11}) ({}_bP_{21}) n_{23}$	$(- {}_aP_{11}) (- {}_aP_{21}) n_{33}$

When 11 coincides with 21, R becomes = 1 and the mean product degenerates into the square of the standard deviation.

This may be summarised in the following table in which the letter, a , b , etc. gives the suffix and the sign gives the sign of the P required.

	P_{11}	P_{12}	P_{21}	P_{22}
n_{11}	$-d$	$-d$	$-d$	$-d$
n_{12}	$+b$	$-d$	$+b$	$-d$
n_{13}	$+b$	$+b$	$+b$	$+b$
n_{21}	$+c$	$+c$	$-d$	$-d$
n_{22}	$-a$	$+c$	$+b$	$-d$
n_{23}	$-a$	$-a$	$+b$	$+b$
n_{31}	$+c$	$+c$	$+c$	$+c$
n_{32}	$-a$	$+c$	$-a$	$+c$
n_{33}	$-a$	$-a$	$-a$	$-a$

Thus the coefficient of n_{32} in $S_{12 \cdot 21}$ is $(+ {}_cP_{12}) (- {}_aP_{21})$.

This table is sufficient for a polychoric table of any size since any two cross points st . $s't'$ in the table, with the planes through them divide it into nine portions or groups of cells, each of which is represented by one of the above cells.

The relations between two superimposed tetrachoric divisions involve the determination of ten constants, four σ 's, σ_{11} , σ_{12} , σ_{21} , σ_{22} , and six R 's, $R_{11 \cdot 12}$, $R_{11 \cdot 21}$, $R_{11 \cdot 22}$, $R_{12 \cdot 21}$, $R_{12 \cdot 22}$, $R_{21 \cdot 22}$. The σ 's follow the example already given, the proper suffixes being attached. The value of $S_{11 \cdot 21}$ has already been given. The remaining five S 's are as follows:

$$S_{11 \cdot 12} = \frac{1}{N^2} \{ {}_aP_{11a}P_{12}n_{11} + {}_cP_{11c}P_{12}(n_{21} + n_{31}) - {}_bP_{11b}P_{12}n_{12} - {}_aP_{11c}P_{12}(n_{22} + n_{32}) \\ {}_bP_{11b}P_{12}n_{13} + {}_aP_{11a}P_{12}(n_{23} + n_{33}) \} \dots\dots\dots(91).$$

$$S_{11 \cdot 22} = \frac{1}{N^2} \{ {}_aP_{11a}P_{22}n_{11} - {}_cP_{11a}P_{22}n_{21} + {}_cP_{11c}P_{22}n_{31} - {}_bP_{11b}P_{22}n_{12} + {}_aP_{11a}P_{22}n_{22} \\ - {}_aP_{11c}P_{22}n_{32} + {}_bP_{11b}P_{22}n_{13} - {}_aP_{11b}P_{22}n_{23} + {}_aP_{11a}P_{22}n_{33} \} \dots\dots\dots(92).$$

$$S_{12 \cdot 21} = \frac{1}{N^2} \{ {}_aP_{12a}P_{21}n_{11} - {}_cP_{12a}P_{21}n_{21} + {}_cP_{12c}P_{21}n_{31} - {}_aP_{12b}P_{21}n_{12} + {}_cP_{12b}P_{21}n_{22} \\ - {}_cP_{12a}P_{21}n_{32} + {}_bP_{12b}P_{21}n_{13} - {}_aP_{12b}P_{21}n_{23} + {}_aP_{12a}P_{21}n_{33} \} \dots\dots\dots(93).$$

$$S_{12 \cdot 22} = \frac{1}{N^2} \{ {}_aP_{12a}P_{22}(n_{11} + n_{12}) - {}_cP_{12a}P_{22}(n_{21} + n_{22}) + {}_cP_{12c}P_{22}(n_{31} + n_{32}) \\ + {}_bP_{12b}P_{22}n_{13} - {}_aP_{12b}P_{22}n_{23} + {}_aP_{12a}P_{22}(n_{33}) \} \dots\dots\dots(94).$$

$$S_{21 \cdot 22} = \frac{1}{N^2} \{ {}_aP_{21a}P_{22}(n_{11} + n_{21}) + {}_cP_{21c}P_{22}n_{31} - {}_bP_{21a}P_{22}(n_{12} + n_{22}) - {}_aP_{21c}P_{22}n_{32} \\ + {}_bP_{21b}P_{22}(n_{13} + n_{23}) + {}_aP_{21a}P_{22}n_{33} \} \dots\dots\dots(95).$$

A more convenient form of the above for actual computation purposes will be found on page 120.

We may now by means of the P 's express the standard deviation r_p in a form consisting of sums of squares.

$$-(\Sigma C_{st}) dr = \Sigma \left(\frac{C_{st}}{\chi_{st}} \delta P_{st} \right) \dots\dots\dots(96),$$

$$\therefore (\Sigma C_{st})^2 \sigma_r^2 = \mathfrak{Z} \left\{ \Sigma \left(\frac{C_{st}}{\chi_{st}} \delta P_{st} \right) \right\}^2 = \Sigma \left(\frac{C_{st}}{\chi_{st}} \right)^2 S_{st, st} + 2 \Sigma \frac{C_{st} C_{s't'}}{\chi_{st} \chi_{s't'}} S_{st, s't'} \\ = \frac{C_{11}}{\chi_{11}} \left(\frac{C_{11}}{\chi_{11}} S_{11, 11} + \frac{C_{12}}{\chi_{12}} S_{11, 12} + \frac{C_{13}}{\chi_{13}} S_{11, 13} + \dots \right) \\ + \frac{C_{12}}{\chi_{12}} \left(\frac{C_{11}}{\chi_{11}} S_{11, 12} + \frac{C_{12}}{\chi_{12}} S_{12, 12} + \frac{C_{13}}{\chi_{13}} S_{12, 13} + \dots \right) + \text{etc.} \dots\dots\dots(97).$$

Now consider the S 's to be expanded in terms of the frequencies and pick out all the coefficients of the frequency n_{st} say. The coefficient of the n_{st} taken from $S_{lm, l'm'}$ say will be : $P_{lm} : P_{l'm'}$ in which the quadrant suffixes will be determined by the relative position of n_{st} to lm and $l'm'$. Let these undetermined P 's be denoted by p . We shall then have as the complete coefficient of n_{st}

$$\frac{C_{11}}{\chi_{11}} \left(\frac{C_{11}}{\chi_{11}} p_{11 \cdot 11} + \frac{C_{12}}{\chi_{12}} p_{11 \cdot 12} + \frac{C_{13}}{\chi_{13}} p_{11 \cdot 13} + \dots \right) \\ + \frac{C_{12}}{\chi_{12}} \left(\frac{C_{11}}{\chi_{11}} p_{11 \cdot 12} + \frac{C_{12}}{\chi_{12}} p_{12 \cdot 12} + \frac{C_{13}}{\chi_{13}} p_{12 \cdot 13} + \dots \right) + \text{etc.}$$

Since we are dealing throughout with the cell n_{st} the quadrantal suffix (i.e. the

a, b , etc.) for any l, m will be the same throughout. Hence we may write the complete coefficient of n_{st} as

$$\begin{aligned} & \frac{C_{11}}{\chi_{11}} p_{11} \left(\frac{C_{11}}{\chi_{11}} p_{11} + \frac{C_{12}}{\chi_{12}} p_{12} + \frac{C_{13}}{\chi_{13}} p_{13} + \dots \right) \\ & + \frac{C_{12}}{\chi_{12}} p_{12} \left(\frac{C_{11}}{\chi_{11}} p_{11} + \frac{C_{12}}{\chi_{12}} p_{12} + \frac{C_{13}}{\chi_{13}} p_{13} + \dots \right) + \text{etc.} \\ & = \left(\frac{C_{11}}{\chi_{11}} p_{11} + \frac{C_{12}}{\chi_{12}} p_{12} + \frac{C_{13}}{\chi_{13}} p_{13} + \dots \right)^2 \dots\dots\dots(98). \end{aligned}$$

In the case of an enneachoric table for example we have, ΣC_{st} being = 1,

$$\begin{aligned} \sigma_r^2 = & \left(-\frac{C_{11}}{\chi_{11}} aP_{11} - \frac{C_{12}}{\chi_{12}} aP_{12} - \frac{C_{21}}{\chi_{13}} aP_{21} - \frac{C_{22}}{\chi_{22}} aP_{22} \right)^2 n_{11} \\ & + \left(\frac{C_{11}}{\chi_{11}} bP_{11} - \frac{C_{12}}{\chi_{12}} aP_{12} + \frac{C_{21}}{\chi_{11}} bP_{21} - \frac{C_{22}}{\chi_{22}} aP_{22} \right)^2 n_{12} \\ & + \left(\frac{C_{11}}{\chi_{11}} bP_{11} + \frac{C_{12}}{\chi_{12}} bP_{12} + \frac{C_{21}}{\chi_{21}} bP_{21} + \frac{C_{22}}{\chi_{22}} bP_{22} \right)^2 n_{13} + \text{etc.} \dots\dots(99), \end{aligned}$$

the P 's being at once written down from the table on page 114.

Or more generally thus:

Since the P of any cell n_{lm} with reference to any cross point (st) is invariable it may be written generally as ${}_{lm}P_{st}$. This notation gives up the recognition of the equality of the P 's in any given quadrant but gains in generality. The quadrantal suffix and sign may be supplied by inspection. We have then the following lemma:

$$\begin{aligned} S_{st, s't'} &= \mathfrak{Z}(\delta P_{st} \delta P_{s't'}) \\ &= {}_{11}P_{st \cdot 11} P_{s't'} n_{11} + {}_{21}P_{st \cdot 21} P_{s't'} n_{21} + \dots + {}_{12}P_{st \cdot 12} P_{s't'} n_{12} + \dots \\ &= \sum_{lm} ({}_{lm}P_{st \cdot lm} P_{s't'} n_{lm}) \dots\dots\dots(100). \end{aligned}$$

The standard deviation of r_d may then be developed as follows:

$$\begin{aligned} (\Sigma C_{st}) dr &= \Sigma (C_{st} dr_{st}) \\ &= \Sigma \left(C_{st} \frac{\delta P_{st}}{\chi_{st}} \right) \dots\dots\dots(101). \end{aligned}$$

$$\begin{aligned} \therefore (\Sigma C_{st})^2 \sigma_r^2 &= \Sigma \left(\frac{C_{st}^2}{\chi_{st}^2} \delta P_{st} \right) + 2 \Sigma \left(\frac{C_{st} C_{s't'}}{\chi_{st} \chi_{s't'}} \delta P_{st} \delta P_{s't'} \right) \\ &= \Sigma_{lm} \left[\left\{ \Sigma_{st} \left(\frac{C_{st}}{\chi_{st}} \right)^2 {}_{lm}P_{st}^2 + 2 \Sigma_{st} \frac{C_{st} C_{s't'}}{\chi_{st} \chi_{s't'}} {}_{lm}P_{st \cdot lm} P_{s't'} \right\} \right] n_{lm} \\ &= \Sigma_{lm} \left\{ \left[\Sigma_{st} \left(\frac{C_{st}}{\chi_{st}} {}_{lm}P_{st} \right) \right]^2 n_{lm} \right\} \dots\dots\dots(102). \end{aligned}$$

More fully written this is

$$\begin{aligned} (\Sigma C_{st})^2 \sigma_r^2 &= \left(\frac{C_{11}}{\chi_{11}} {}_{11}P_{11} + \frac{C_{12}}{\chi_{12}} {}_{11}P_{12} + \frac{C_{13}}{\chi_{13}} {}_{11}P_{13} + \dots \right)^2 n_{11} \\ &+ \left(\frac{C_{11}}{\chi_{11}} {}_{12}P_{11} + \frac{C_{12}}{\chi_{12}} {}_{12}P_{12} + \frac{C_{13}}{\chi_{13}} {}_{12}P_{13} + \dots \right)^2 n_{12} + \dots \dots(103). \end{aligned}$$

§ 9. THE STANDARD DEVIATION OF POLYCHORIC r IN SPECIAL CASES.

The value of the standard deviation when $r = 0$ is of interest and may be got as follows.

Assuming $r = 0$ throughout:

$$A_{st} = \mathfrak{E}(k_t) = \frac{m_{\cdot t}}{N},$$

$$B_{st} = \mathfrak{E}(h_s) = \frac{m_{s \cdot}}{N},$$

and writing $m'_{s \cdot} = N - m_{s \cdot}$ and $m'_{\cdot t} = N - m_{\cdot t}$, then

$${}_a P_{st} = \frac{m_{s \cdot} m_{\cdot t}}{N},$$

$${}_b P_{st} = \frac{m'_{s \cdot} m_{\cdot t}}{N},$$

$${}_c P_{st} = \frac{m_{s \cdot} m'_{\cdot t}}{N},$$

$${}_d P_{st} = \frac{m'_{s \cdot} m'_{\cdot t}}{N}.$$

Substituting these values in the S 's we have after reduction

$$S_{11 \cdot 11} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 1} m'_{1 \cdot} m'_{\cdot 1} \dots\dots\dots (104),$$

$$S_{12 \cdot 12} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 2} m'_{1 \cdot} m'_{\cdot 2} \dots\dots\dots (105),$$

$$S_{21 \cdot 21} = \frac{1}{N^3} m_{2 \cdot} m_{\cdot 1} m'_{2 \cdot} m'_{\cdot 1} \dots\dots\dots (106),$$

$$S_{22 \cdot 22} = \frac{1}{N^3} m_{2 \cdot} m_{\cdot 2} m'_{2 \cdot} m'_{\cdot 2} \dots\dots\dots (107),$$

$$S_{11 \cdot 12} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 1} m'_{1 \cdot} m'_{\cdot 2} \dots\dots\dots (108),$$

$$S_{11 \cdot 21} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 1} m'_{2 \cdot} m'_{\cdot 1} \dots\dots\dots (109),$$

$$S_{12 \cdot 21} = S_{11 \cdot 22} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 1} m'_{2 \cdot} m'_{\cdot 2} \dots\dots\dots (110),$$

$$S_{12 \cdot 22} = \frac{1}{N^3} m_{1 \cdot} m_{\cdot 2} m'_{2 \cdot} m'_{\cdot 2} \dots\dots\dots (111),$$

$$S_{21 \cdot 22} = \frac{1}{N^3} m_{2 \cdot} m_{\cdot 1} m'_{2 \cdot} m'_{\cdot 2} \dots\dots\dots (112).$$

From these values we get the σ 's and R 's,

$$\sigma_{11} = \frac{1}{N^2 H_1 K_1} \sqrt{\frac{m_{1 \cdot} m_{\cdot 1} m'_{1 \cdot} m'_{\cdot 1}}{N}} \left(= \frac{1}{q_{11}} \right) \dots\dots\dots (113),$$

$$\sigma_{12} = \frac{1}{N^2 H_1 K_2} \sqrt{\frac{m_{1.} m_{.2} m'_{1.} m'_{.2}}{N}} \left(= \frac{1}{q_{12}} \right) \dots\dots\dots (114),$$

$$\sigma_{21} = \frac{1}{N^2 H_2 K_1} \sqrt{\frac{m_{2.} m_{.1} m'_{2.} m'_{.1}}{N}} \left(= \frac{1}{q_{21}} \right) \dots\dots\dots (115),$$

$$\sigma_{22} = \frac{1}{N^2 H_2 K_2} \sqrt{\frac{m_{2.} m_{.2} m'_{2.} m'_{.2}}{N}} \left(= \frac{1}{q_{22}} \right) \dots\dots\dots (116).$$

Since $m_{1.} m'_{.1} = m_{1.} (N - m_{1.}) = N \sigma_{m_1}^2$.

$$\sigma_{11} = \frac{1}{N \sqrt{N}} \frac{\sigma_{m_1} \cdot \sigma_{m'_{.1}}}{H_1 K_1},$$

and similarly for the others

$$R_{11.12} = R_{21.22} = \sqrt{\frac{m_{.1} m'_{.2}}{m'_{.1} m_{.2}}} \quad (= \epsilon) \dots\dots\dots (117),$$

$$R_{11.21} = R_{12.22} = \sqrt{\frac{m_{1.} m'_{2.}}{m'_{1.} m_{2.}}} \quad (= \epsilon') \dots\dots\dots (118),$$

$$R_{11.22} = R_{12.21} = \sqrt{\frac{m'_{1.} m'_{.1} m'_{2.} m'_{.2}}{m'_{1.} m'_{.1} m_{2.} m_{.2}}} \quad (= \epsilon \epsilon') \dots\dots\dots (119).$$

With these values we have

$$\Delta = \begin{vmatrix} 1 & q_{11} & q_{12} & q_{21} & q_{22} \\ q_{11} & 1 & \epsilon & \epsilon' & \epsilon \epsilon' \\ q_{12} & \epsilon & 1 & \epsilon \epsilon' & \epsilon' \\ q_{21} & \epsilon' & \epsilon \epsilon' & 1 & \epsilon \\ q_{22} & \epsilon \epsilon' & \epsilon' & \epsilon & 1 \end{vmatrix} \dots\dots\dots (120).$$

Δ_{0000} is symmetrical with respect to the centre of the square, hence*

$$= \begin{vmatrix} 1 + \epsilon \epsilon' & \epsilon + \epsilon' \\ \epsilon + \epsilon' & 1 + \epsilon \epsilon' \end{vmatrix} \cdot \begin{vmatrix} 1 - \epsilon \epsilon' & \epsilon - \epsilon' \\ \epsilon - \epsilon' & 1 - \epsilon \epsilon' \end{vmatrix} = \{(1 - \epsilon^2) (1 - \epsilon'^2)\}^2 = \frac{N^3 n_{22}}{m'_{1.} m'_{.1} m_{2.} m_{.2}} \dots\dots\dots (121).$$

The remaining minors are easily reduced and we have after reduction and substitution

$$\frac{\Delta_{0011}}{\sigma_{11}} = - (1 - \epsilon^2) (1 - \epsilon'^2) N^5 \frac{H_1}{m_{1.} m'_{.1}} \left(H_1 - \frac{m_{1.}}{m_{2.}} H_2 \right) \frac{K_1}{m_{.1} m'_{.1}} \left(K_1 - \frac{m_{.1}}{m_{.2}} K_2 \right) \dots\dots\dots (122),$$

$$\frac{\Delta_{0012}}{\sigma_{12}} = (1 - \epsilon^2) (1 - \epsilon'^2) N^5 \frac{H_1}{m_{1.} m'_{.1}} \left(H_1 - \frac{m_{1.}}{m_{2.}} H_2 \right) \frac{K_2}{m_{.2} m'_{.2}} \left(\frac{m'_{.2}}{m'_{.1}} K_1 - K_2 \right) \dots\dots\dots (123),$$

$$\frac{\Delta_{0021}}{\sigma_{21}} = (1 - \epsilon^2) (1 - \epsilon'^2) \frac{H_2}{m_{2.} m'_{2.}} \left(\frac{m'_{2.}}{m'_{1.}} H_1 - H_2 \right) \frac{K_1}{m_{.1} m'_{.1}} \left(K_1 - \frac{m_{.1}}{m_{.2}} K_2 \right) \dots\dots\dots (124),$$

$$\frac{\Delta_{0022}}{\sigma_{22}} = - (1 - \epsilon^2) (1 - \epsilon'^2) \frac{H_2}{m_{2.} m'_{2.}} \left(\frac{m'_{2.}}{m'_{1.}} H_1 - H_2 \right) \frac{K_2}{m_{.2} m'_{.2}} \left(\frac{m'_{.2}}{m'_{.1}} K_1 - K_2 \right) \dots\dots\dots (125).$$

* See Scott and Mathews, *Theory of Determinants* (2nd ed.), p. 89.

Summing these four latter expressions we have

$$\Delta - \Delta_{0000} = - (1 - \epsilon^2) (1 - \epsilon'^2) N^5 \left\{ \frac{H_1^2}{m_{1.} m'_{1.}} - \frac{2H_1 H_2}{m_{2.} m'_{1.}} + \frac{H_2^2}{m_{2.} m'_{2.}} \right\} \\ \times \left\{ \frac{K_1^2}{m_{.1} m'_{.1}} - \frac{2K_1 K_2}{m_{.2} m'_{.1}} + \frac{K_2^2}{m_{.2} m'_{.2}} \right\} \dots\dots\dots (126),$$

$$\therefore \sigma_r^2 = \frac{\Delta_{0000}}{\Delta_{0000} - \Delta} \\ = \frac{n_{22} / m'_{1.} m'_{.1} m_{2.} m_{.2}}{N^2 \left\{ \frac{H_1^2}{m_{1.} m'_{1.}} - \frac{2H_1 H_2}{m_{2.} m'_{1.}} + \frac{H_2^2}{m_{2.} m'_{2.}} \right\} \times \left\{ \frac{K_1^2}{m_{.1} m'_{.1}} - \frac{2K_1 K_2}{m_{.2} m'_{.1}} + \frac{K_2^2}{m_{.2} m'_{.2}} \right\}} \\ = \frac{n_{22}}{N^2 \left\{ H_1^2 \frac{m_{2.}}{m_{1.}} - 2H_1 H_2 + H_2^2 \frac{m'_{1.}}{m'_{2.}} \right\} \times \left\{ K_1^2 \frac{m_{.2}}{m_{.1}} - 2K_1 K_2 + K_2^2 \frac{m'_{.1}}{m'_{.2}} \right\}} \\ \dots\dots\dots (127).$$

When the table is symmetrically divided in both categories and $r = 0$ we have $m_{2.} = m'_{1.}$, $H_1 = H_2$, $K_1 = K_2$, etc. and the above reduces to

$$\sigma_r^2 = \frac{n_{22}}{N^2 H^2 K^2 \cdot 4 \left(\frac{m_{2.}}{m_{1.}} - 1 \right) \left(\frac{m_{.2}}{m_{.1}} - 1 \right)} = \frac{n_{22}}{4 N^2 H^2 K^2 \frac{n_{2.}}{n_{1.}} \cdot \frac{n_{.2}}{n_{.1}}} = \frac{n_{11}}{4 N^2 H^2 K^2}$$

and
$$\sigma_r = \frac{\sqrt{n_{11}}}{2 N H K} \dots\dots\dots (128).$$

§ 10. COMPARISON OF THE STANDARD DEVIATIONS OF POLYCHORIC r AND ENNEACHORIC r .

We may now compare the standard deviations of r_e and r_p .

$$-(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}) \delta r_e = \delta P_{11} - \delta P_{12} - \delta P_{21} + \delta P_{22} \dots\dots\dots (129),$$

$$-(C_{11} + C_{12} + C_{21} + C_{22}) \delta r_p = \frac{C_{11}}{\chi_{11}} \delta P_{11} - \frac{C_{12}}{\chi_{12}} \delta P_{12} - \frac{C_{21}}{\chi_{21}} \delta P_{21} + \frac{C_{22}}{\chi_{22}} \delta P_{22} \\ \dots\dots\dots (130),$$

from which it appears as before (p. 108) that the enneachoric r is equivalent to a polychoric r in which the weights of the r 's are χ_{11} , $-\chi_{12}$, $-\chi_{21}$, χ_{22} , i.e.

$$r_e = \frac{\chi_{11} r_{11} - \chi_{12} r_{12} - \chi_{21} r_{21} + \chi_{22} r_{22}}{\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22}} \dots\dots\dots (131).$$

Hence also the standard deviation of the enneachoric r may be written

$$\sigma_r^2 = (-{}_a P_{11} + {}_a P_{12} + {}_a P_{21} - {}_a P_{22})^2 n_{11} + \text{etc.} \dots\dots\dots (132).$$

Upon expansion this reduces to

$$\sigma_r^2 = \left[-{}_a P_{11} + N (A_{11} + B_{11} - 1) + {}_a P_{12} - N (A_{12} + B_{12} - 1) \right]^2 n_{11} + \text{etc.} \\ = \left[N \{ (A_{22} - A_{21}) - (A_{12} - A_{11}) + (B_{22} - B_{12}) \} \right]^2 n_{11} + \text{etc.} \\ = \left[N \left\{ a_{22} - a_{12} + \beta_{22} - \beta_{21} - \frac{P_{11} - P_{12} - P_{21} + P_{22}}{N} \right\} \right]^2 n_{11} + \text{etc.} \dots (133)$$

(using α , β in the sense of p. 99), which is identical with the corresponding coefficient in σ_r^2 as given in equation (30).

§ 11. FORMULAE FOR COMPUTATION.

The forms found for S are not convenient for computation. The following have been found more expeditious. Various other formulae are also collected for reference:

$$\chi_{st} = Nz(h_s, k_t, r) = \frac{N}{2\pi\sqrt{1-r^2}} e^{-\frac{h_s^2 + k_t^2 - 2rh_s k_t}{2(1-r^2)}} \dots\dots\dots(134),$$

$$A_{st} = \mathfrak{E}\left(\frac{k_t - rh_s}{\sqrt{1-r^2}}\right),$$

$$B_{st} = \mathfrak{E}\left(\frac{h_s - rk_t}{\sqrt{1-r^2}}\right),$$

$$\Pi_{st} = A_{st} + B_{st} - 1,$$

$$P_{st} = A_{st}m_{s\cdot} + B_{st}m_{\cdot t} - m_{st},$$

when no quadrantal suffix is used a is understood.

$$S_{11\cdot 11} = \Pi_{11}^2 a_{11} + A_{11}^2 c_{11} + B_{11}^2 b_{11} - \frac{P_{11}^2}{N} \dots\dots\dots(135),$$

and similarly for $S_{12\cdot 12}$, etc.,

$$\begin{aligned} S_{11\cdot 12} = & \Pi_{11}\Pi_{12}n_{11} + B_{11}B_{12}(n_{21} + n_{31}) \\ & + A_{11}\Pi_{12}n_{12} - \frac{P_{11}P_{12}}{N} \dots\dots\dots(136), \\ & + A_{11}A_{12}n_{13} \end{aligned}$$

$$\begin{aligned} S_{11\cdot 21} = & \Pi_{11}\Pi_{21}n_{11} + B_{11}\Pi_{21}n_{21} + B_{11}B_{21}n_{31} \\ & + A_{11}\Pi_{22}n_{12} - \frac{P_{11}P_{21}}{N} \dots\dots\dots(137), \\ & + A_{11}A_{22}n_{13} \end{aligned}$$

$$\begin{aligned} S_{11\cdot 22} = & \Pi_{11}\Pi_{22}n_{11} + B_{11}\Pi_{22}n_{21} + B_{11}B_{22}n_{31} \\ & + A_{11}\Pi_{22}n_{12} - \frac{P_{11}P_{22}}{N} \dots\dots\dots(138), \\ & + A_{11}A_{22}n_{13} \end{aligned}$$

$$\begin{aligned} S_{12\cdot 21} = & \Pi_{12}\Pi_{21}n_{11} + B_{12}\Pi_{21}n_{21} + B_{12}B_{21}n_{31} \\ & + \Pi_{12}A_{21}n_{12} + B_{12}A_{21}n_{22} - \frac{P_{12}P_{21}}{N} \dots\dots\dots(139), \\ & + A_{12}A_{21}n_{13} \end{aligned}$$

$$\begin{aligned} S_{12\cdot 22} = & \Pi_{12}\Pi_{22}(n_{11} + n_{12}) + B_{12}\Pi_{22}(n_{21} + n_{22}) + B_{12}B_{22}(n_{31} + n_{32}) \\ & + A_{12}A_{22}n_{13} - \frac{P_{12}P_{22}}{N} \dots\dots\dots(140), \end{aligned}$$

$$\begin{aligned} S_{21\cdot 22} = & \Pi_{21}\Pi_{22}(n_{11} + n_{21}) + B_{21}B_{22}n_{31} \\ & + A_{21}\Pi_{22}(n_{12} + n_{22}) - \frac{P_{21}P_{22}}{N} \dots\dots\dots(141), \\ & + A_{21}A_{22}(n_{13} + n_{23}) \end{aligned}$$

$$\chi_{st}^2 \sigma_{st}^2 = S_{st\cdot st\cdot}$$

$$\therefore \sigma_{st} = \frac{\sqrt{S_{st\cdot st\cdot}}}{\chi_{st}}.$$

$$\chi_{st}\chi_{s't'}\sigma_{st}\sigma_{s't'}R_{st\cdot s't'} = S_{st\cdot s't'} \dots\dots\dots(142),$$

$$\begin{aligned} \therefore R_{st\cdot s't'} &= \frac{S_{st\cdot s't'}}{\chi_{st}\sigma_{st}\chi_{s't'}\sigma_{s't'}} \\ &= \frac{S_{st\cdot s't'}}{\sqrt{S_{st\cdot st\cdot}S_{s't'\cdot s't'}}} \dots\dots\dots(143). \end{aligned}$$

In place of calculating σ and R it will be found easier to employ the S 's directly by writing the equations for C in the form

$$\left. \begin{aligned} C_{11} \frac{S_{11 \cdot 11}}{\chi_{11}^2} + C_{12} \frac{S_{11 \cdot 12}}{\chi_{11} \chi_{12}} + C_{13} \frac{S_{11 \cdot 13}}{\chi_{11} \chi_{13}} + \dots = \lambda \\ C_{11} \frac{S_{11 \cdot 12}}{\chi_{11} \chi_{12}} + C_{12} \frac{S_{12 \cdot 12}}{\chi_{12}^2} + C_{13} \frac{S_{12 \cdot 13}}{\chi_{12} \chi_{13}} + \dots = \lambda \end{aligned} \right\} + \text{etc.} \dots\dots\dots(144).$$

Eliminate the λ by subtracting one equation from each of the others; put $C_{11} = 1$ and solve by successive elimination for the remaining C 's. This is preferable to using the determinant as it is at least no more laborious and lends itself to various checks for accuracy. The λ should be determined from each of the equations as a further check. Then we have

$$\sigma_r^2 = \frac{\lambda}{\sum C} \dots\dots\dots(145).$$

By putting $C_{11} = \chi_{11}$, $C_{12} = -\chi_{12}$, $C_{21} = -\chi_{21}$, $C_{22} = \chi_{22}$, we may derive the enneachoric standard deviation from the polychoric r in a form convenient for computation in terms of the S 's,

$$(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})^2 \sigma_{re}^2 = S_{11 \cdot 11} + S_{12 \cdot 12} + S_{21 \cdot 21} + S_{22 \cdot 22} + 2(S_{11 \cdot 22} + S_{12 \cdot 21}) - 2(S_{11 \cdot 12} + S_{11 \cdot 21} + S_{12 \cdot 22} + S_{21 \cdot 22}) \dots\dots\dots(146).$$

§ 12. COMPARATIVE RESULTS OF VARIOUS METHODS OF FINDING r FROM A 3×3 TABLE.

In testing the methods developed in this paper upon actual material it was thought desirable to try them side by side with all the other methods of finding the correlation coefficient so that some indication could be got of their comparative accuracy. Each of the tables was therefore dealt with by nine methods which are indicated in § 13. These tables were selected at the beginning of the investigation, and had the course which the research has taken been foreseen probably a different selection might have been made. Two of them, I and III, are normal tables with an arbitrary population of 1000. In Table I the frequencies have been taken to the nearest integer and in III to the nearest two places of decimals, so that any irregularity in them is due to the roughness of the approximation to the true figures. In the r_ψ , we have an additional lack of approximation in taking r_ψ from the curve* for determining r_ψ and also in r_ψ , r_ϕ and r_e from finding the class index correlation from a small number of marginal groups. In II and IV we have actual samples.

A rough test of the value of the various methods may be made by finding the mean square deviation of the calculated from the "observed" value of r , each constituent being merely weighted with its total frequency, regarding the product moment values of r as the "observed" value.

Thus let $n_1, n_2 =$ total frequency in Tables I, II; $R_1, R_2 =$ product moment value of the correlation coefficient in Tables I, II; $r_1, r_2 =$ correlation coefficient calculated by one of the methods, then writing

$$(n_1 + n_2 + \dots) \Sigma^2 = n_1 (R_1 - r_1)^2 + n_2 (R_2 - r_2)^2 + \dots,$$

we shall have in Σ^2 a measure of the goodness of the various methods. This gives the following values of Σ^2 .

* *Tables for Statisticians and Biometricians*, p. lvii and p. 65.

*Mean weighted square deviation of calculated from "observed" or product
moment values of r .*

	Σ^2	Σ^2 (omitting H)
Mean contingency r_ψ	·00138	·00102
Mean square contingency r_ϕ	·00089	·00060
Enneachoric r r_e	·00364	·00036
Polychoric r r_p	·00004	·00002
Tetrachoric r r_t	·00018	·00016
Mean tetrachoric r r_m	·00005	·00003
Mean weighted tetrachoric r r_w	·00002	·00002
Three row η from mean dispersion* ... η_m	·00020	·00019
Three row η from "individual" dispersion η_n	·00151	·00144
Marginal centroids r_c	·00215	·00255

I have given the value of Σ^2 including and omitting Table H, which gives very anomalous results, as yet unexplained. Broadly the best results are given by r_p , r_m and r_w , and, Table H aside, the best result is by r_p . In the case of r_ψ the results are not quite satisfactory. The figure given was arrived at by taking the mean of the raw figure from the curve and the same corrected for broad categories as suggested in *Tables for Statisticians and Biometricians*. An attempt was made to find an empirical formula which would give better results with the tables here described, but the result was not worthy of record. With three row η , although strictly the method is quite inapplicable to 3×3 tables, it may be useful to notice that when so applied the best results on the whole were got from assuming the distribution to be homoscedastic and using the mean dispersion of the arrays. This was largely due to several of the tables being divided so that some of the arrays contained very small frequencies which had therefore large probable errors, giving an undue effect on the result when squared. When such small frequencies are avoided the results appear to be about equally good. Of course our theory fails, as we have already pointed out, when any cell frequency is of the same order as its variation.

Comparing the probable errors of r_m , r_w and r_p (tabulated for convenience in the Appendix on page 133) it will be seen that on the whole they are in descending order of magnitude. They differ very little from each other and, considering the labour involved in finding r_p , r_m , would in most cases give a result with a sufficiently low probable error.

The method of marginal centroids as already known is unsuited for tables with so few categories.

An interesting and important relation which is not shown in the tables of numerical results (§ 13) is the degree of correlation between r_{11} , r_{12} , etc., viz. $R_{11 \cdot 12}$, $R_{11 \cdot 21}$, etc. These are collected in the table on p. 123.

All the enneachoric tables are arranged so that reading from n_{11} to the right, and downwards, r is positive so that the values R may be compared among each other. It will be seen on examination that $R_{11 \cdot 12}$, $R_{11 \cdot 21}$, $R_{12 \cdot 22}$, $R_{21 \cdot 22}$ are on the whole greater than $R_{11 \cdot 22}$ and $R_{12 \cdot 21}$ and of the two latter $R_{12 \cdot 21}$ is usually the greater.

* See § 13, 8.

	$R_{11 \cdot 12}$	$R_{11 \cdot 21}$	$R_{12 \cdot 22}$	$R_{21 \cdot 22}$	$R_{11 \cdot 22}$	$R_{12 \cdot 21}$
A	·3418	·3041	·3196	·3450	·0608	·1466
B	·6977	·6040	·6132	·7050	·4048	·5231
C	·4880	·4550	·4751	·4821	·1872	·2614
D	·5100	·5678	·6038	·4907	·2620	·3221
E	·2180	·4953	·5378	·1813	·0830	·1217
F	·4252	·4422	·4133	·4307	·1209	·2392
G	·6633	·6723	·6565	·6701	·3414	·4769
H	·4395	·6592	·6567	·4367	·2391	·3219
K	·3842	·3488	·3656	·3882	·1124	·1632
L	·5014	·4831	·4798	·5162	·2276	·2611
M	·5282	·2820	·2961	·5213	·0865	·2128
Pairs of brothers	·8203	·8203	·8732	·8732	·8837	·8756

With regard to the computation of r_p it will be seen from example appended to Table A that the amount of labour involved in dealing even with a 3×3 table is considerable and will rapidly increase with the number of cells, and it is very desirable that some short method of approximating to the weights (C 's) of the r 's be devised. For the present it may be of interest to give here the C 's for the various tables used.

	C_{11}	C_{12}	C_{21}	C_{22}
A	1	·51333	·38587	·71892
B	1	·26872	·19654	·55240
C	1	·27000	·26993	·34266
D	1	·65737	·15786	·85445
E	1	3·79397	·16186	3·81496
F	1	·56452	·43667	·72594
G	1	−·02973	−·13835	·69769
H	1	·72919	·33838	·79708
K	1	·59665	·48857	·66355
L	1	·39951	·34626	·27119
M	1	·36146	·38718	·74467
Pairs of brothers	1	−·26399	−·26399	·82793

The case of Table G, with negative weight for r_{12} and r_{21} is suggestive and needs further study. The table has the characteristic that the mean is in n_{11} and the marginal frequencies are decreasing in magnitude and nearly equal in both sets of categories. The table “Pairs of brothers” which is accompanied by similar weights is taken from *Biometrika*, vol. III, 1904, p. 182, and is given below. It compares the athletic capacities of pairs of brothers.

First Brother

Second Brother		Athletic	Betwixt	Non-athletic	Total
	Athletic	906	20	140	1066
	Betwixt	20	76	9	105
	Non-athletic	140	9	370	519
	Total	1066	105	519	1690

$$r_{11} = .8046 \pm .0126,$$

$$r_{12} = .7190 \pm .0162,$$

$$r_{21} = .7190 \pm .0162,$$

$$r_{22} = .8028 \pm .0132.$$

$$C_{11} = 1, \quad C_{12} = -.26399, \quad C_{21} = -.26399, \quad C_{22} = .82793.$$

$$r_p = .8382 \pm .0122.$$

These negative weights require further investigation, particularly the conditions for the existence of zero weights, but it is clear that certain divisions are to be avoided in determining r from a 3×3 table.

On the whole C_{11} , C_{22} , C_{12} , C_{21} are in descending order of magnitude.

§ 13. PRÉCIS OF THE METHODS OF FINDING THE COEFFICIENT OF CORRELATION.

1. r_ψ . Mean contingency, corrected for class index correlation.
2. r_ϕ . Mean square contingency, corrected for class index correlation and where necessary for the number of cells.
3. r_e . By selecting the central cell, the method first described in this paper. As its use treats any table as virtually 3×3 , it may be called enneachoric r .
4. r_p . By weighting the r 's so that the P.E. shall be a minimum, the second method described in this paper. As it is applicable to tables of any size it may be called polychoric r .
5. $r_{11}, r_{12}, r_{21}, r_{22}$. Tetrachoric r of the various quadrants. The probable errors were calculated by the complete formula (P.E.) and also by the approximate method (A.P.E.). (*Tables for Statisticians and Biometricians*, p. xl.)
6. r_m . The unweighted mean of $r_{11}, r_{12}, r_{21}, r_{22}$.
7. r_w . The mean of the r_{11} , etc., weighted by the reciprocals of the squares of their standard deviation.
8. $\eta_{k_1}, \eta_{k_2}, \eta_{h_1}, \eta_{h_2}$. Three row η calculated from each of the dividing planes as planes of reference with a class index correction on the foot of the columns. Since the standard deviation may be found in this case from the individual arrays or, assuming the distribution sufficiently homoscedastic, may be given the mean value $\sigma\sqrt{1-\eta^2}$, I have used both methods for the purpose of comparison. These are distinguished by the headings "individual dispersion" and "mean dispersion" respectively*.
9. r_c . By marginal centroids.

The probable error of r_m and r_w was obtained as follows:

Let the correlation coefficients	$r_{11}, r_{12}, \dots,$
have the S.D.'s	$\sigma_{11}, \sigma_{12}, \dots,$
and the weights	$t_{12}, t_{12}, \dots$

* The probable error of Biserial (or three row η) has now been given (*Biometrika*, Vol. ix, part iv), but too late for use in the present paper.

$$\text{Then } r_t = \frac{\Sigma t_{11} r_{11}}{\Sigma t_{11}}, \quad (dr_t)^2 = \frac{\Sigma t_{11}^2 (dr_{11})^2 + 2 \Sigma t_{11} t_{12} dr_{11} dr_{12}}{(\Sigma t_{11})^2},$$

$$\sigma_{r_t}^2 = \frac{\Sigma t_{11}^2 \sigma_{11}^2 + 2 \Sigma t_{11} t_{12} \sigma_{11} \sigma_{12} R_{11 \cdot 12}}{(\Sigma t_{11})^2} \dots \dots \dots (147).$$

When $t_{11} = t_{12} = \dots = 1$ we have the mean r , r_m , and if there are l r 's

$$\sigma_{r_m}^2 = \frac{\Sigma \sigma_{11}^2 + 2 \Sigma \sigma_{11} \sigma_{12} R_{11 \cdot 12}}{l^2} \dots \dots \dots (148),$$

which for convenient computation may be written

$$= \frac{\Sigma \frac{S_{11 \cdot 11}}{\chi_{11}^2} + 2 \Sigma \frac{S_{11 \cdot 12}}{\chi_{11} \chi_{12}}}{l^2}.$$

In finding the mean weighted r we may regard r_{11} as the mean of t_{11} uncorrelated values of r of equal weight each having the s.d. σ_0 . Hence $\sigma_{11} = \frac{\sigma_0}{\sqrt{t_{11}}}$ and $t_{11} = \frac{\sigma_0^2}{\sigma_{11}^2}$, i.e. the weights are proportional to the reciprocals of the squares of the s.d.'s. Putting this value in (147) we have

$$\sigma_{r_w}^2 = \frac{\Sigma \left(\frac{1}{\sigma_{11}^2} \right) + 2 \Sigma \frac{R_{11 \cdot 12}}{\sigma_{11} \sigma_{12}}}{\Sigma \left(\frac{1}{\sigma_{11}^2} \right)^2} \dots \dots \dots (149).$$

§ 14. DETAILS OF TABLES AND SUMMARY OF NUMERICAL RESULTS.

I. The first table examined was taken from Pearson and Heron's paper "On Theories of Association," *Biometrika*, vol. ix, p. 220, Table XIV, and is a Gaussian surface for $r = .5$ adjusted to give whole units in the cells.

	1	2	3	4	5 + 6	7	8	Total
1	7	20	5	2	—	—	—	34
2	21	145	79	36	10	9	1	301
3	6	94	85	54	19	22	4	284
4	2	32	39	31	12	17	4	137
5 + 6	—	18	28	25	11	18	5	105
7	—	11	22	24	12	22	7	98
8	—	2	6	8	5	13	7	41
Total	36	322	264	180	69	101	28	1000

The frequency in heavy type contains the mean of the surface.

A. Table I divided so that the mean falls in cell n_{22} .

	1 + 2	3 + 4	5 + 6 + 7 + 8	Total
1 + 2	193	122	20	335
3 + 4	134	209	78	421
5 + 6 + 7 + 8	31	113	100	244
Total	358	444	198	1000

$r_\psi = .482$		
$r_\phi = .4840 \pm .0170$		A.P.E.
$r_p = .50346 \pm .02094$	$\left. \begin{array}{l} r_{11} = .498 \pm .02872 \quad (.0290) \\ r_{12} = .510 \pm .03210 \quad (.0303) \\ r_{21} = .508 \pm .03505 \quad (.0321) \\ r_{22} = .504 \pm .03259 \quad (.0340) \end{array} \right\}$	
$r_e = .48594 \pm .04918$		
$r_m = .5050 \pm .0246$		
$r_w = .5045 \pm .0211$		
$r_c = .5145$		
	Mean dispersion	Individual dispersion
η_{k_1}	.5031	.4950
η_{k_2}	.5057	.4975
η_{h_1}	.5045	.4955
η_{h_2}	.5058	.4942

I here insert as an illustration of the new method the constants required in finding r_p for the above table, and the calculation of $S_{12 \cdot 21}$, P_{11} and Π_{11} , and the equations to find the C 's.

Table A	$h_1 k_1$	$h_1 k_2$	$h_2 k_1$	$h_2 k_2$
r	.498	.510	.508	.504
h	-.36381	-.36381	.84879	.84879
rh	-.18118	-.18554	.43117	.42779
Ek	.3733945	.3733945	.2782707	.2782707
k	-.42615	.69349	-.42615	.69349
rk	-.21222	.35368	-.21648	.34952
Ek	.3643145	.3136735	.3643145	.3136735
$h - rk$	-.15159	-.71749	1.06527	.49927
$\frac{h - rk}{\sqrt{1 - r^2}}$	-.1748086	-.8341218	1.236735	.5780570
$E \frac{h - rk}{\sqrt{1 - r^2}}$	.3928931	.2817266	.1856865	.3375594
$B = \frac{h - rk}{\sqrt{1 - r^2}}$	.4306151	.2021063	.8919072	.7183872
$k - rh$	-.24497	.87903	-.85732	.26570
$\frac{k - rh}{\sqrt{1 - r^2}}$	-.2824913	1.021920	-.9953132	.3076287
$E \frac{k - rh}{\sqrt{1 - r^2}}$	.3833377	.2366676	.2431048	.3805048
$A = \frac{k - rh}{\sqrt{1 - r^2}}$	.3887835	.8465906	.1597920	.6208174
χ	165.0602	102.7353	78.53720	122.5923
Π	-.1806014	.0486969	.0516992	.3392046
P	90.44050	128.8716	111.9422	383.0020
$S_{11}/\chi : \chi : *$	.001812696	.000692554	.0006728030	.0001333663
$S_{12}/\chi : \chi :$	—	.002265285	.0003626002	.0007349840
$S_{21}/\chi : \chi :$	—	—	.002700292	.0008662932
$S_{22}/\chi : \chi :$	—	—	—	.002335123
$C :$	1	.51333	.3859	.71892

* The suffix : indicates that appropriate suffix is to be taken from the column.

Calculation of $S_{12 \cdot 21}$.

Π_{12}	.0486969			B_{12}	.2021063	
Π_{21}	.0516992			A_{21}	.1597920	
n_{11}	193			n_{22}	209	6.749650
			.485895			
B_{12}	.2021063			A_{12}	.8465906	
Π_{21}	.0516992			A_{21}	.1597920	
n_{21}	122			n_{13}	31	4.193630
			1.274745			17.351825
B_{12}	.2021063					
B_{21}	.8919072			P_{12}	128.8716	
n_{21}	20			P_{21}	111.9422	
			3.605201	$\div N$	1000	14.426170
						2.925655
Π_{12}	.0486969					
A_{21}	.1597920			χ_{12}	102.7353	
n_{12}	134			χ_{21}	78.5370	$\div 8068.543$
			1.042704			= .0003626002

Calculation of P_{11} .

A_{11}	.3887835	
m_{11}	358	139.1845
B_{11}	.4306151	
m_{11}	335	144.2560
		283.4405
		193
$P_{11} =$		90.4405

Calculation of Π_{11} .

A_{11}	.3887835
B_{11}	.4306151
	.8193986
	1
$\Pi_{11} = -$	.1806014

Equations to find C ,

$\cdot 001812696C_{11} + \cdot 000692554C_{12} + \cdot 000672803C_{21} + \cdot 000133363C_{22} = \lambda,$
 $\cdot 000692554C_{11} + \cdot 002265285C_{12} + \cdot 000362600C_{21} + \cdot 000734984C_{22} = \lambda,$
 $\cdot 000672803C_{11} + \cdot 000362600C_{12} + \cdot 002700292C_{21} + \cdot 000866293C_{22} = \lambda,$
 $\cdot 000133366C_{11} + \cdot 000734984C_{12} + \cdot 000866293C_{21} + \cdot 002335123C_{22} = \lambda.$

The solution of these equations gives the first row of figures in the C table on page 123.

B. Table I divided so that the mean falls in cell n_{11} .

	1 + 2 + 3	4	5 + 6 + 7 + 8	Total
1 + 2 + 3	462	92	65	619
4	73	31	33	137
5 + 6 + 7 + 8	87	57	100	244
Total	622	180	198	1000

$r_\psi = .533$
 $r_\phi = .510 \pm .0161$
 $r_p = .5007 \pm .0250$
 $r_e = .5073 \pm .1437$
 $r_m = .5010 \pm .0254$
 $r_w = .5008 \pm .0253$
 $r_c = .5445$

$\left. \begin{array}{l} r_{11} = .499 \pm .028 \quad (.028) \\ r_{12} = .501 \pm .030 \quad (.030) \\ r_{21} = .500 \pm .031 \quad (.032) \\ r_{22} = .504 \pm .033 \quad (.034) \end{array} \right\}$

A.P.E.

	Mean dispersion	Individual dispersion
η_{k_1}	.4921	.4873
η_{k_2}	.4917	.4763
η_{h_1}	.4858	.4802
η_{h_2}	.4881	.4671

C. Table I divided so that the mean falls in cell n_{11} .

	1 + 2 + 3	4 + 5 + 6	7 + 8	Total
1 + 2 + 3	462	121	36	619
4 + 5 + 6	119	79	44	242
7 + 8	41	49	49	139
Total	622	249	129	1000

$r_\psi = .537$
 $r_\phi = .5183 \pm .0202$
 $r_p = .4988 \pm .0241$
 $r_e = .5055 \pm .0652$
 $r_m = .4985 \pm .0253$
 $r_w = .4973 \pm .0311$
 $r_c = .5480$

$\left. \begin{array}{l} r_{11} = .499 \pm .028 \quad (.028) \\ r_{12} = .490 \pm .035 \quad (.035) \\ r_{21} = .505 \pm .035 \quad (.035) \\ r_{22} = .500 \pm .039 \quad (.043) \end{array} \right\}$

A.P.E.

	Mean dispersion	Individual dispersion
η_{k_1}	.4895	.5041
η_{k_2}	.4934	.4615
η_{h_1}	.4846	.4730
η_{h_2}	.4947	.4496

D. Table I divided so that the mean falls in cell n_{12} .

	1 + 2 + 3	4	5 + 6 + 7 + 8	Total
1 + 2	277	38	20	335
3	185	54	45	284
4 + 5 + 6 + 7 + 8	160	88	133	381
Total	622	180	198	1000

$$\begin{aligned}
 r_\psi &= .524 \\
 r_\phi &= .482 \pm .0631 && \text{A.P.E.} \\
 r_p &= .5018 \pm .0235 \\
 r_e &= .5173 \pm .2330 \\
 r_m &= .5030 \pm .0239 \\
 r_w &= .5025 \pm .0236 \\
 r_c &= .5098
 \end{aligned}
 \left\{
 \begin{aligned}
 r_{11} &= .501 \pm .030 & (.030) \\
 r_{12} &= .499 \pm .028 & (.029) \\
 r_{21} &= .508 \pm .035 & (.032) \\
 r_{22} &= .504 \pm .031 & (.032)
 \end{aligned}
 \right.$$

	Mean dispersion	Individual dispersion
η_{k_1}	.5211	.4798
η_{k_2}	.4885	.4907
η_{h_1}	.4956	.4709
η_{h_2}	.5115	.4553

E. Table I divided so that the frequency of n_{22} differs very little from the frequency of a table with the same marginal frequencies but of zero correlation. The mean is in cell n_{23} .

	1 + 2	3	4 + 5 + 6 + 7 + 8	Total
1	27	5	2	34
2	166	79	56	301
3 + 4 + 5 + 6 + 7 + 8	165	180	320	665
Total	358	264	378	1000

Here $\frac{301 \times 264}{1000} = 79.464$ so that the constant term in the equation for r is a small quantity and any error of sampling will have an excessive weight. It will be found as one might expect that the P.E. of r_e is very large.

The very large value of η_{h_2} is due to the column having the marginal total 378, for the frequency 2 in it is the nearest whole number to a true value and being so small, a small absolute difference makes a large fractional value resulting in a large difference between the true and apparent standard deviations of this particular array. Actually the method applied is inapplicable to a frequency of this order.

$$\begin{aligned}
 r_\psi &= .502 \\
 r_\phi &= .4827 \pm .0246 && \text{A.P.E.} \\
 r_p &= .4991 \pm .0245 \\
 r_e &= .4658 \pm .4371 \\
 r_m &= .4995 \pm .0327 \\
 r_w &= .4995 \pm .0247 \\
 r_c &= .5065
 \end{aligned}
 \left\{
 \begin{aligned}
 r_{11} &= .500 \pm .056 & (.054) \\
 r_{12} &= .498 \pm .029 & (.029) \\
 r_{21} &= .500 \pm .072 & (.054) \\
 r_{22} &= .500 \pm .030 & (.029)
 \end{aligned}
 \right.$$

	Mean dispersion	Individual dispersion
η_{k_1}	.4862	.4906
η_{k_2}	.5169	.4991
η_{h_1}	.4950	.5022
η_{h_2}	.4966	.7150

II. The second table examined was taken from Macdonell's paper "On Criminal Anthropometry," *Biometrika*, vol. I, p. 216. The original table is too extensive to be given here, but may be found in *loc. cit.* The horizontal categories are the heights of 3000 criminals in feet and inches, and the vertical categories the lengths of their left middle fingers in millimetres. The correlation coefficient found by the product moment method is $\cdot6608 \pm \cdot0069$.

F. Table II divided so that the mean falls in cell n_{22} .

	$55\frac{9}{16}''-64\frac{9}{16}''$	$64\frac{9}{16}''-66\frac{9}{16}''$	$66\frac{9}{16}''-77''$	Total
9.4-11.3 mm.	682	270	101	1053
11.4-11.7 mm.	282	351	286	919
11.8-13.5 mm.	90	299	639	1028
Total	1054	920	1026	3000

$$r_{\psi} = \cdot6635$$

$$r_{\phi} = \cdot6170 \pm \cdot0075$$

A.P.E.

$$r_p = \cdot6544 \pm \cdot0101$$

$$r_e = \cdot6316 \pm \cdot0301$$

$$r_m = \cdot6530 \pm \cdot0101$$

$$r_w = \cdot6538 \pm \cdot0101$$

$$r_c = \cdot6911$$

$$r_{11} = \cdot667 \pm \cdot013 \quad (\cdot014)$$

$$r_{12} = \cdot670 \pm \cdot014 \quad (\cdot013)$$

$$r_{21} = \cdot644 \pm \cdot015 \quad (\cdot014)$$

$$r_{22} = \cdot631 \pm \cdot014 \quad (\cdot014)$$

Mean
dispersion

Individual
dispersion

 η_{k_1} $\cdot6477$ $\cdot6295$ η_{k_2} $\cdot6548$ $\cdot6306$ η_{h_1} $\cdot6647$ $\cdot6151$ η_{h_2} $\cdot6345$ $\cdot6510$

G. Table II divided so that the mean falls in cell n_{11} .

	$55\frac{9}{16}''-65\frac{9}{16}''$	$65\frac{9}{16}''-66\frac{9}{16}''$	$66\frac{9}{16}''-77''$	Total
9.4-11.5 mm.	1122	176	216	1514
11.6-11.7 mm.	191	96	171	458
11.8-13.5 mm.	203	186	639	1028
Total	1516	458	1026	3000

$$r_{\psi} = \cdot731$$

$$r_{\phi} = \cdot6426 \pm \cdot0077$$

A.P.E.

$$r_p = \cdot6613 \pm \cdot0108$$

$$r_e = \cdot6808 \pm \cdot0390$$

$$r_m = \cdot6553 \pm \cdot0111$$

$$r_w = \cdot6573 \pm \cdot0112$$

$$r_c = \cdot7182$$

$$r_{11} = \cdot680 \pm \cdot012 \quad (\cdot013)$$

$$r_{12} = \cdot668 \pm \cdot013 \quad (\cdot013)$$

$$r_{21} = \cdot642 \pm \cdot014 \quad (\cdot014)$$

$$r_{22} = \cdot631 \pm \cdot014 \quad (\cdot014)$$

	Mean dispersion	Individual dispersion
η_{k_1}	·6553	·6421
η_{k_2}	·6473	·6546
η_{h_1}	·6657	·6669
η_{h_2}	·6277	·7113

H. Table II divided so that the mean falls in cell n_{12} .

	$55\frac{9}{16}''-65\frac{9}{16}''$	$65\frac{9}{16}''-66\frac{9}{16}''$	$66\frac{9}{16}''-77''$	Total
9·4-11·3 mm.	840	112	101	1053
11·4-11·7 mm.	473	160	286	919
11·8-13·5 mm.	203	186	639	1028
Total	1516	458	1026	3000

$r_\psi = \cdot669$
 $r_\phi = \cdot6172 \pm \cdot0088$
 $r_p = \cdot6479 \pm \cdot0107$
 $r_e = \cdot5162 \pm \cdot0438$
 $r_m = \cdot6478 \pm \cdot0108$
 $r_w = \cdot6591 \pm \cdot0107$
 $r_c = \cdot7920$

$r_{11} = \cdot648 \pm \cdot014$
 $r_{12} = \cdot668 \pm \cdot013$
 $r_{21} = \cdot644 \pm \cdot015$
 $r_{22} = \cdot631 \pm \cdot014$

A.P.E.
(·014)
(·013)
(·014)
(·014)

	Mean dispersion	Individual dispersion
η_{k_1}	·6539	·6279
η_{k_2}	·6472	·6156
η_{h_1}	·6479	·6103
η_{h_2}	·6345	·6418

III. The third table examined was taken from Pearson and Heron's paper "On Theories of Association," *Biometrika*, vol. ix, p. 219, Table XIII, and is a normal surface having $r = \cdot3$. The values of the frequencies to two places of decimals were used.

	1	2	3	4	5 + 6	7	8	Total
1	4·04	17·16	7·55	3·30	0·91	0·92	0·12	34
2	17·41	123·59	79·76	46·64	14·61	17·67	3·32	301
3	8·86	93·00	78·31	52·04	19·20	26·40	6·19	284
4	2·83	37·73	37·24	27·51	10·95	16·31	4·43	137
5 + 6	1·62	25·21	27·75	22·09	9·26	14·64	4·43	105
7	1·02	19·50	24·47	21·39	9·58	16·36	5·68	98
8	0·22	5·81	8·92	9·03	4·49	8·70	3·83	41
Total	36	322	264	180	69	101	28	1000

K. Table III divided so that the mean falls in cell n_{22} .

	1 + 2	3 + 4	5 + 6 + 7 + 8	Total
1 + 2	162.20	135.25	37.55	335
3 + 4	142.42	195.10	83.48	421
5 + 6 + 7 + 8	53.38	113.65	76.97	244
Total	358	444	198	1000

$$r_{\psi} = .3088$$

$$r_{\phi} = .2960 \pm .0199 \quad \text{A.P.E.}$$

$$r_p = .3000 \pm .0246 \quad \left. \begin{array}{l} r_{11} = .300 \pm .033 \quad (.034) \\ r_{12} = .300 \pm .036 \quad (.035) \\ r_{21} = .300 \pm .038 \quad (.037) \\ r_{22} = .300 \pm .038 \quad (.039) \end{array} \right\}$$

$$r_e = .3000 \pm .0991$$

$$r_m = .3000 \pm .0249$$

$$r_w = .3000 \pm .0247$$

$$r_c = .3025$$

	Mean dispersion	Individual dispersion		Mean dispersion	Individual dispersion
η_{k_1}	.3011	.2988	η_{h_1}	.2998	.2997
η_{k_2}	.2999	.3003	η_{h_2}	.2989	.2989

L. Table III divided so that the mean falls in cell n_{11} .

	1 + 2 + 3	4 + 5 + 6	7 + 8	Total
1 + 2 + 3	429.68	134.70	54.62	619
4 + 5 + 6	132.38	69.81	39.81	242
7 + 8	59.94	44.49	34.57	139
Total	622	249	129	1000

$$r_{\psi} = .330$$

$$r_{\phi} = .3095 \pm .0225 \quad \text{A.P.E.}$$

$$r_p = .3000 \pm .0282 \quad \left. \begin{array}{l} r_{11} = .300 \pm .032 \quad (.033) \\ r_{12} = .300 \pm .039 \quad (.040) \\ r_{21} = .300 \pm .040 \quad (.041) \\ r_{22} = .300 \pm .046 \quad (.050) \end{array} \right\}$$

$$r_e = .3000 \pm .1031$$

$$r_m = .3000 \pm .0326$$

$$r_w = .3000 \pm .0285$$

$$r_c = .3152$$

	Mean dispersion	Individual dispersion
η_{k_1}	.2965	.2975
η_{k_2}	.2910	.2898
η_{h_1}	.2953	.2975
η_{h_2}	.2909	.2871

IV. The fourth table examined was from Pearson and Lee's paper "On the Distribution of Frequency (Variation and Correlation) of Barometric Heights at Divers Stations," *Phil. Trans. A*, 1897, vol. 190, p. 453, Table IX. The original

table is too extensive for reproduction and may be found in *loc. cit.* A condensed form of it will be found in *Biometrika*, vol. ix, 1913, p. 223, Table XVIII.

This was selected as an example of a very skew distribution. The correlation coefficient found by Product Moments is $\cdot780$ (*Biometrika*, vol. ix, p. 223).

M. Table IV divided so as to give a reasonably large frequency in the cell n_{22} . The mean falls in the cell n_{21} .

	30.1'' and over	30''-29.8''	29.7'' and under	Total
29.9'' and over	1086.5	412	43	1541.5
29.8''-29.7''	144.5	275	103	522.5
29.6'' and under	56.5	323	478.5	858.0
Total	1287.5	1010	624.5	2922

$$r_{\psi} = \cdot789$$

$$r_{\phi} = \cdot7504 \pm \cdot0210 \quad \text{A.P.E.}$$

$$r_p = \cdot7864 \pm \cdot0077 \quad \left. \begin{array}{l} r_{11} = \cdot780 \pm \cdot010 \quad (\cdot010) \\ r_{12} = \cdot787 \pm \cdot011 \quad (\cdot011) \\ r_{21} = \cdot795 \pm \cdot012 \quad (\cdot011) \\ r_{22} = \cdot785 \pm \cdot011 \quad (\cdot012) \end{array} \right\}$$

$$r_e = \cdot7745 \pm \cdot0151$$

$$r_m = \cdot7877 \pm \cdot0078$$

$$r_w = \cdot7858 \pm \cdot0077$$

$$r_c = \cdot8770$$

	Mean dispersion	Individual dispersion		Mean dispersion	Individual dispersion
η_{k_1}	$\cdot7857$	$\cdot7116$	η_{h_1}	$\cdot7962$	$\cdot7417$
η_{k_2}	$\cdot7812$	$\cdot6951$	η_{h_2}	$\cdot8065$	$\cdot6841$

Appendix.

Probable errors of r_m , r_w , r_p .

	P.E. of arithmetic mean (r_m)	P.E. of weighted mean (r_w)	P.E. of polychoric r (r_p)
A	$\cdot0246$	$\cdot0211$	$\cdot0209$
B	$\cdot0254$	$\cdot0253$	$\cdot0250$
C	$\cdot0253$	$\cdot0311$	$\cdot0241$
D	$\cdot0239$	$\cdot0236$	$\cdot0235$
E	$\cdot0327$	$\cdot0247$	$\cdot0245$
F	$\cdot0101$	$\cdot0101$	$\cdot0101$
G	$\cdot0111$	$\cdot0112$	$\cdot0108$
H	$\cdot0108$	$\cdot0107$	$\cdot0107$
K	$\cdot0249$	$\cdot0247$	$\cdot0246$
L	$\cdot0326$	$\cdot0285$	$\cdot0282$
M	$\cdot0078$	$\cdot0077$	$\cdot0077$

My thanks are due to Professor Pearson, who suggested the enquiry, for his ever ready help and advice throughout the work. I have also to thank Miss Alison Robertson for assistance in reading the proofs.

ON A FORMULA FOR THE PRODUCT-MOMENT COEFFICIENT OF ANY ORDER OF A NORMAL FREQUENCY DISTRIBUTION IN ANY NUMBER OF VARIABLES.

By L. ISSERLIS, D.Sc.

1. In *Biometrika*, Vol. XI, Part III, I have shown that for a normal frequency distribution in four variables, if

$$p_{xyzt} = \frac{SSSS \{n_{xyzt} xyz\}}{N}$$

denotes the product-moment coefficient of the distribution about the means of the four variables and q_{xyzt} is the *reduced* moment, i.e.

$$q_{xyzt} = p_{xyzt} / \sigma_x \sigma_y \sigma_z \sigma_t,$$

then

$$q_{xyzt} = r_{xy}r_{zt} + r_{yz}r_{xt} + r_{zx}r_{yt} \dots\dots\dots(1).$$

In this result any two or more variables may be made identical leading to a variety of results for moment coefficients of distributions containing fewer than four variables but of total order four, for example identifying t with x we obtain

$$q_{x^2yz} = r_{yz} + 2r_{xy}r_{xz} \dots\dots\dots(2),$$

and putting $y = z = t = x$ we find $q_{x^4} = 3$; of course $q_{xy} = r_{xy}$ and q_{x^2} is merely β_2 .

I suggested that (1) was probably capable of generalisation, and I now propose to prove a general theorem which gives immediately the value of the mixed moment coefficient of any order in each variable for a normal frequency distribution in any number of variables.

2. Consider a normal distribution, total population N . Let $N_{12\dots n}$ denote the frequency of the group in which the characters differ by $x_1, x_2, \dots x_n$ from the mean values for the whole population and let

$$p_{1^{l_1} 2^{l_2} \dots n^{l_n}} = S (N_{12\dots n} x_1^{l_1} x_2^{l_2} \dots x_n^{l_n}) / N \dots\dots\dots(3),$$

denote the moment coefficient of the most general kind about the mean values of the characters. The corresponding reduced moment will be

$$q_{1^{l_1} 2^{l_2} \dots n^{l_n}} = p_{1^{l_1} 2^{l_2} \dots n^{l_n}} / \sigma_1^{l_1} \sigma_2^{l_2} \dots \sigma_n^{l_n} \dots\dots\dots(4).$$

Then for *normal distributions*,

$$\text{if } n \text{ be odd, } q_{12\dots n} = 0 \dots\dots\dots(5),$$

$$\text{and if } n \text{ be even, } q_{12\dots n} = S (r_{ab}r_{cd} \dots r_{hk}) \dots\dots\dots(6),$$

where the summation on the right-hand side extends to every possible selection of $n/2$ pairs $ab, cd, \dots hk$, that can be formed out of the n suffixes 1, 2, 3, $\dots n$; equation (1) is thus a particular case of (6).

Equation (6) is the theorem it is proposed to prove. The value of $q_{1^{l_1} 2^{l_2} \dots n^{l_n}}$ is at once found for given numerical values of the indices $l_1, l_2, \dots l_n$ by writing down (5) for $l_1 + l_2 + \dots + l_n$ variables and identifying the values of l_1 of them with that of the first and so on.

For example if we require the value of $q_{1^2 2^2 3^2}$ we commence with,

$$\begin{aligned} q_{123456} &= S(r_{ab}r_{cd}r_{ef}) \\ &= r_{12}(r_{34}r_{56} + r_{35}r_{46} + r_{36}r_{45}) + r_{13}(r_{24}r_{56} + r_{25}r_{46} + r_{26}r_{45}) \\ &\quad + r_{14}(r_{23}r_{56} + r_{25}r_{36} + r_{26}r_{35}) + r_{15}(r_{23}r_{46} + r_{24}r_{36} + r_{26}r_{34}) \\ &\quad + r_{16}(r_{23}r_{45} + r_{24}r_{35} + r_{25}r_{34}) \dots\dots\dots(7). \end{aligned}$$

Identifying 4 with 1, 5 with 2 and 6 with 3 we find at once

$$q_{1^2 2^2 3^2} = 1 + 2r_{12}^2 + 2r_{23}^2 + 2r_{31}^2 + 8r_{12}r_{23}r_{31} \dots\dots\dots(8).$$

3. We note first that q_1^n which in the more usual notation for distributions in one variable is $\mu_n/\mu_2^{n/2}$ is known to have the value $1.3.5 \dots (n-1)$ when n is even. As regards $S(r_{ab}r_{cd} \dots r_{hk})$, if all the n variables are made identical, each term becomes unity and the number of terms is the same as the number of ways of breaking up an even number (n) of objects into $(n/2)$ pairs. This last number is clearly

$$\frac{n!}{2!} \frac{n-2!}{n-2!} \frac{n-4!}{n-4!} \dots \frac{4!}{2!} \frac{2!}{2!} / (n/2)!$$

which also reduces to $1.3.5 \dots (n-1)$; thus equation (6) is correct for this particular case.

Secondly let us consider the value of $q_{1^{n-1}2}$. The mean value of x_2 for a given value of x_1 is $r_{12}\sigma_2 x_1/\sigma_1$, let

$$x_2 = r_{12} \frac{\sigma_2}{\sigma_1} x_1 + X_2.$$

Then the distribution of X_2 for a given value of x_1 is itself normal and its k th moment is zero for an odd k and

$$1.3.5 \dots (k-1) (\sigma_2)^{k/2}$$

for an even k where σ_2 is the standard deviation of 2 within the x_1 array so that $\sigma_2^2 = (1 - r_{12}^2) \sigma_2^2$.

$$\begin{aligned} q_1^{n-1} &= \frac{1}{\sigma_1^{n-1} \sigma_2} \text{Mean value } (x_1^{n-1} x_2) \\ &= \frac{1}{\sigma_1^{n-1} \sigma_2} \text{Mean} \left\{ x_1^{n-1} \text{Mean} \left(r_{12} \frac{\sigma_2}{\sigma_1} x_1 + X_2 \right) \right\} \\ &= r_{12} q_1^n = 1.3.5 \dots (n-1) r_{12} \dots\dots\dots(9). \end{aligned}$$

The method employed in the original proof of equation (1) is not convenient for generalisation and we will now prove the equation

$$q_{1234} = r_{12}r_{34} + r_{13}r_{24} + r_{14}r_{23}$$

by the method that leads to the general case.

Putting as above

$$x_2 = r_{12} \frac{\sigma_2}{\sigma_1} x_1 + X_2,$$

$$x_3 = r_{13} \frac{\sigma_3}{\sigma_1} x_1 + X_3,$$

$$x_4 = r_{14} \frac{\sigma_4}{\sigma_1} x_1 + X_4,$$

we have

$$\begin{aligned} p_{1234} &= \text{Mean of } (x_1 x_2 x_3 x_4) \\ &= \text{Mean of } \{x_1 (\text{Mean of } x_2 x_3 x_4 \text{ for a given value of } x_1)\} \\ &= \text{Mean of } \left[x_1 \left\{ \text{Mean of } \left(r_{12} \frac{\sigma_2}{\sigma_1} x_1 + X_2 \right) \left(r_{13} \frac{\sigma_3}{\sigma_1} x_1 + X_3 \right) \left(r_{14} \frac{\sigma_4}{\sigma_1} x_1 + X_4 \right) \right\} \right]. \end{aligned}$$

Now for normal distributions (and if the original distribution is normal, so is that within the x_1 array), Mean $X_2 = 0$, Mean $X_2 X_3 X_4 = 0$, while

$$\begin{aligned} \text{Mean } X_2 X_3 &= ({}_1\sigma_2) ({}_1\sigma_3) {}_1r_{23} \\ &= \sqrt{1 - r_{12}^2} \sigma_2 \sqrt{1 - r_{13}^2} \frac{\sigma_3 (r_{23} - r_{12} r_{13})}{\sqrt{1 - r_{12}^2} \sqrt{1 - r_{13}^2}} \\ &= (r_{23} - r_{12} r_{13}) \sigma_2 \sigma_3 \dots\dots\dots(10). \end{aligned}$$

Hence

$$\begin{aligned} p_{1234} &= \text{Mean of } \left[x_1 \left\{ r_{12} r_{13} r_{14} \sigma_2 \sigma_3 \sigma_4 \frac{x_1^3}{\sigma_1^3} + r_{12} \sigma_2 \frac{x_1}{\sigma_1} (r_{34} - r_{13} r_{14}) \sigma_3 \sigma_4 \right. \right. \\ &\quad \left. \left. + r_{13} \sigma_3 \frac{x_1}{\sigma_1} (r_{24} - r_{12} r_{14}) \sigma_2 \sigma_4 \right. \right. \\ &\quad \left. \left. + r_{14} \sigma_4 \frac{x_1}{\sigma_1} (r_{23} - r_{12} r_{13}) \sigma_2 \sigma_3 \right\} \right], \end{aligned}$$

or dividing by $\sigma_1 \sigma_2 \sigma_3 \sigma_4$,

$$\begin{aligned} q_{1234} &= r_{12} r_{13} r_{14} q_{1^3} + q_{1^2} \{r_{12} (r_{34} - r_{13} r_{14}) + r_{13} (r_{24} - r_{12} r_{14})\} + r_{14} (r_{23} - r_{12} r_{13}) \\ &= r_{12} r_{34} + r_{23} r_{14} + r_{14} r_{23}, \end{aligned}$$

since $q_{1^2} = 1$ and $q_{1^3} = 3$. Thus our formula is established for the case of four variables.

4. We will establish the case for n variables by induction, and it will be convenient to denote by ${}_1q_{234\dots n}$ the value of the reduced product-moment coefficient for the variables 2, 3, 4, ... n within the x_1 array so that

$${}_1q_{234\dots n} = \frac{\text{Mean value of } (X_2 X_3 \dots X_n)}{({}_1\sigma_2) ({}_1\sigma_3) \dots ({}_1\sigma_n)},$$

where $X_2, X_3, \dots, X_n$ denote as before the deviations of the variables from their means within the x_1 array. Of course when n is even,

${}_1q_{234\dots n}$ is zero since $n - 1$ is now odd.

Let n be even and assume that our formula has been proved true for all even values of n up to $n - 2$ inclusive, then

$$\begin{aligned} p_{123\dots n} &= \text{Mean } (x_1 x_2 x_3 \dots x_n) \\ &= \text{Mean } \left\{ x_1 \left(r_{12} \frac{x_1}{\sigma_1} + X_2 \right) \left(r_{13} \frac{x_1}{\sigma_1} + X_3 \right) \dots \left(r_{1n} \frac{x_1}{\sigma_1} + X_n \right) \right\} \\ &= r_{12} r_{13} \dots r_{1n} \sigma_2 \sigma_3 \dots \sigma_n \text{Mean } (x_1^n) / \sigma_1^{n-1} \\ &\quad + S \{ (r_{1a} r_{1b} r_{1c} \dots) (\sigma_a \sigma_b \sigma_c \dots) \text{Mean } (X_a X_b) \} \text{Mean } (x_1^{n-2}) / \sigma_1^{n-3} \\ &\quad + S \{ (r_{1a} r_{1b} r_{1c} \dots) (\sigma_a \sigma_b \sigma_c \dots) \text{Mean } (X_a X_b X_\gamma X_\delta) \} \text{Mean } (x_1^{n-4}) / \sigma_1^{n-5} \\ &\quad + \dots \\ &\quad + S \{ r_{1a} \sigma_a \text{Mean } (X_a X_\beta \dots X_\gamma) \} \text{Mean } (x_1^2) / \sigma_1 \dots\dots\dots(11), \end{aligned}$$

the summations in each line extending to all possible permutations of the suffixes 2, 3, 4, ... n . The last line for example being

$$\frac{\text{Mean}(x_1^2)}{\sigma_1} \{r_{12}\sigma_2 \text{Mean}(X_3X_4 \dots X_n) + r_{13}\sigma_3 \text{Mean}(X_2X_4X_5 \dots X_n) + \dots + r_{1n}\sigma_n \text{Mean}(X_2X_3 \dots X_{n-1})\}.$$

Now we have seen that $\text{Mean}(X_2X_3) = (r_{23} - r_{12}r_{13})\sigma_2\sigma_3$. Similarly,

$$\begin{aligned} \text{Mean}(X_2X_3X_4X_5) &= ({}_1\sigma_2)({}_1\sigma_3)({}_1\sigma_4)({}_1\sigma_5)({}_1q_{2345}) \\ &= ({}_1\sigma_2)({}_1\sigma_3)({}_1\sigma_4)({}_1\sigma_5)[({}_1r_{23})({}_1r_{45}) + ({}_1r_{35})({}_1r_{24}) + ({}_1r_{25})({}_1r_{34})] \\ &= (r_{23} - r_{12}r_{13})(r_{45} - r_{14}r_{15}) + (r_{35} - r_{13}r_{15})(r_{24} - r_{12}r_{14}) \\ &\quad + (r_{25} - r_{12}r_{15})(r_{34} - r_{13}r_{14}), \end{aligned}$$

and our assumption of the truth of equation (6) up to $(n-2)$ variables will enable us to write down the mean value of every product of X 's occurring in (11).

Dividing by $\sigma_1\sigma_2 \dots \sigma_n$ we have, remembering that $\text{Mean } x_1^n/\sigma_1^n$ is $1.3.5 \dots (n-1)$

$$\begin{aligned} q_{123 \dots n} &= (r_{12}r_{13} \dots r_{1n}) 1.3.5 \dots (n-1) \\ &\quad + S \{r_{1a}r_{1b}r_{1c} \dots (r_{a\beta} - r_{1a}r_{1\beta})\} 1.3.5 \dots (n-3) \\ &\quad + S \{r_{1a}r_{1b}r_{1c} \dots S'[(r_{a\beta} - r_{1a}r_{1\beta})(r_{\gamma\delta} - r_{1\gamma}r_{1\delta})]\} 1.3.5 \dots (n-5) \\ &\quad + \dots \\ &\quad + S \{r_{1a}S'[(r_{a\beta} - r_{1a}r_{1\beta})(r_{\gamma\delta} - r_{1\gamma}r_{1\delta})(r_{\epsilon\rho} - r_{1\epsilon}r_{1\rho}) \dots]\} 1 \dots (12), \end{aligned}$$

where S' refers to permutations of $a\beta\gamma \dots$ only, and S to permutations of all the suffixes $a, b, c, \dots a, \beta, \gamma \dots$, i.e. all the suffixes 2, 3, 4, ... n .

It is clear that when the right-hand member of (12) is completely expanded no terms can survive which contain as a factor more than one correlation coefficient with suffix unity. This is easily verified in simple cases, and if in the general case a term $r_{1a}^\lambda r_{bc}r_{de} \dots$ survived, this term would reduce to r_{1a}^λ when we identified the characters $a, 2, 3, \dots n$, which contradicts the value $1.3.5 \dots (n-1)r_{1a}$ we have already found for it (equation (9)).

The value of the right-hand member is therefore easily found by neglecting all terms containing more than one such factor.

Hence on the assumption that (5) is true for all values of n up to $(n-2)$ we find

$$q_{123 \dots n} = S \{r_{1a}S'(r_{a\beta}r_{\gamma\delta}r_{\epsilon\rho} \dots)\},$$

but this is exactly the formula we wished to establish for it is obvious that $S(r_{ab}r_{cd} \dots r_{hk})$ where $abc \dots k$ is a permutation of $12 \dots n$ is equivalent to

$$S \{r_{1a}S'(r_{a\beta}r_{\gamma\delta} \dots)\}$$

where $a, \alpha, \beta, \gamma \dots$ is a permutation of $2, 3, 4, \dots n$. Thus our formula which has been proved true for 4 variables is seen by induction to be true in general.

5. Formula (6) can be exhibited as a multiple definite integral: Let Δ denote the determinant whose k th row consists of the elements

$$(r_{1k}, r_{2k}, \dots r_{k-1,k}, \dots r_{k+1,k}, \dots r_{nk})$$

and let Δ_{hk} denote the cofactor of the element in the h th row and k th column.

Let

$$\chi^2 = \Sigma \left(\frac{\Delta_{hh}}{\Delta} \frac{x_h^2}{\sigma_h^2} + 2\Sigma \Delta_{hk} \frac{x_h x_k}{\sigma_h \sigma_k} \right),$$

and

$$z = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma_1 \sigma_2 \dots \sigma_n \sqrt{\Delta}} e^{-\frac{1}{2}\chi^2},$$

then

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_1 x_2 \dots x_n z dx_1 dx_2 \dots dx_n = S(r_{ab} r_{cd} \dots r_{uv}) \dots (13),$$

where $a, b, c, d, \dots u, v$ are the suffixes 1, 2, 3, ... n in any possible order.

It is clear that (13) will enable us to write down the value of the multiple integral $\int_n P e^{-Q} dx_1 \dots dx_n$ where P is any polynomial in $x_1, x_2, \dots x_n$ on Q a positive quadratic form.

In fact, let $\Sigma a_{pp} x_p^2 + 2\Sigma a_{pq} x_p x_q$, ($a_{pq} = a_{qp}$) be a positive, definite, quadratic form, then

$$\begin{aligned} W &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \exp - \frac{1}{2} (\Sigma a_{pp} x_p^2 + 2\Sigma a_{pq} x_p x_q) dx_1 dx_2 \dots dx_n \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} \sigma_1 \sigma_2 \dots \sigma_n \sqrt{\Delta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}}{\sigma_1^{a_1} \sigma_2^{a_2} \dots \sigma_n^{a_n}} \\ &\quad \exp - \frac{1}{2\Delta} \left(\Sigma \Delta_{pp} \frac{x_p^2}{\sigma_p^2} + 2\Sigma \Delta_{pq} \frac{x_p x_q}{\sigma_p \sigma_q} \right) dx_1 dx_2 \dots dx_n \end{aligned}$$

$= \Sigma [r_{ab} r_{cd} \dots r_{hk}]$ where $abc \dots hk$ is any permutation of the $\alpha_1 + \alpha_2 + \dots + \alpha_n$ suffixes of which α_1 are equal to 1, α_2 are equal to 2 and so on.

Let D denote the determinant of the quadratic form and D_{pq} the cofactor of a_{pq} the two multiple integrals will be identical if

$$\begin{aligned} 1 &= \sigma_1^2 \sigma_2^2 \dots \sigma_{p-1}^2 \sigma_{p+1}^2 \dots \sigma_n^2 \Delta D_{pp}, \\ r_{pq} &= \sigma_1^2 \sigma_2^2 \dots \sigma_p \sigma_q \dots \sigma_n^2 \Delta D_{pq}. \end{aligned}$$

Hence $r_{pq}^2 = [D_{pq}]^2 / D_{pp} D_{qq}$ and $\sigma_p^2 = D_{pp} / D$ while $\Delta = D^{n-1} / D_{11} D_{22} \dots D_{nn}$,

so that

$$W = \frac{(2\pi)^{\frac{n}{2}}}{D^{\frac{m}{2} + \frac{1}{2}}} \Sigma D_{ab} D_{cd} \dots D_{hk} \dots (13'),$$

where $a, b, \dots h, k$ is a permutation as above, and $m = \alpha_1 + \alpha_2 + \dots + \alpha_n$ is even. $W = 0$ when m is odd.

As an illustration of this result:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (Mx^2 y^2 z^2 + Nx^2 yz) \\ &\quad \exp - \frac{1}{2} (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) dx dy dz \\ &= \frac{(2\pi)^{3/2}}{\Delta^{7/2}} M (8FGH + 2AF^2 + 2BG^2 + 2CH^2) + \frac{(2\pi)^{3/2}}{\Delta^{5/2}} N (2GH + AF), \end{aligned}$$

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

A cognate result is discussed by Mr Arthur Black in the *Transactions of the Cambridge Philosophical Society**. Black's integral is $\int_n V e^{-U} dx_1 \dots dx_w$ where V and U are any *quadratic* functions, the only restriction on U being that it should be essentially positive. Other particular cases have been dealt with in the paper previously quoted, and for the case of two variables several results are given by Mr H. E. Soper†.

For reference we add a table of values of the reduced product-moment coefficients that occur frequently in formulae for probable errors and similar work.

$$\begin{aligned}
 q_{1^4} &= 3. \\
 q_{1^3 2} &= 3r_{12}. \\
 q_{1^2 2^2} &= 1 + 2r_{12}^2. \\
 q_{1^2 23} &= r_{23} + 2r_{12}r_{13}. \\
 q_{1^6} &= 15. \\
 q_{1^5 2} &= 15r_{12}. \\
 q_{1^4 2^2} &= 3 + 12r_{12}^2. \\
 q_{1^3 2^3} &= 9r_{12} + 6r_{12}^3. \\
 q_{1^3 2^2 3} &= 3(r_{13} + 2r_{23}r_{12} + 2r_{13}r_{12}^2). \\
 q_{1^4 23} &= 3(r_{23} + 12r_{12}r_{13}). \\
 q_{1^2 2^2 3^2} &= 1 + 2r_{23}^2 + 2r_{31}^2 + 2r_{12}^2 + 8r_{12}r_{23}r_{31}. \\
 q_{1^5} &= 105, \quad q_{1^4 2} = 105r_{12}, \quad q_{1^6 2^2} = 15(6r_{12}^2 + 1). \\
 q_{1^5 2^3} &= 15(4r_{12}^3 + 3r_{12}). \\
 q_{1^4 2^4} &= 3(8r_{12}^4 + 24r_{12}^2 + 3). \\
 q_{1^\lambda 23} &= 1.3 \dots \lambda - 1(r_{23} + \lambda r_{12}r_{13}), \quad \lambda \text{ even.} \\
 q_{1^\lambda 2^2 3} &= 1.3.5 \dots \lambda[(\lambda - 1)r_{12}^2 r_{13} + r_{13} + 2r_{12}r_{23}], \quad \lambda \text{ odd.}
 \end{aligned}$$

For the case of two variables we add the following formula which is easily proved by the methods employed in this paper.

$$\begin{aligned}
 q_{1^u 2^v} &= \psi(u+v)r^v + \binom{v}{2}\psi(2)\psi(u+v-2)r^{v-2}(1-r^2) \\
 &\quad + \binom{v}{4}\psi(4)\psi(u+v-4)r^{v-4}(1-r^2)^2 + \dots \ddagger
 \end{aligned}$$

the series terminating. Here

$$\psi(2m) = 1.3.5 \dots (2m-1)$$

and

$$\binom{v}{m} = \frac{v(v-1) \dots (v-m+1)}{m!}.$$

* Vol. xvi, 1898, pp. 219—227.

† *Biometrika*, vol. ix, p. 101.

‡ This is virtually the formula (xxxii) employed by H. E. Soper, *l.c.a.* corrected for some misprints.

ON THE MATHEMATICAL EXPECTATION OF THE MOMENTS OF FREQUENCY DISTRIBUTIONS.

BY PROFESSOR AL. A. TCHOUPROFF of Petrograd.

INTRODUCTION

I

(1) One of my pupils, O. Anderson, in a brief exposition* of his researches on the Variate Difference Correlation Method in *Biometrika* (1914), draws attention to the superiority of the method of mathematical expectation over the methods usually employed by English statisticians. The small popularity enjoyed by the method of mathematical expectation in England is not of course accidental.

English scientific tradition rejects the concept of "mathematical probability."

From the time of R. L. Ellis and of the first edition of John Stuart Mill's *System of Logic*, the logician's basis of probability has, in England, been the notion of empirical frequency. English mathematicians have followed the lead of the writers on logic in their preference for the idea of statistical frequency, and the method of mathematical expectation has naturally shared the fate of the concept of mathematical probability on which it rests.

Notwithstanding its deep-rooted historical basis, English statisticians should break with this tradition. The substitution of statistical frequency for mathematical probability does not obviate the logical difficulties in laying the foundations for a statistical study of Causation, but merely shifts them elsewhere. The gain from the point of view of philosophical representation is sufficiently doubtful, while from the purely mathematical point of view the rejection of the ideas of mathematical probability and mathematical expectation is accompanied by very substantial disadvantages. Verbal formulation becomes very complicated, leading to loss of economy of attention: it is continually necessary to speak of "the statistical frequencies which would become established if the number of occurrences were infinitely great." The absence of a sharp distinction in terminology between statistical frequency in the exact meaning of the term and those quasi-empirical

* Anderson's research was carried out under my supervision in the statistical seminary attached to the Economics Department of the Petrograd Polytechnic Institute; the results he obtained were to have been published *in extenso* in the *Proceedings (Students' Section) of the Economics Department*, but the War drew Mr Anderson away from his scientific pursuits to other work of a more practical character and the complete publication of his researches had to be postponed.

"frequencies which would become established in an indefinitely great number of occurrences" often fails to make the very statement of the problem clear to the reader, and occasionally it would appear, to the author: when reading published papers one not infrequently feels that the author does not give himself a full account as to what it is he is really calculating.

Little harm follows so long as the problems dealt with are comparatively simple. But at the present time there are problems waiting for solution which are so complex that the slightest obscurity in their formulation threatens to become a source of error in the final deductions.

When we start with "mathematical probability" and "mathematical expectation" as a foundation we substantially simplify the mathematical exposition. The logical analysis of the conclusions to which we are led is not injuriously affected by the substitution of one set of terms for the other during the calculations.

(2) If the variable magnitude X can take the values $\xi_1, \xi_2, \dots, \xi_k$ with probabilities $p_1, p_2, \dots, p_k$, I call the system of values $\xi_1, \xi_2, \dots, \xi_k$ and the values $p_1, p_2, \dots, p_k$ associated with them "*the law of distribution of the values of the variable X .*" The law of distribution of values lies at the base of empirical "*frequency curves*," just as the mathematical probability of an event lies at the base of its statistically established frequency.

Denoting by the symbol EX the mathematical expectation of the variable magnitude X , we have as is well known:

$$EX = \sum_{i=1}^k p_i \xi_i,$$

where

$$\sum_{i=1}^k p_i = 1.$$

I call the variable magnitudes $X, Y, Z, \dots$ *mutually independent*, if the law of distribution of each of them remains one and the same whatever values are given to the others. In this case EX remains constant for all possible values of the variables $Y, Z, \dots$

If the law of distribution of X does not remain the same for different values of $Y, Z, \dots$, the variables $X, Y, Z, \dots$ are mutually dependent. The mathematical expectation of the variable X on the supposition that Y has received the value η_i , Z the value ζ_j , etc., I denote by $E^{(\eta_i, \zeta_j, \dots)}X$ and call it the "*conditional mathematical expectation of X on the supposition that the remaining variables have received definite values.*"

It follows from the definitions that

$$E(X + Y + Z + \dots) = EX + EY + EZ + \dots$$

both in the case when the variables are mutually independent, and when they are correlated, and that

$$E(XYZ\dots) = (EX)(EY)(EZ)\dots,$$

if the variables are mutually independent.

In the case in which X and Y are correlated we have :

$$EXY = \sum_{i=1}^k p_i \xi_i E^{(\xi_i)} Y,$$

$$EY = \sum_{i=1}^k p_i E^{(\xi_i)} Y.$$

II

(1) In investigations in the theory of probability we frequently have to deal with expressions of the type: $N(N-1)(N-2)\dots(N-k+1)$. Following the example of Capelli*, I use the slightly modified notation :

$$\left. \begin{aligned} N(N-1)(N-2)\dots(N-k+1) &= N^{[-k]} \\ N(N+1)(N+2)\dots(N+k-1) &= N^{[+k]} \end{aligned} \right\} \dots\dots\dots(1).$$

$$\text{Let } \left. \begin{aligned} N^k &= \sum_{i=1}^k \alpha_{k,i} N^{[-k]} \\ N^{[-k]} &= \sum_{j=0}^{k-1} (-1)^j \beta_{k,j} N^{k-j} \end{aligned} \right\} \dots\dots\dots(2).$$

The coefficients I have denoted by α, β are beginning to play an important part in the theory of finite differences† and are of the first importance in all investigations into the law of large numbers. Their properties were first studied systematically in Chapter III of Cramp's well-known work, *Analyse des réfractions astronomiques et terrestres*; some of their properties were discovered by investigators studying Bernoulli's numbers; recently they have received the attention of the Italian mathematical school associated with Césaro and Capelli. The methods I employ to solve fundamental problems of mathematical statistics are directly founded on certain properties of the α, β coefficients. In view of the fact that I shall later on frequently make use of these methods, I state here, without proof, those properties of the coefficients α and β that I shall have to quote in the present paper‡.

(2) We have :

$$\left. \begin{aligned} \alpha_{k,1} &= 1 \\ \alpha_{k,k} &= 1 \\ \alpha_{k,i} &= i\alpha_{k-1,i} + \alpha_{k-1,i-1} \end{aligned} \right\} \dots\dots\dots(3).$$

$$\alpha_{k,i} = \frac{i^k - C_i^1(i-1)^k + \dots + (-1)^h C_i^h(i-h)^k + \dots + (-1)^{i-1} C_i^{i-1}}{1.2.3\dots i} = \frac{\Delta^i 0^i}{1.2.3\dots i} \dots\dots(4).$$

* Vide Capelli: "Instituzione di analisi" and the same author's "L'analisi algebrica e l'interpretazione fattoriale delle potenze." (*Giornale di matematica di Battaglini*, Vol. xxxi.)

† Cf. A. A. Markoff, *Calculus of Finite Differences* (2nd edition).

‡ Readers interested in the proofs of these properties, many of them established for the first time by myself, will find a complete analysis in my paper in the *Proceedings of the Petrograd Polytechnic Institute*.

Putting
$$\frac{k(k-1)\dots(k-h+1)}{1.2.3\dots h} = C_k^h,$$

i.e. denoting by C_k^h the number of combinations of k elements h at a time, we may express $\alpha_{k,k-n}$ in the form :

$$\alpha_{k,k-n} = \sum_{i=0}^{n-1} A_{n,i} C_k^{2n-i} \dots\dots\dots(5),$$

where the coefficients $A_{n,i}$ are independent of k and are defined by the relations :

$$\left. \begin{aligned} A_{n,0} &= (2n-1) A_{n-1,0} \\ A_{n,j} &= (n-j) A_{n-1,j-1} + (2n-j-1) A_{n-1,j} \\ A_{n,n-1} &= A_{n-1,n-2} = 1 \end{aligned} \right\} \dots\dots\dots(6).$$

Hence

$$\left. \begin{aligned} A_{n,0} &= 1.3.5\dots(2n-1) \\ A_{n,1} &= 1.3.5\dots(2n-1) \frac{1}{3} [n-1] \\ A_{n,2} &= 1.3.5\dots(2n-3) \left\{ \frac{1}{9} [n-1]^{[-3]} + \frac{1}{6} [n-1]^{[-2]} \right\} \\ A_{n,3} &= 1.3.5\dots(2n-3) \left\{ \frac{1}{81} [n-2]^{[-4]} + \frac{1}{18} [n-2]^{[-3]} + \frac{1}{30} [n-2]^{[-2]} \right\} \\ A_{n,4} &= 1.3.5\dots(2n-5) \left\{ \frac{1}{486} [n-2]^{[-6]} + \frac{1}{54} [n-2]^{[-5]} + \frac{1}{360} [n-2]^{[-4]} \right. \\ &\quad \left. + \frac{1}{90} [n-2]^{[-3]} \right\} \\ A_{n,5} &= 1.3.5\dots(2n-5) \left\{ \frac{1}{7290} [n-3]^{[-7]} + \frac{1}{486} [n-3]^{[-6]} + \frac{1}{120} [n-3]^{[-5]} \right. \\ &\quad \left. + \frac{1}{108} [n-3]^{[-4]} + \frac{1}{630} [n-3]^{[-3]} \right\} \end{aligned} \right\} \dots\dots\dots(7).$$

The coefficient $A_{n,n-i}$ can easily be expressed in an independent form.

Putting
$$1.2.3\dots i = i!,$$

we shall have :

$$A_{n,n-i} = (n+i)! \sum \frac{1}{i_1! i_2! \dots i_f! [h_1!]^{i_1} [h_2!]^{i_2} \dots [h_f!]^{i_f}} \dots\dots\dots(8),$$

where the summation extends to all possible positive integer values of $i_1, i_2, \dots, i_f$, satisfying the relation : $i_1 + i_2 + \dots + i_f = i$, and to all integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions

$$2 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 i_1 + h_2 i_2 + \dots + h_f i_f = n + i.$$

Introducing the notation

$$\nabla_{(x)}^{j^c} f(x) = \nabla^j f(x-j) = \sum_{l=0}^j (-1)^l C_j^l f(x-l) \dots\dots\dots(9),$$

and noting that

$$\nabla_{(k)}^j C_k^h = C_{k-j}^{h-j},$$

we find from (5) :

$$\nabla_{(k)}^j \alpha_{k,k-n} = \sum_{i=0}^{n-1} A_{n,i} C_{k-j}^{2n-i-j}.$$

When $h > 0$ and $2n - h < k$ we have :

$$\left. \begin{aligned} \nabla_{(k)}^{2n+h} \alpha_{k, k-n} &= 0 \\ \nabla_{(k)}^{2n-h} \alpha_{k, k-n} &= \sum_{i=0}^h \text{ (or } 2n-k) A_{n, i} C_{k-2n+h}^{h-i} \\ \nabla_{(k)}^{2n} \alpha_{k, k-n} &= A_{n, 0} = 1.3.5 \dots (2n-1) \\ \nabla_{(k)}^k \alpha_{k, k-n} &= A_{n, 2n-k} \\ \nabla_{(k)}^{k+h} \alpha_{k, k-n} &= 0 \end{aligned} \right\} \dots\dots\dots(10).$$

(3) Of the properties of the β coefficients it is essential to note the following :

$$\left. \begin{aligned} \beta_{k, 0} &= 1 \\ \beta_{k, j} &= \beta_{k-1, j} + (k-1) \beta_{k-1, j-1} \\ \beta_{k, k-1} &= 1.2.3 \dots (k-1) \end{aligned} \right\} \dots\dots\dots(11),$$

$$\beta_{k, j} = \sum_{i=0}^{j-1} B_{j, i} C_k^{2j-i} \dots\dots\dots(12),$$

where the coefficients $B_{j, i}$ are independent of k and are determined by the relations

$$\left. \begin{aligned} B_{j, 0} &= 1.3.5 \dots (2j-1) \\ B_{j, i} &= (2j-i-1) [B_{j-1, i-1} + B_{j-1, i}] \\ B_{j, j-1} &= 1.2.3 \dots j \end{aligned} \right\} \dots\dots\dots(13).$$

Hence we have :

$$\left. \begin{aligned} B_{j, 0} &= 1.3.5 \dots (2j-1) \\ B_{j, 1} &= 1.3.5 \dots (2j-1) \frac{2}{3} [j-1] \\ B_{j, 2} &= 1.3.5 \dots (2j-3) \left\{ \frac{4}{3} [j-1]^{[-3]} + [j-1]^{[-2]} \right\} \\ B_{j, 3} &= 1.3.5 \dots (2j-3) \left\{ \frac{8}{15} [j-2]^{[-4]} + \frac{2}{3} [j-2]^{[-3]} + \frac{4}{5} [j-2]^{[-2]} \right\} \\ B_{j, 4} &= 1.3.5 \dots (2j-5) \left\{ \frac{8}{243} [j-2]^{[-6]} + \frac{4}{9} [j-2]^{[-5]} + \frac{47}{30} [j-2]^{[-4]} \right. \\ &\quad \left. + \frac{4}{3} [j-2]^{[-3]} \right\} \\ B_{j, 5} &= 1.3.5 \dots (2j-5) \left\{ \frac{16}{3645} [j-3]^{[-7]} + \frac{8}{81} [j-3]^{[-6]} + \frac{31}{45} [j-3]^{[-5]} \right. \\ &\quad \left. + \frac{76}{45} [j-3]^{[-4]} + \frac{8}{7} [j-3]^{[-3]} \right\} \end{aligned} \right\} \dots\dots\dots(14).$$

From (12) we find, when $h > 0$,

$$\left. \begin{aligned} \nabla_{(k)}^f \beta_{k, j} &= \sum_{i=0}^{j-1} B_{j, i} C_{k-f}^{2j-i-f} \\ \nabla_{(k)}^{2j+h} \beta_{k, j} &= 0 \\ \nabla_{(k)}^{2j-h} \beta_{k, j} &= \sum_{i=0}^h \text{ (or } 2j-k) B_{j, i} C_{k-2j+h}^{h-i} \\ \nabla_{(k)}^{2j} \beta_{k, j} &= B_{j, 0} = 1.3.5 \dots (2j-1) \\ \nabla_{(k)}^k \beta_{k, j} &= B_{j, 2j-k} \\ \nabla_{(k)}^{k+h} \beta_{k, j} &= 0 \end{aligned} \right\} \dots\dots\dots(15).$$

(4) Further, it is important to note certain relations connecting the coefficients α and β :

$$\sum_{i=0}^{n-k} (-1)^i \alpha_{n,k+i} \beta_{k+i,i} = 0 \dots\dots\dots(16),$$

$$\alpha_{n,n-k} \beta_{n-k,m} = \sum_{i=0}^{k+m-1} C_n^{2k+2m-i} R_{k,m,i} \dots\dots\dots(17),$$

where the coefficients $R_{k,m,i}$ are independent of n and are determined by the relations:

$$\left. \begin{aligned} R_{0,m,i} &= B_{m,i} \\ R_{k,0,i} &= A_{k,i} \\ R_{k,m,0} &= (2k+2m-1) [R_{k-1,m,0} + R_{k,m-1,0}] \\ R_{k,m,i} &= (2k+2m-1-i) [R_{k-1,m,i} + R_{k,m-1,i}] \\ &\quad + (k+2m-i) [R_{k-1,m,i-1} + R_{k,m-1,i-1}] - R_{k,m-1,i-1} \end{aligned} \right\} \dots\dots\dots(18).$$

From (18) we find

$$\left. \begin{aligned} R_{k,s-k,0} &= 1 \cdot 3 \cdot 5 \dots (2s-1) C_s^k \\ R_{k,s-k,1} &= \frac{s-1}{3} \cdot 1 \cdot 3 \cdot 5 \dots (2s-1) \{C_s^k + C_{s-1}^k\} \end{aligned} \right\} \dots\dots\dots(19),$$

and, in general,
$$R_{k,s-k,h} = \sum_{j=0}^h r_{s,h,j} C_{s-j}^k \dots\dots\dots(20),$$

where the coefficients $r_{s,h,j}$ are independent of k and are determined by the relations:

$$\left. \begin{aligned} r_{s,h,0} &= A_{s,h} \\ \sum_{j=0}^h r_{s,h,j} &= B_{s,h} \\ r_{s,h,j} &= (2s-h-1) r_{s-1,h,j} + (s+j-h) r_{s-1,h-1,j} + (s-j) r_{s-1,h-1,j-1} \end{aligned} \right\} \dots\dots\dots(21).$$

Putting

$$\left. \begin{aligned} t_{s,h,l} &= \sum_{j=0}^l \{ [C_{2h-l-j}^{l-j} + C_{2h-l-j-1}^{l-j-1}] r_{s,2h,j} \} \\ t'_{s,h,l} &= \sum_{j=0}^l C_{2h-l-j}^{l-j} r_{s,2h+1,j} \end{aligned} \right\} \dots\dots\dots(22),$$

we find further:

$$\left. \begin{aligned} R_{k,s-k,2h} &= \sum_{l=0}^h t_{s,h,l} C_{s-2h+l}^{s-k-l} \\ R_{k,s-k,2h+1} &= \sum_{l=0}^h t'_{s,h,l} [C_{s-2h+l}^{s-k-l} + C_{s-2h+l-1}^{s-k-l-1}] \end{aligned} \right\} \dots\dots\dots(23),$$

where

$$\left. \begin{aligned}
 t_{s,0,0} &= 1.3.5 \dots (2s-1) \\
 t_{s,1,0} &= 1.3.5 \dots (2s-3) \left\{ \frac{1}{9} [s-1]^{[-3]} + \frac{1}{6} [s-1]^{[-2]} \right\} \\
 t_{s,1,1} &= 1.3.5 \dots (2s-3) \left\{ \frac{4}{9} [s-1]^{[-3]} + [s-1]^{[-2]} \right\} \\
 t_{s,2,0} &= 1.3.5 \dots (2s-5) \left\{ \frac{1}{486} [s-2]^{[-6]} + \frac{1}{54} [s-2]^{[-5]} + \frac{1}{360} [s-2]^{[-4]} + \frac{1}{90} [s-2]^{[-3]} \right\} \\
 t_{s,2,1} &= 1.3.5 \dots (2s-5) \left\{ \frac{4}{243} [s-2]^{[-6]} + \frac{5}{27} [s-2]^{[-5]} + \frac{4}{90} [s-2]^{[-4]} + \frac{1}{3} [s-2]^{[-3]} \right\} \\
 t_{s,2,2} &= 1.3.5 \dots (2s-5) \left\{ \frac{8}{243} [s-2]^{[-6]} + \frac{4}{9} [s-2]^{[-5]} + \frac{4}{30} [s-2]^{[-4]} + \frac{4}{3} [s-2]^{[-3]} \right\} \\
 t'_{s,0,0} &= 1.3.5 \dots (2s-1) \frac{1}{3} [s-1] \\
 t'_{s,1,0} &= 1.3.5 \dots (2s-3) \left\{ \frac{1}{81} [s-2]^{[-4]} + \frac{1}{18} [s-2]^{[-3]} + \frac{1}{30} [s-2]^{[-2]} \right\} \\
 t'_{s,1,1} &= 1.3.5 \dots (2s-3) \left\{ \frac{4}{81} [s-2]^{[-4]} + \frac{1}{3} [s-2]^{[-3]} + \frac{2}{3} [s-2]^{[-2]} \right\} \\
 t'_{s,2,0} &= 1.3.5 \dots (2s-5) \left\{ \frac{1}{7290} [s-3]^{[-7]} + \frac{1}{486} [s-3]^{[-6]} + \frac{1}{120} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{1}{108} [s-3]^{[-4]} + \frac{1}{30} [s-3]^{[-3]} \right\} \\
 t'_{s,2,1} &= 1.3.5 \dots (2s-5) \left\{ \frac{4}{3645} [s-3]^{[-7]} + \frac{5}{243} [s-3]^{[-6]} + \frac{3}{270} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{1}{90} [s-3]^{[-4]} + \frac{2}{21} [s-3]^{[-3]} \right\} \\
 t'_{s,2,2} &= 1.3.5 \dots (2s-5) \left\{ \frac{8}{3645} [s-3]^{[-7]} + \frac{4}{81} [s-3]^{[-6]} + \frac{3}{90} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{3}{45} [s-3]^{[-4]} + \frac{4}{7} [s-3]^{[-3]} \right\}
 \end{aligned} \right\} \dots (24),$$

$$\left. \begin{aligned}
 t'_{s,2,0} &= 1.3.5 \dots (2s-5) \left\{ \frac{1}{7290} [s-3]^{[-7]} + \frac{1}{486} [s-3]^{[-6]} + \frac{1}{120} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{1}{108} [s-3]^{[-4]} + \frac{1}{30} [s-3]^{[-3]} \right\} \\
 t'_{s,2,1} &= 1.3.5 \dots (2s-5) \left\{ \frac{4}{3645} [s-3]^{[-7]} + \frac{5}{243} [s-3]^{[-6]} + \frac{3}{270} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{1}{90} [s-3]^{[-4]} + \frac{2}{21} [s-3]^{[-3]} \right\} \\
 t'_{s,2,2} &= 1.3.5 \dots (2s-5) \left\{ \frac{8}{3645} [s-3]^{[-7]} + \frac{4}{81} [s-3]^{[-6]} + \frac{3}{90} [s-3]^{[-5]} \right. \\
 &\quad \left. + \frac{3}{45} [s-3]^{[-4]} + \frac{4}{7} [s-3]^{[-3]} \right\}
 \end{aligned} \right\} \dots (25).$$

From (17) we find, when $h > 0$,

$$\left. \begin{aligned}
 \nabla_{(n)}^j \alpha_{n,n-k} \beta_{n-k,m} &= \sum_{i=0}^{k+m-1} R_{k,m,i} C_{n-j}^{2k+2m-i-j} \\
 \nabla_{(n)}^{2k+2m+h} \alpha_{n,n-k} \beta_{n-k,m} &= 0 \\
 \nabla_{(n)}^{2k+2m-h} \alpha_{n,n-k} \beta_{n-k,m} &= \sum_{i=0}^h R_{k,m,i} C_{n-2k-2m+h}^{h-i} \\
 \nabla_{(n)}^{2k+2m} \alpha_{n,n-k} \beta_{n-k,m} &= R_{k,m,0} = 1.3.5 \dots (2k+2m-1) C_{k+m}^k \\
 \nabla_{(n)}^n \alpha_{n,n-k} \beta_{n-k,m} &= R_{k,m,n-2k-2m} \\
 \nabla_{(n)}^{n+h} \alpha_{n,n-k} \beta_{n-k,m} &= 0
 \end{aligned} \right\} \dots (26).$$

(5) In Chapter IV we shall have to deal with more complicated expressions of type:

$$\nabla_{(r_1)}^{r_1} \nabla_{(r_2)}^{r_2} \dots \nabla_{(r_k)}^{r_k} \alpha_{r_1, r_1-h_1} \alpha_{r_2, r_2-h_2} \dots \alpha_{r_k, r_k-h_k} \beta_{r_1+r_2+\dots+r_k-h_1-h_2-\dots-h_k, f-h_1-h_2-\dots-h_k}.$$

In my previously quoted paper they are not considered as I met them for the first time in connection with the problems considered in the fourth Chapter of the present paper. My discussion of these expressions has not so far led me to results which may be considered final, and I shall merely indicate the method by which their fundamental properties may be established.

Putting $r_1 + r_2 + \dots + r_k = R$, $h_1 + h_2 + \dots + h_k = H$, let us replace α and β by their values from (5) and (12). Noting that, as is well known,

$$[x+y]^{[-m]} = \sum_{i=0}^m C_m^i x^{[-i]} y^{[-(m-i)]},$$

we find :

$$\begin{aligned}
 & \nabla_{(r_1)}^{r_1} \nabla_{(r_2)}^{r_2} \dots \nabla_{(r_k)}^{r_k} \alpha_{r_1, r_1-h_1} \alpha_{r_2, r_2-h_2} \dots \alpha_{r_k, r_k-h_k} \beta_{R-H, f-H} \\
 &= \sum_{l_1=0}^{h_1-1} \sum_{l_2=0}^{h_2-1} \dots \sum_{l_k=0}^{h_k-1} \sum_{j=0}^{f-H-1} \frac{A_{h_1, l_1} A_{h_2, l_2} \dots A_{h_k, l_k} B_{f-H, j}}{(2h_1-l_1)! (2h_2-l_2)! \dots (2h_k-l_k)! (2f-2H-j)!} \\
 & \quad \times \nabla_{(r_1)}^{r_1} \nabla_{(r_2)}^{r_2} \dots \nabla_{(r_k)}^{r_k} r_1^{[-(2h_1-l_1)]} r_2^{[-(2h_2-l_2)]} \dots \\
 & \quad r_k^{[-(2h_k-l_k)]} [r_1-h_1+r_2-h_2+\dots+r_k-h_k]^{[-(2f-2H-j)]} \\
 &= S_l^{f-H-1} \frac{A_{h_1, l_1} A_{h_2, l_2} \dots A_{h_k, l_k} B_{f-H, j}}{(2h_1-l_1)! (2h_2-l_2)! \dots (2h_k-l_k)!} S_g \frac{1}{g_1! g_2! \dots g_{k-1}! (2f-2H-j-g)!} \\
 & \quad \times \nabla_{(r_1)}^{r_1} r_1^{[-(2h_1-l_1)]} [r_1-h_1]^{[-g_1]} \nabla_{(r_2)}^{r_2} r_2^{[-(2h_2-l_2)]} [r_2-h_2]^{[-g_2]} \dots \\
 & \quad \nabla_{(r_{k-1})}^{r_{k-1}} r_{k-1}^{[-(2h_{k-1}-l_{k-1})]} [r_{k-1}-h_{k-1}]^{[-(g_{k-1})]} \nabla_{(r_k)}^{r_k} r_k^{[-(2h_k-l_k)]} [r_k-h_k]^{[-(2f-2H-j-g)]} \\
 & \dots (27),
 \end{aligned}$$

where S_l denotes

$$\sum_{l_1=0}^{h_1-1} \sum_{l_2=0}^{h_2-1} \dots \sum_{l_k=0}^{h_k-1},$$

S_g denotes

$$\sum_{g_1=0}^{2f-2H-j} \sum_{g_2=0}^{2f-2H-j-g_1} \dots \sum_{g_{k-1}=0}^{2f-2H-j-g_1-\dots-g_{k-2}}$$

and

$$g = g_1 + g_2 + \dots + g_{k-1}.$$

If we note that $\nabla_{(r_1)}^{r_1} r_1^{[-(2h_1-l_1)]} [r_1-h_1]^{[-g_1]} = 0$, when $r_1 > 2h_1-l_1+g_1$, etc., we see without difficulty that, when $2f < R$, the sum we are discussing is equal to zero. If $2f = R$, then the only non-vanishing term in the sum is the one corresponding to $l_1=l_2=\dots=l_k=j=0$ and $g_1=r_1-2h_1$, $g_2=r_2-2h_2$, ..., $g_{k-1}=r_{k-1}-2h_{k-1}$, $2f-2H-g=r_k-2h_k$, and the sum reduces to

$$C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1, 0} A_{h_2, 0} \dots A_{h_k, 0} B_{\frac{R+1}{2}-H, 0} \dots (28).$$

If $2f = R+1$, there are three types of non-vanishing terms: (1) terms for which $l_1=l_2=\dots=l_k=j=0$, and for which, of the quantities $g_1, g_2, \dots, g_{k+1}$, $2f-2H-g$, one, e.g. g_i , is equal to r_i-2h_i+1 , and each of the rest is equal to $r-2h$; (2) terms, for which $l_1=l_2=\dots=l_k=0$, $j=1$ and all the quantities $g_i=r_i-2h_i$; (3) terms, for which $j=0$, one of the quantities l , e.g. $l_i=1$, and the other quantities l vanish, $g_i=r_i-2h_i+1$, and the remaining quantities g are each equal to $r-2h$.

Noting that

$$\nabla^r X^{[-2h]} [X-h]^{[-k]}$$

becomes $r!h(r-2h+1)$ when $k=r-2h+1$ and $X=r$, we can without difficulty reduce the sum we are considering, for the case $2f=R+1$ to:

$$\begin{aligned}
 & H \cdot C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1, 0} A_{h_2, 0} \dots A_{h_k, 0} B_{\frac{R+1}{2}-H, 0} \\
 & + C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1, 0} A_{h_2, 0} \dots A_{h_k, 0} B_{\frac{R+1}{2}-H, 1} \\
 & + C_{r_1}^{2h_1-1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1, 1} A_{h_2, 0} \dots A_{h_k, 0} B_{\frac{R+1}{2}-H, 0} \\
 & \quad + C_{r_1}^{2h_1} C_{r_2}^{2h_2-1} \dots C_{r_k}^{2h_k} A_{h_1, 0} A_{h_2, 1} \dots A_{h_k, 0} B_{\frac{R+1}{2}-H, 0} + \dots \\
 & + C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_{k-1}}^{2h_{k-1}} C_{r_k}^{2h_k-1} A_{h_1, 0} A_{h_2, 0} \dots A_{h_{k-1}, 0} A_{h_k, 1} B_{\frac{R+1}{2}-H, 0}
 \end{aligned} \quad (29).$$

CHAPTER I

I

Consider a variable magnitude X , admitting the values $\xi_1, \xi_2, \dots, \xi_k$ with probabilities $p_1, p_2, \dots, p_k$. Let us make N experiments, and suppose that the law of distribution of the values of the variable remains unaltered, and that the separate experiments are independent of one another. Denoting by X_i the value taken by the variable in the i th experiment and by n_j the number of times, out of the N experiments, that the variable X takes the value ξ_j , let

$$\left. \begin{aligned} p_j' &= \frac{n_j}{N} \\ m_r &= EX^r = Ex_i^r = \sum_{j=1}^k p_j \xi_j^r \\ \mu_r &= E(X - m_1)^r = E(X_i - m_1)^r = \sum_{j=1}^k p_j (\xi_j - m_1)^r \\ X_{(N)} &= \frac{1}{N} \sum_{i=1}^N X_i = \frac{1}{N} \sum_{j=1}^k n_j \xi_j = \sum_{j=1}^k p_j' \xi_j \\ m_{r, (N)} &= EX_{(N)}^r \\ \mu_{r, (N)} &= E[X_{(N)} - m_1]^r \end{aligned} \right\} \dots\dots\dots(1).$$

We have, whatever be the law of distribution of the variable X :

$$\begin{aligned} \sum_{j=1}^k p_j &= \sum_{j=1}^k p_j' = 1, \\ \sum_{j=1}^k n_j &= N, \\ m_{1, (N)} &= m_1, \\ \mu_{1, (N)} &= \mu_1 = 0, \\ \mu_{0, (N)} &= \mu_0 = 1. \end{aligned}$$

We find further, without difficulty:

$$\left. \begin{aligned} \mu_2 &= m_2 - m_1^2 \\ \mu_3 &= m_3 - 3m_2 m_1 + 2m_1^3 \\ \mu_4 &= m_4 - 4m_3 m_1 + 6m_2 m_1^2 - 3m_1^4 \\ \mu_r &= m_r - C_r^1 m_{r-1} m_1 + \dots + (-1)^h C_r^h m_{r-h} m_1^h + \dots \\ &\quad + (-1)^{r-2} C_r^{r-2} m_2 m_1^{r-2} + (-1)^{r-1} (r-1) m_1^r \\ \mu_{r, (N)} &= m_{r, (N)} - C_r^1 m_{r-1, (N)} m_1 + \dots + (-1)^h C_r^h m_{r-h, (N)} m_1^h + \dots \\ &\quad + (-1)^{r-2} C_r^{r-2} m_{2, (N)} m_1^{r-2} + (-1)^{r-1} (r-1) m_1^r \end{aligned} \right\} \dots\dots\dots(2).$$

Conversely, expressing the quantities m_r in terms of μ_r , we find :

$$\left. \begin{aligned} m_r &= E \{ (X - m_1)^r + m_1 \}^r = m_1^r + \sum_{h=2}^{r-1} C_r^h m_1^{r-h} \mu_h + \mu_r \\ m_{r, (N)} &= m_1^r + \sum_{h=2}^{r-1} C_r^h m_1^{r-h} \mu_{h, (N)} + \mu_{r, (N)} \\ \sum_{h=0}^r (-1)^h C_r^h m_{r-h} m_{h, (N)} &= \sum_{h=0}^r (-1)^h C_r^h \mu_{r-h} \mu_{h, (N)} \end{aligned} \right\} \dots\dots\dots (3).$$

II

(1) Noting that

$$X_{(N)}^r = \frac{1}{N^r} \left[\sum_{i=1}^N X_i \right]^r,$$

we write $\left[\sum_{i=1}^N X_i \right]^r$ in the form

$$\sum_j \sum_{i_1 i_2} \dots \sum_{i_j} \frac{r!}{r_1! r_2! \dots r_j!} X_{i_1}^{r_1} X_{i_2}^{r_2} \dots X_{i_j}^{r_j},$$

where, as is known, the summation with regard to j extends from 1 to the smaller of the two integers r and N ; the summation with regard to $i_1, i_2, \dots, i_j$ extends to all integral and unequal values of $i_1, i_2, \dots, i_j$ from 1 to N , and the last summation extends to all positive integer or zero values of $r_1, r_2, \dots, r_j$ satisfying the relation $r_1 + r_2 + \dots + r_j = r$.

Passing to mathematical expectations we find hence

$$\begin{aligned} m_{r, (N)} &= EX_{(N)}^r = \frac{1}{N^r} \sum_j \sum_{i_1 i_2} \dots \sum_{i_j} \frac{r!}{r_1! r_2! \dots r_j!} EX_{i_1}^{r_1} EX_{i_2}^{r_2} \dots EX_{i_j}^{r_j} \\ &= \frac{1}{N^r} \sum_j \frac{N^{[-j]}}{1.2.3 \dots j} \sum \frac{r!}{r_1! r_2! \dots r_j!} m_{r_1} m_{r_2} \dots m_{r_j}, \end{aligned}$$

where the summation with regard to j extends to all positive integer values from 1 to the smaller of the two numbers r and N , while the second sum extends to all positive integer values of $r_1, r_2, \dots, r_j$, satisfying the relation :

$$r_1 + r_2 + \dots + r_j = r.$$

If $r \leq N$, we have consequently :

$$m_{r, (N)} = \frac{1}{N^r} \sum_{j=1}^r R_{r,j} N^{[-j]} \dots\dots\dots (4),$$

where the $R_{r,j}$ coefficients are independent of N and are defined by the relation :

$$R_{r,j} = \frac{1}{1.2.3 \dots j} \sum \frac{r!}{r_1! r_2! \dots r_j!} m_{r_1} m_{r_2} \dots m_{r_j},$$

and the summation extends over the ranges specified above.

Hence we find :

$$R_{r,r-h} = \sum_i \frac{r^{[-(h+i)]}}{i!} m_1^{r-h-i} \sum \frac{1}{h_1! h_2! \dots h_i!} m_{h_1} m_{h_2} \dots m_{h_i},$$

where the summation with regard to i extends to all integral values from 1 to the smaller of the two numbers h and $r-h$, and the second summation extends to all positive integer values not less than 2 of $h_1, h_2, \dots, h_i$, satisfying the condition :

$$h_1 + h_2 + \dots + h_i = h + i,$$

$$\text{or} \quad R_{r,r-h} = \sum_i r^{[-(h+i)]} m_1^{r-h-i} \sum \frac{m_{h_1}^{j_1} m_{h_2}^{j_2} \dots m_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \dots (5),$$

where the summation for i extends to all integer values from 1 to the smaller of the numbers h and $r-h$, and the second summation extends to all positive integer values of $j_1, j_2, \dots, j_f$, satisfying the equation :

$$j_1 + j_2 + \dots + j_f = i,$$

and to all positive integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions :

$$2 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = h + i.$$

We find hence :

$$\left. \begin{aligned} R_{r,r} &= m_1^r \\ R_{r,r-1} &= C_r^2 m_1^{r-2} m_2 \\ R_{r,r-2} &= 1.3 C_r^4 m_1^{r-4} m_2^2 + C_r^3 m_1^{r-3} m_3 \\ R_{r,r-3} &= 1.3.5 C_r^6 m_1^{r-6} m_2^3 + 10 C_r^5 m_1^{r-5} m_3 m_2 + C_r^4 m_1^{r-4} m_4 \\ R_{r,r-4} &= 1.3.5.7 C_r^8 m_1^{r-8} m_2^4 + 105 C_r^7 m_1^{r-7} m_2^2 m_3 \\ &\quad + C_r^6 m_1^{r-6} [15m_2 m_4 + 10m_3^2] + C_r^5 m_1^{r-5} m_5 \end{aligned} \right\} \dots (6),$$

and on the other hand

$$\left. \begin{aligned} R_{r,1} &= m_r \\ R_{r,2} &= \frac{1}{1.2} \sum_{h=1}^{r-1} C_r^h m_h m_{r-h} \end{aligned} \right\} \dots (7),$$

and so on.

(2) The calculation of the coefficients $R_{r,r-h}$ may also be effected by another method.

$$\text{Noting that} \quad E \left[\sum_{i=1}^N X_i \right]^{r+1} = N E \left\{ X_1 \left[\sum_{i=1}^N X_i \right]^r \right\},$$

we find

$$\left. \begin{aligned} m_{r+1, (N)} &= \frac{1}{N^r} E \left\{ X_1 \left[\sum_{i=1}^N X_i \right]^r \right\} = \frac{1}{N^r} E \left\{ X_1 \sum_{h=0}^r C_r^h X_1^{r-h} \left[\sum_{i=2}^N X_i \right]^h \right\} \\ &= \frac{1}{N^r} \left\{ m_{r+1} + \sum_{h=1}^{r-1} C_r^h m_{r+1-h} (N-1)^h m_{h, (N-1)} + (N-1)^r m_1 m_{r, (N-1)} \right\} \\ &= \frac{1}{N^r} \left\{ m_{r+1} + \sum_{h=1}^r C_r^h (N-1)^h m_{r+1-h} m_{h, (N-1)} \right\} \end{aligned} \right\} \dots (8).$$

Substituting here (see (4))

$$(N-1)^h m_{h, (N-1)} = \sum_{j=1}^h [N-1]^{[-j]} R_{h,j}$$

we find :

$$\begin{aligned} m_{r+1, (N)} &= \frac{1}{N^r} \left\{ m_{r+1} + \sum_{h=1}^r C_r^h m_{r+1-h} \sum_{j=1}^h [N-1]^{[-j]} R_{h,j} \right\} \\ &= \frac{1}{N^{r+1}} \left\{ N m_{r+1} + \sum_{j=1}^r N^{[-(j+1)]} \sum_{h=j}^r C_r^h m_{r+1-h} R_{h,j} \right\}. \end{aligned}$$

On the other hand

$$m_{r+1, (N)} = \frac{1}{N^{r+1}} \sum_{i=1}^{r+1} N^{[-i]} R_{r+1,i}.$$

Hence

$$\left. \begin{aligned} R_{r+1,1} &= m_{r+1} \\ R_{r+1,i} &= \sum_{h=i-1}^r C_r^h m_{r+1-h} R_{h,i-1} \\ R_{r+1,r+1} &= m_1 R_{r,r} = m_1^{r+1} \end{aligned} \right\} \dots \dots \dots (9).$$

(3) Putting (see Introduction (2))

$$N^{[-j]} = \sum_{f=0}^{j-1} (-1)^f \beta_{j,f} N^{j-f},$$

we find from (4)

$$\left. \begin{aligned} m_{r, (N)} &= \frac{1}{N^r} \sum_{j=1}^r \sum_{f=0}^{j-1} (-1)^f N^{j-f} \beta_{j,f} R_{r,j} \\ &= \frac{1}{N^r} \sum_{h=1}^r N^h \sum_{j=h}^r (-1)^{j-h} \beta_{j,j-h} R_{r,j} \\ &= m_1^r + \sum_{i=1}^{r-1} \frac{1}{N^i} \sum_{h=0}^i (-1)^h \beta_{r-i+h,h} R_{r,r-i+h} \\ &= m_1^r + \frac{1}{N} C_r^2 m_1^{r-2} \mu_2 + \frac{1}{N^2} \{ 1 + 3 C_r^4 m_1^{r-4} \mu_2^2 + C_r^3 m_1^{r-3} \mu_3 \} + \dots \end{aligned} \right\} \dots (10).$$

When $r=2, 3, 4$ we find :

$$\left. \begin{aligned} m_{2, (N)} &= m_1^2 + \frac{1}{N} \mu_2 \\ m_{3, (N)} &= m_1^3 + \frac{3}{N} m_1 \mu_2 + \frac{1}{N^2} \mu_3 \\ m_{r, (N)} &= m_1^4 + \frac{6}{N} m_1^2 \mu_2 + \frac{1}{N^2} [4m_1 \mu_3 + 3\mu_2^2] + \frac{1}{N^3} [\mu_4 - 3\mu_2^2] \end{aligned} \right\} \dots (11).$$

III

(1) If $m_1 = 0$, then

$$\mu_r = E(X - m_1)^r = EX^r = m_r$$

and

$$\mu_{r, (N)} = m_{r, (N)}.$$

Putting $m_1 = 0$ in the formulae of § II, we may, consequently, replace in them the quantities m by the corresponding quantities μ . But when $m = 0$, $R_{r, r-h} = 0$, if $h < r/2$; but if $h \geq r/2$, then $R_{r, r-h}$ reduces (cf. (5)) to

$$T_{r, r-h} = r! \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \dots \dots \dots (12),$$

where the summation extends to all positive integer values of $j_1, j_2, \dots, j_f$, satisfying the condition $j_1 + j_2 + \dots + j_f = r - h$, and to all positive integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions:

$$2 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = r.$$

Hence

$$\left. \begin{aligned} \mu_{2r, (N)} &= \frac{1}{N^{2r}} \sum_{h=r}^{2r-1} N^{[-(2r-h)]} T_{2r, 2r-h} = \frac{1}{N^{2r}} \sum_{h=0}^{r-1} N^{[-(r-h)]} T_{2r, r-h} \\ \mu_{2r+1, (N)} &= \frac{1}{N^{2r+1}} \sum_{h=r+1}^{2r} N^{[-(2r+1-h)]} T_{2r+1, 2r+1-h} = \frac{1}{N^{2r+1}} \sum_{h=0}^{r-1} N^{[-(r-h)]} T_{2r+1, r-h} \end{aligned} \right\} (13),$$

or

$$\mu_{r, (N)} = \frac{1}{N^r} \sum_{h=0}^{\text{Ent.} \left(\frac{r}{2} \right) - 1} N^{\{ -[\text{Ent.} \left(\frac{r}{2} \right) - h] \}} T_{r, \text{Ent.} \left(\frac{r}{2} \right) - h} \dots \dots \dots (14),$$

where $\text{Ent.} \left(\frac{r}{2} \right)$ denotes the greatest integer in $\frac{r}{2}$.

If we now put

$$N^{[-(r-h)]} = \sum_{i=0}^{r-h-1} (-1)^i \beta_{r-h, i} N^{r-h-i},$$

we find after some transformations:

$$\left. \begin{aligned} \mu_{2r, (N)} &= \sum_{i=0}^{r-1} \frac{1}{N^{r+i}} \sum_{h=0}^i (-1)^h \beta_{r-i+h, h} T_{2r, r-i+h} \\ \mu_{2r+1, (N)} &= \sum_{i=0}^{r-1} \frac{1}{N^{r+1+i}} \sum_{h=0}^i (-1)^h \beta_{r-i+h, h} T_{2r+1, r-i+h} \end{aligned} \right\} \dots \dots \dots (15),$$

or

$$\mu_{r, (N)} = \sum_{i=0}^{\text{Ent.} \left(\frac{r}{2} \right) - 1} \frac{1}{N^{\text{Ent.} \left(\frac{r+1}{2} \right) + i}} \sum_{h=0}^i (-1)^h \beta_{\text{Ent.} \left(\frac{r}{2} \right) - i + h, h} T_{r, \text{Ent.} \left(\frac{r}{2} \right) - i + h} \dots (16).$$

When r is even,

$$\left. \begin{aligned} T_{2r,r} &= 1.3.5 \dots (2r-1) \mu_2^r, \\ T_{2r,r-h} &= (2r)! \sum_i \frac{\mu_2^{r-h-i}}{2^{r-h-i} (r-h-i)!} \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \\ &= 1.3.5 \dots (2r-1) \sum_i r^{[-(h+i)]} 2^{h+i} \mu_2^{r-h-i} \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \end{aligned} \right\} (17),$$

where the summation for i extends to all integer values from 1 to the smaller of the numbers $2h$ and $r-h$, and the last summation to all positive integer values of $j_1, j_2, \dots, j_f$, satisfying the condition:

$$j_1 + j_2 + \dots + j_f = i,$$

and to all positive integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions:

$$3 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = 2h + 2i.$$

When r is odd,

$$\left. \begin{aligned} T_{2r+1,r} &= 1.3.5 \dots (2r+1) \frac{r}{3} \mu_2^{r-1} \mu_3, \\ T_{2r+1,r-h} &= (2r+1)! \sum_i \frac{\mu_2^{r-h-i}}{2^{r-h-i} (r-h-i)!} \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \\ &= 1.3.5 \dots (2r+1) \sum_i r^{[-(h+i)]} 2^{h+i} \mu_2^{r-h-i} \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \end{aligned} \right\} (18),$$

where the summation for i extends to all integer values from 1 to the smaller of the numbers $2h+1$ and $r-h$, and the last summation extends to all positive integer values of $j_1, j_2, \dots, j_f$, satisfying the condition:

$$j_1 + j_2 + \dots + j_f = i,$$

and to all integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions:

$$3 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = 2h + 2i + 1.$$

The coefficients T may also be calculated by means of recurrence formulae. Putting $m_1 = 0$ in (8) and replacing the m 's by μ 's, we find:

$$\mu_{r+1, (N)} = \frac{1}{N^r} \left\{ \mu_{r+1} + \sum_{h=2}^{r-1} C_r^h \mu_{r+1-h} (N-1)^h \mu_{h, (N-1)} \right\},$$

and hence (or directly from (9))

$$\left. \begin{aligned} T_{r+1,1} &= \mu_{r+1} \\ T_{r+1,i} &= \sum_{h=i-1}^{r-1} C_r^h \mu_{r+1-h} T_{h,i-1} \end{aligned} \right\} \dots \dots \dots (19).$$

We find in this way:

$$\left. \begin{aligned} T_{2r,r} &= 1.3.5 \dots (2r-1) \mu_2^r \\ T_{2r,r-1} &= 1.3.5 \dots (2r-1) \left\{ \frac{1}{6} r^{[-2]} \mu_2^{r-2} \mu_4 + \frac{1}{9} r^{[-3]} \mu_2^{r-3} \mu_3^2 \right\} \\ T_{2r,r-2} &= 1.3.5 \dots (2r-1) \left\{ \frac{1}{90} r^{[-3]} \mu_2^{r-3} \mu_6 + \frac{1}{4} r^{[-4]} \mu_2^{r-4} \mu_4^2 \right. \\ &\quad \left. + \frac{1}{45} r^{[-4]} \mu_2^{r-4} \mu_3 \mu_5 + \frac{1}{54} r^{[-5]} \mu_2^{r-5} \mu_3^2 \mu_4 + \frac{1}{480} r^{[-6]} \mu_2^{r-6} \mu_3^4 \right\} \end{aligned} \right\} \dots (20),$$

$$\left. \begin{aligned} T_{2r+1,r} &= 1.3.5 \dots (2r+1) \frac{1}{3} r \mu_2^{r-1} \mu_3 \\ T_{2r+1,r-1} &= 1.3.5 \dots (2r+1) \left\{ \frac{1}{30} r^{[-2]} \mu_2^{r-2} \mu_5 + \frac{1}{18} r^{[-3]} \mu_2^{r-3} \mu_3 \mu_4 \right. \\ &\quad \left. + \frac{1}{81} r^{[-4]} \mu_2^{r-4} \mu_3^3 \right\} \\ T_{2r+1,r-2} &= 1.3.5 \dots (2r+1) \left\{ \frac{1}{630} r^{[-3]} \mu_2^{r-3} \mu_7 + \frac{1}{270} r^{[-4]} \mu_2^{r-4} \mu_3 \mu_6 \right. \\ &\quad + \frac{1}{180} r^{[-4]} \mu_2^{r-4} \mu_4 \mu_5 + \frac{1}{135} r^{[-5]} \mu_2^{r-5} \mu_3^2 \mu_5 + \frac{1}{108} r^{[-5]} \mu_2^{r-5} \mu_3 \mu_4^2 \\ &\quad \left. + \frac{1}{81} r^{[-6]} \mu_2^{r-6} \mu_3^3 \mu_4 + \frac{1}{7290} r^{[-7]} \mu_2^{r-7} \mu_3^5 \right\} \end{aligned} \right\} \dots (21).$$

Hence:

$$\left. \begin{aligned} \mu_{2r,(N)} &= 1.3.5 \dots (2r-1) \left\{ \frac{1}{N^r} \mu_2^r \right. \\ &\quad + \frac{1}{N^{r+1}} \left[\frac{1}{6} r^{[-2]} \mu_2^{r-2} \mu_4 + \frac{1}{9} r^{[-3]} \mu_2^{r-3} \mu_3^2 - \frac{1}{2} r^{[-2]} \mu_2^r \right] \\ &\quad + \frac{1}{N^{r+2}} \left[\frac{1}{90} r^{[-3]} \mu_2^{r-3} \mu_6 + \frac{1}{45} r^{[-4]} \mu_2^{r-4} \mu_3 \mu_5 + \frac{1}{72} r^{[-4]} \mu_2^{r-4} \mu_4^2 \right. \\ &\quad + \frac{1}{54} r^{[-5]} \mu_2^{r-5} \mu_3^2 \mu_4 - \frac{r-1}{12} r^{[-3]} \mu_2^{r-2} \mu_4 + \frac{1}{480} r^{[-6]} \mu_2^{r-6} \mu_3^4 \\ &\quad \left. - \frac{(r-1)(r-2)}{18} r^{[-3]} \mu_2^{r-3} \mu_3^2 + \frac{3r-1}{24} r^{[-3]} \mu_2^r \right] + \dots \left. \right\} \end{aligned} \right\} \dots (22),$$

$$\left. \begin{aligned} \mu_{2r+1,(N)} &= 1.3.5 \dots (2r+1) \left\{ \frac{1}{N^{r+1}} \frac{1}{3} r \mu_2^{r-1} \mu_3 \right. \\ &\quad + \frac{1}{N^{r+2}} \left[\frac{1}{30} r^{[-2]} \mu_2^{r-2} \mu_5 + \frac{1}{18} r^{[-3]} \mu_2^{r-3} \mu_3 \mu_4 \right. \\ &\quad \left. \left. + \frac{1}{81} r^{[-4]} \mu_2^{r-4} \mu_3^3 - \frac{r}{6} r^{[-2]} \mu_2^{r-1} \mu_3 \right] \right. \\ &\quad + \frac{1}{N^{r+3}} \left[\frac{1}{630} r^{[-3]} \mu_2^{r-3} \mu_7 + \frac{1}{270} r^{[-4]} \mu_2^{r-4} \mu_3 \mu_6 + \frac{1}{180} r^{[-4]} \mu_2^{r-4} \mu_4 \mu_5 \right. \\ &\quad + \frac{1}{135} r^{[-5]} \mu_2^{r-5} \mu_3^2 \mu_5 - \frac{r-1}{60} r^{[-3]} \mu_2^{r-2} \mu_5 + \frac{1}{108} r^{[-5]} \mu_2^{r-5} \mu_3 \mu_4^2 \\ &\quad + \frac{1}{81} r^{[-6]} \mu_2^{r-6} \mu_3^3 \mu_4 - \frac{(r-1)(r-2)}{36} r^{[-3]} \mu_2^{r-3} \mu_3 \mu_4 + \frac{1}{7290} r^{[-7]} \mu_2^{r-7} \mu_3^5 \\ &\quad \left. \left. - \frac{(r-1)(r-2)(r-4)}{162} r^{[-3]} \mu_2^{r-4} \mu_3^3 + \frac{r(3r-1)}{72} r^{[-3]} \mu_2^{r-1} \mu_3 \right] + \dots \right\} \end{aligned} \right\} \dots (23).$$

On the other hand,

$$\left. \begin{aligned} T_{2r,1} &= \mu_{2r} \\ T_{2r,2} &= (2r)! \left\{ \frac{\mu_r^2}{2[r!]^2} + \sum_{h=2}^{r-1} \frac{\mu_h \mu_{2r-h}}{h! (2r-h)!} \right\} = \sum_{h=2}^{r-1} C_{2r}^h \mu_h \mu_{2r-h} + C_{2r-1}^{r-1} \mu_r^2 \end{aligned} \right\} \dots (24),$$

$$\left. \begin{aligned} T_{2r+1,1} &= \mu_{2r+1} \\ T_{2r+1,2} &= \sum_{h=2}^r C_{2r+1}^h \mu_h \mu_{2r+1-h} \end{aligned} \right\} \dots (25).$$

For $r = 2, 3, 4, 5, 6, 7, 8$ we find hence :

$$\begin{aligned}
 \mu_{2,(N)} &= \frac{1}{N} \mu_2 \\
 \mu_{3,(N)} &= \frac{1}{N^2} \mu_3 \\
 \mu_{4,(N)} &= \frac{1}{N^4} \{N\mu_4 + 3N^{[-2]} \mu_2^2\} = \frac{3}{N^2} \mu_2^2 + \frac{1}{N^3} [\mu_4 - 3\mu_2^2] \\
 \mu_{5,(N)} &= \frac{1}{N^5} \{N\mu_5 + 10N^{[-2]} \mu_3\mu_2\} = \frac{10}{N^3} \mu_3\mu_2 + \frac{1}{N^4} [\mu_5 - 10\mu_3\mu_2] \\
 \mu_{6,(N)} &= \frac{1}{N^6} \{N\mu_6 + N^{[-2]} [15\mu_4\mu_2 + 10\mu_3^2] + 15N^{[-3]} \mu_2^3\} \\
 &= \frac{15}{N^3} \mu_2^3 + \frac{5}{N^4} [3\mu_4\mu_2 + 2\mu_3^2 - 9\mu_2^3] + \frac{1}{N^5} [\mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3] \\
 \mu_{7,(N)} &= \frac{1}{N^7} \left\{ N\mu_7 + N^{[-2]} [21\mu_5\mu_2 + 35\mu_4\mu_3] + 105N^{[-3]} \mu_3\mu_2^2 \right\} \dots(26). \\
 &= \frac{105}{N^4} \mu_3\mu_2^2 + \frac{7}{N^5} [3\mu_5\mu_2 + 5\mu_4\mu_3 - 63\mu_3\mu_2^2] \\
 &\quad + \frac{1}{N^6} [\mu_7 - 21\mu_5\mu_2 - 35\mu_4\mu_3 + 210\mu_3\mu_2^2] \Big\} \\
 \mu_{8,(N)} &= \frac{1}{N^8} \{N\mu_8 + N^{[-2]} [28\mu_6\mu_2 + 56\mu_5\mu_3 + 35\mu_4^2] \\
 &\quad + N^{[-3]} [210\mu_4\mu_2^2 + 280\mu_3^2\mu_2] + 105N^{[-4]} \mu_2^4\} \\
 &= \frac{105}{N^4} \mu_2^4 + \frac{70}{N^5} [3\mu_4\mu_2^2 + 4\mu_3^2\mu_2 - 9\mu_2^4] \\
 &\quad + \frac{7}{N^6} [4\mu_6\mu_2 + 8\mu_5\mu_3 + 5\mu_4^2 - 90\mu_4\mu_2^2 - 120\mu_3^2\mu_2 + 165\mu_2^4] \\
 &\quad + \frac{1}{N^7} [\mu_8 - 28\mu_6\mu_2 - 56\mu_5\mu_3 - 35\mu_4^2 + 420\mu_4\mu_2^2 + 560\mu_3^2\mu_2 - 630\mu_2^4] \Big\}
 \end{aligned}$$

(2) Noting that

$$\begin{aligned}
 X_{(N)} - m_1 &= \frac{1}{N} \sum_{i=1}^N (X_i - m_1) = \frac{1}{N} \left\{ \sum_{i=1}^h (X_i - m_1) + \sum_{i=h+1}^N (X_i - m_1) \right\} \\
 &= \frac{1}{N} \{h[X_{(h)} - m_1] + (N-h)[X_{(N-h)} - m_1]\},
 \end{aligned}$$

we find

$$\begin{aligned}
 [X_{(N)} - m_1]^r &= \frac{1}{N^r} \sum_{i=0}^r C_r^i h^{r-i} (N-h)^i [X_{(h)} - m_1]^{r-i} [X_{(N-h)} - m_1]^i, \\
 \mu_{r,(N)} &= \frac{1}{N^r} \sum_{i=0}^r C_r^i h^{r-i} (N-h)^i \mu_{r-i,(h)} \mu_{i,(N-h)}, \\
 N^r \mu_{r,(N)} &= h^r \mu_{r,(h)} + \sum_{i=2}^{r-2} C_r^i h^{r-i} \mu_{r-i,(h)} (N-h)^i \mu_{i,(N-h)} + (N-h)^r \mu_{r,(N-h)}.
 \end{aligned}$$

Putting $h = 1$, we obtain

$$N^r \mu_{r,(N)} = \mu_r + \sum_{i=2}^{r-2} C_r^i (N-1)^i \mu_{r-i} \mu_{i,(N-1)} + (N-1)^r \mu_{r,(N-1)}.$$

Hence :

$$N^r \mu_{r,(N)} - (N-1)^r \mu_{r,(N-1)} = \mu_r + \sum_{i=2}^{r-2} C_r^i (N-1)^i \mu_{r-i} \mu_{i,(N-1)},$$

and
$$N^r \mu_{r,(N)} = N \mu_r + \sum_{i=2}^{r-2} C_r^i \mu_{r-1} \sum_{j=1}^{N-1} (N-j)^i \mu_{i,(N-j)} \dots\dots\dots(27).$$

If we give r in turns the values 2, 3, 4, ..., we find the relations (26).

IV

(1) From the relations (15) and (17) we find :

$$\frac{\mu_{2r,(N)}}{\mu_2^r(N)} = 1.3.5 \dots (2r-1) \left\{ 1 + \sum_{i=1}^{r-1} \frac{1}{N^i} \sum_{h=0}^i (-1)^h \beta_{r-i+h,h} \frac{T_{2r,r-i+h}}{1.3.5 \dots (2r-1) \mu_2^r} \right\}.$$

As N increases the ratio $\frac{\mu_{2r,(N)}}{\mu_2^r(N)}$ consequently tends to the limit $1.3.5 \dots (2r-1)$, if

$$\sum_{i=1}^{r-1} \frac{1}{N^i} \sum_{h=0}^i (-1)^h \beta_{r-i+h,h} \frac{T_{2r,r-i+h}}{1.3.5 \dots (2r-1) \mu_2^r}$$

tends to zero.

But if this last expression tends to a limit different from zero as N increases, then the limit to which $\frac{\mu_{2r,(N)}}{\mu_2^r(N)}$ tends cannot be equal to $1.3.5 \dots (2r-1)$.

The quantity $\frac{1}{1.3.5 \dots (2r-1)} \beta_{r-i+h,h}$ does not become infinitely great for any value of r and is independent both of the value of N and of the law of distribution of values of X . In order that $\frac{\mu_{2r,(N)}}{\mu_2^r}$ should tend to $1.3.5 \dots (2r-1)$ it appears then to be a sufficient condition that $\frac{1}{N^i \mu_2^r} T_{2r,r-i+h}$ should tend to zero for $i = 1, 2, 3, \dots, r-1$. A sufficient condition for this, in its turn, is that expressions of type

$$\frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{N^i \mu_2^{i-h+l}}$$

should tend to zero, when the quantities $j_1, j_2, \dots, j_f$, are connected by the relation: $j_1 + j_2 + \dots + j_f = l$, and the quantities $h_1, h_2, \dots, h_f$ satisfy the relation :

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = 2(i-h+l),$$

and l can take all integral values between 1 and $2(i-h)$. Finally this condition is satisfied, if expressions of type

$$\frac{N \mu_i}{[N \mu_2]^{i/2}}$$

tend to zero, as N increases, when $i = 3, 4, 5, \dots, 2r-1, 2r$. Noting that when these conditions are satisfied the fraction $\frac{\mu_{2r-1, (N)}}{[\mu_{2, (N)}]^{r-\frac{1}{2}}}$ tends to zero, we arrive at the well-known result (cf. A. A. Markoff, *Theory of Probability*, 3rd Edition, pp. 329-330).

The probability of the fulfilment of the inequality

$$t_1 < \frac{1}{\sqrt{\frac{2}{N} \mu_2}} \left[\frac{1}{N} \sum_{i=1}^N (X_i - m_1) \right] < t_2$$

tends with increasing N to the limit $\frac{1}{\sqrt{\pi}} \int_{t_1}^{t_2} e^{-t^2} dt$, whatever be the law of distribution of values of the variable X , if only it satisfies the condition, that $\frac{N\mu_i}{[N\mu_2]^{i/2}}$ should tend with increasing N to zero, when $i = 3, 4, 5, \dots, \infty$, and if at the same time the law of distribution of values remains unaltered, and the separate experiments are mutually independent*.

(2) From (22) we find :

$$\frac{\mu_{2r, (N)}}{1.3.5 \dots (2r-1) \mu_{2, (N)}^r} = 1 + \frac{1}{N} \left\{ \frac{r(r-1)}{2} \left(\frac{\mu_4}{3\mu_2^2} - 1 \right) + \frac{r(r-1)(r-2)}{9} \frac{\mu_3^2}{\mu_2^3} \right\} + \dots$$

Thus we see that $\frac{\mu_{2r, (N)}}{\mu_{2, (N)}^r}$ tends the more slowly to $1.3.5 \dots (2r-1)$, the more the law of distribution of values deviates from the Gauss-Laplace law, and the greater r is. If $\mu_4 > 3\mu_2^2$, then for sufficiently large values of N ,

$$\frac{\mu_{2r, (N)}}{\mu_{2, (N)}^r} > 1.3.5 \dots (2r-1).$$

* The condition " $\frac{N\mu_i}{[N\mu_2]^{i/2}}$ tends, with increasing N , to zero for $i = 3, 4, 5, \dots, \infty$," while sufficient, is, as is well known, not necessary. From the form to which Liapounoff succeeded in reducing the condition (see *Proc. Imp. Acad. Sci.* VIII series, vol. XII) it follows, among other consequences, that the law of distribution of values of $X_{(N)}$ tends with increasing N to the Gaussian, if $\frac{1}{N} \frac{\mu_4}{\mu_2^2}$ tends to zero. Noting that

$$\frac{\mu_{4, (N)}}{\mu_{2, (N)}^2} - 3 = \frac{1}{N} \left[\frac{\mu_4}{\mu_2^2} - 3 \right],$$

we see that this condition is satisfied if $\frac{\mu_{4, (N)}}{\mu_{2, (N)}^2}$ tends with increasing N to 3. It is in this way that Liapounoff's results justify arguments based on the examination of the two moments only $\mu_{3, (N)}$ and $\mu_{4, (N)}$ in deciding the question, whether the law of distribution of the values of $X_{(N)}$ tends to the Gaussian with increasing N or not. The assumption usually underlying such a procedure, viz. that the law of distribution is the Gauss-Laplace law, if $\frac{\mu_{3, (N)}}{\mu_{2, (N)}^3} = 0$ and $\frac{\mu_{4, (N)}}{\mu_{2, (N)}^2} = 3$, is clearly inexact: the coincidence of the values of two or even more moments does not guarantee the identity of the laws, but merely compresses the possible divergence between them into limits which become narrower as the number of coinciding moments increases (cf. the investigation by Chebysheff "On the Integral..., forming Approximations to the Value of an Integral" in *Oeuvres*, t. II, and the related papers by A. A. Markoff; cf. also T. T. Stieltjes, "Recherches sur les fractions continues" in *Annales de la faculté de Toulouse* (1894).

V

Equations (15), (16), (17) and (18) hold for all laws of distribution of the variable X . If the law fulfils the conditions

$$\mu_{2i+1} = 0 \text{ for } i = 0, 1, 2, \dots, r,$$

then $\mu_{2r+1, (N)} = 0$, for all the coefficients $T_{2r+1, r-h}$ then vanish. As regards the coefficients $T_{2r, r-h}$, they take the form:

$$T_{2r, r-h} = 1.3.5 \dots (2r-1) \sum_{i=1}^h r^{[-(h+i)]} 2^{h+i} \mu_2^{r-h-i} \\ \times \sum \frac{\mu_{2h_1}^{j_1} \mu_{2h_2}^{j_2} \dots \mu_{2h_f}^{j_f}}{j_1! j_2! \dots j_f! [(2h_1)!]^{j_1} [(2h_2)!]^{j_2} \dots [(2h_f)!]^{j_f}}$$

where the last summation extends to all positive integer values $j_1, j_2, \dots, j_f$, satisfying the condition: $j_1 + j_2 + \dots + j_f = i$, and to all integer values of $h_1, h_2, \dots, h_f$, satisfying the conditions:

$$2 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = h + i.$$

If the law of distribution of values of X also satisfies the conditions:

$$\mu_{2i} = 1.3.5 \dots (2i-1) \mu_2^i \text{ for } i = 1, 2, 3, \dots, r,$$

then (cf. Introduction (5), (8) and (16))

$$T_{2r, r-h} = 1.3.5 \dots (2r-1) \sum_{i=1}^h r^{[-(h+i)]} 2^{h+i} \mu_2^r \times \\ \sum \frac{[1.3.5 \dots (2h_1-1)]^{j_1} [1.3.5 \dots (2h_2-1)]^{j_2} \dots [1.3.5 \dots (2h_f-1)]^{j_f}}{j_1! j_2! \dots j_f! [(2h_1)!]^{j_1} [(2h_2)!]^{j_2} \dots [(2h_f)!]^{j_f}} \\ = 1.3.5 \dots (2r-1) \mu_2^r \sum_{i=1}^h r^{[-(h+i)]} \sum \frac{1}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \\ = 1.3.5 \dots (2r-1) \mu_2^r \alpha_{r, r-h},$$

and

$$\mu_{2r, (N)} = 1.3.5 \dots (2r-1) \mu_2^r \sum_{i=0}^{r-1} \frac{1}{N^{r+i}} \sum_{h=0}^i (-1)^i \alpha_{r, r-i+h} \beta_{r-i+h, h} \\ = 1.3.5 \dots (2r-1) \mu_2^r \frac{1}{N^r} = 1.3.5 \dots (2r-1) \mu_{2r, (N)}^r.$$

Thus in the case when the law of distribution of the values of X for $i \leq r$ gives values of μ_i answering the Gauss-Laplace law, then the values $X_{(N)}$ follow a law of distribution giving for $\mu_{i, (N)}$ for $i \leq r$ the same values as the law of Gauss. Consequently if the law of distribution of values of X is Gaussian, then the law of distribution of values of $X_{(N)}$ is Gaussian also for all values of N .

CHAPTER II

I

$$(1) \quad \text{Putting} \quad \frac{1}{N} \sum_{i=1}^N (X_i - m_1)^r = \mu'_r \dots\dots\dots(1),$$

we have :

$$E\mu'_r = \mu_r.$$

Noting that μ'_r is the arithmetic mean of the mutually independent quantities

$$(X_1 - m_1)^r, (X_2 - m_1)^r, \dots (X_N - m_1)^r,$$

and that

$$E(X_1 - m_1)^r = \mu_r,$$

while

$$E(X_1 - m_1)^{kr} = \mu_{kr},$$

we find from formulae (4) and (5) of Chapter I, when we put $r = m$, and replace $m_{r(N)}$ by $E\mu'_r{}^m$ and m_h by μ_{hr} :

$$\left. \begin{aligned} E\mu'_r{}^m &= \frac{1}{N^m} \sum_{h=1}^m R_{[m,h]} N^{[-h]} = \frac{1}{N^m} \sum_{h=0}^{m-1} R_{[m,m-h]} N^{[-(m-h)]} \\ &= \frac{1}{N^m} \sum_{i=1}^m N^i \sum_{h=0}^{m-i} (-1)^{m-h-i} \beta_{m-h,m-h-i} R_{[m,m-h]} \end{aligned} \right\} \dots\dots\dots(2),$$

where

$$R_{[m,m-h]} = \sum_{i=1}^h m^{[-(h+i)]} \mu_r^{m-h-i} \sum \frac{\mu_{h_1}^{j_1} \mu_{h_2}^{j_2} \dots \mu_{h_f}^{j_f}}{j_1! j_2! \dots j_f! [h_1!]^{j_1} [h_2!]^{j_2} \dots [h_f!]^{j_f}} \dots(3),$$

where the last summation extends to all positive integer values of $j_1, j_2, \dots j_f$, satisfying the condition: $j_1 + j_2 + \dots + j_f = i$, and to all integer values of $h_1, h_2, \dots h_f$, satisfying the conditions :

$$2 \leq h_1 < h_2 < \dots < h_f,$$

$$h_1 j_1 + h_2 j_2 + \dots + h_f j_f = h + i.$$

Hence :

$$\left. \begin{aligned} R_{[m,m]} &= \mu_r^m \\ R_{[m,m-1]} &= C_m^2 \mu_r^{m-2} \mu_{2r} \\ R_{[m,m-2]} &= 3C_m^4 \mu_r^{m-4} \mu_{2r}^2 + C_m^3 \mu_r^{m-5} \mu_{3r} \\ R_{[m,m-3]} &= 15C_m^6 \mu_r^{m-6} \mu_{2r}^3 + 10C_m^5 \mu_r^{m-5} \mu_{2r} \mu_{3r} + C_m^4 \mu_r^{m-4} \mu_{4r} \\ R_{[m,m-4]} &= 105C_m^8 \mu_r^{m-8} \mu_{2r}^4 + 105C_m^7 \mu_r^{m-7} \mu_{2r}^2 \mu_{3r}^2 + 15C_m^6 \mu_r^{m-6} \mu_{2r} \mu_{4r} \\ &\quad + 10C_m^6 \mu_r^{m-6} \mu_{3r}^2 + C_m^5 \mu_r^{m-5} \mu_{5r} \end{aligned} \right\} (4),$$

and, on the other hand

$$\left. \begin{aligned} R_{[m,1]} &= \mu_{mr} \\ R_{[m,2]} &= \frac{1}{2} \sum_{h=1}^{m-1} C_m^h \mu_{hr} \mu_{(m-h)r} \\ R_{[m,i]} &= \sum_{h=i-1}^{m-1} C_m^h \mu_{(m-h)r} R_{h,i-1} \end{aligned} \right\} \dots\dots\dots(5).$$

Substituting in (2), we find:

$$E\mu_r'^m = \mu_r^m + \frac{1}{N} \frac{m(m-1)}{2} \mu_r^{m-2} [\mu_{2r} - \mu_r^2] \\ + \frac{1}{N^2} \left\{ \frac{m^{[-4]}}{8} \mu_r^{m-4} [\mu_{2r} - \mu_r^2]^2 + \frac{m^{[-3]}}{6} \mu_r^{m-3} [\mu_{3r} - 3\mu_{2r}\mu_r + 2\mu_r^3] \right\} + \dots \quad (6).$$

(2) $E\mu_r'^m$ may also be calculated by means of recurrence formulae. Put

$$EN^m \mu_r'^m = E \left[\sum_{i=1}^N (X_i - m_1)^r \right]^m = v_{r, (N)}^{(m)} \dots\dots\dots (7).$$

From relation (8) of the first chapter we find:

$$v_{r, (N)}^{(m)} = N \left\{ \mu_{mr} + \sum_{h=1}^{m-1} C_{m-1}^h \mu_{(m-h)r} v_{r, (N-1)}^{(h)} \right\} \dots\dots\dots (8).$$

Noting that $v_{r, (N)}^{(h)} = N\mu_r^h$, we find hence:

$$\left. \begin{aligned} v_{r, (N)}^{(2)} &= N\mu_{2r} + N^{[-2]} \mu_r^2 = N[\mu_{2r} - \mu_r^2] + N^2 \mu_r^2 \\ v_{r, (N)}^{(3)} &= N\mu_{3r} + N^{[-2]} 3\mu_{2r}\mu_r + N^{[-3]} \mu_r^3 \\ v_{r, (N)}^{(4)} &= N\mu_{4r} + N^{[-2]} [4\mu_{3r}\mu_r + 3\mu_{2r}^2] + N^{[-3]} 6\mu_{2r}\mu_r^2 + N^{[-4]} \mu_r^4 \end{aligned} \right\} \dots\dots (9),$$

$$\text{and, in general,} \quad v_{r, (N)}^{(m)} = \sum_{i=1}^m N^{[-i]} D_{r, i}^{(m)} \dots\dots\dots (10),$$

where

$$\left. \begin{aligned} D_{r, 1}^{(m)} &= \mu_{mr} \\ D_{r, m}^{(m)} &= \mu_r^m \\ D_{r, i}^{(m)} &= \sum_{h=i-1}^{m-1} C_{m-1}^h \mu_{(m-h)r} D_{r, i-1}^{(h)} \\ D_{r, m-1}^{(m)} &= (m-1) \mu_{2r} D_{r, m-2}^{(m-2)} + \mu_r D_{r, m-2}^{(m-1)} = (m-1) \mu_{2r} \mu_r^{m-2} + \mu_r D_{r, m-2}^{(m-1)} \\ &= C_m^2 \mu_{2r} \mu_r^{m-2} \\ D_{r, (m)}^{m-2} &= \mu_{3r} \sum_{j=0}^{m-3} C_{m-1-j}^2 \mu_r^j \mu_{(m-3-j)r} + 3C_m^4 \mu_{2r}^2 \mu_r^{m-4} \end{aligned} \right\} \quad (11),$$

and so on.

Substituting in (10), we obtain

$$\left. \begin{aligned} v_{r, (N)}^{(m)} &= N^m E \mu_r'^m = N^{[-m]} \mu_r^m + N^{[-(m-1)]} C_m^2 \mu_{2r} \mu_r^{m-2} \\ &+ N^{[-(m-2)]} \left\{ \mu_{3r} \sum_{j=0}^{m-3} C_{m-1-j}^2 \mu_r^j \mu_{(m-3-j)r} + 3C_m^4 \mu_{2r}^2 \mu_r^{m-4} \right\} + \dots \\ &+ N^{[-2]} \sum_{h=1}^{m-1} C_{m-1}^h \mu_{(m-h)r} \mu_r^h + N\mu_{mr} \end{aligned} \right\} \dots\dots (12).$$

(3) When $r=2$, in the case when the law of distribution is Gaussian, we find from (8):

$$v_{2,(N)}^{(m)} = N \left\{ 1.3.5 \dots (2m-1) \mu_2^m + \sum_{h=1}^{m-1} C_{m-1}^h 1.3.5 \dots (2m-2h-1) \mu_2^{m-h} v_{2,(N-1)}^{(h)} \right\}.$$

Hence noting that $v_{2,(N-1)}^{(1)} = (N-1) \mu_2$, we find:

$$v_{2,(N)}^{(2)} = N(N+2) \mu_2^2,$$

$$v_{2,(N)}^{(3)} = N(N+2)(N+4) \mu_2^3,$$

$$v_{2,(N)}^{(4)} = N(N+2)(N+4)(N+6) \mu_2^4.$$

Let us assume that for all values $i \leq m$,

$$v_{2,(N)}^{(i)} = N(N+2)(N+4) \dots (N+2i-2) \mu_2^i.$$

In this case:

$$\begin{aligned} v_{2,(N)}^{(m+1)} &= N \mu_2^{m+1} \left\{ 1.3.5 \dots (2m+1) \right. \\ &\quad \left. + \sum_{h=1}^m C_m^h 1.3.5 \dots (2m-2h+1) (N-1)(N+1) \dots (N+2h-3) \right\}, \\ v_{2,(N)}^{(m)} &= N \mu_2^m \left\{ 1.3.5 \dots (2m-1) \right. \\ &\quad \left. + \sum_{h=1}^{m-1} C_{m-1}^h 1.3.5 \dots (2m-2h-1) (N-1)(N+1) \dots (N+2h-3) \right\}, \end{aligned}$$

and, consequently,

$$\begin{aligned} (N+2m) \mu_2 v_{2,(N)}^{(m)} &= N \mu_2^{m+1} \left\{ 1.3.5 \dots (2m-1)(2m+1) + (N-1) 1.3.5 \dots (2m-1) \right. \\ &\quad \left. + \sum_{h=1}^{m-1} C_{m-1}^h 1.3.5 \dots (2m-2h-1)(2m-2h+1)(N-1)(N+1) \dots (N+2h-3) \right. \\ &\quad \left. + \sum_{h=1}^{m-1} C_{m-1}^h 1.3.5 \dots (2m-2h-1)(N-1)(N+1) \dots (N+2h-1) \right\} \\ &= N \mu_2^{m+1} \left\{ 1.3.5 \dots (2m+1) \right. \\ &\quad \left. + \sum_{h=1}^m C_m^h 1.3.5 \dots (2m-2h+1)(N-1)(N+1) \dots (N+2h-3) \right\} \\ &= v_{2,(N)}^{(m+1)}. \end{aligned}$$

Thus when the law of distribution of values of X is Gaussian,

$$E \mu_{2,(N)}'^m = \frac{1}{N^m} N(N+2)(N+4) \dots (N+2m-2) \mu_2^m \text{ for } m = 0, 1, 2, 3, \dots \infty \quad (13).$$

(4) When $r = 3$, in the case of a Gaussian distribution of the values of X ,

$$v_{3,(N)}^{(m)} = N \left\{ \mu_{3m} + \sum_{h=1}^{m-1} C_{m-1}^h \mu_3^{(m-h)} v_{3,(N-1)}^{(h)} \right\}.$$

Noting that $v_{3,(N)}^{(1)} = 0$, if the distribution of the values of X be Gaussian, we find:

$$v_{3,(N)}^{(2)} = N \cdot 1 \cdot 3 \cdot 5 \cdot \mu_2^3,$$

$$v_{3,(N)}^{(3)} = 0 \text{ and, in general, } v_{3,(N)}^{(2i+1)} = 0,$$

$$v_{3,(N)}^{(4)} = N \mu_2^6 \cdot 1 \cdot 3 \cdot 5 \cdot 9 [5N + 72],$$

$$v_{3,(N)}^{(6)} = N \mu_2^9 \cdot 3 \cdot 5 \cdot 9 \cdot 15 [25N^2 + 1080N + 15912].$$

When $r = 4$, for the case of a Gaussian distribution of the values of X , we have:

$$v_{4,(N)}^{(m)} = N \left\{ 1 \cdot 3 \cdot 5 \dots (4m-1) \mu_2^{2m} + \sum_{h=1}^{m-1} C_{m-1}^h \cdot 1 \cdot 3 \cdot 5 \dots (4m-4h-1) \mu_2^{2(m-h)} v_{4,(N-1)}^{(h)} \right\},$$

$$v_{4,(N-1)}^{(1)} = (N-1) \mu_4 = 3(N-1) \mu_2^2,$$

$$v_{4,(N)}^{(2)} = N \mu_2^4 \cdot 3 [3N + 32],$$

$$v_{4,(N)}^{(3)} = N \mu_2^6 \cdot 27 [N^2 + 32N + 352].$$

II

Replacing r by m and $\mu_{r,(N)}$ by $E(\mu'_r - \mu_r)^m$ in formulae (22) and (23) of Chapter I, and replacing μ_h by $\sum_{k=0}^h (-1)^k C_h^k \mu_r^k \mu_{(h-k)r}$ and putting

$$\sum_{k=0}^h (-1)^k C_h^k \mu_r^k \mu_{(h-k)r} = X_h,$$

we find:

$$\left. \begin{aligned} E(\mu'_r - \mu_r)^{2m} &= 1 \cdot 3 \cdot 5 \dots (2m-1) \left\{ \frac{1}{N^m} X_2^m \right. \\ &+ \frac{1}{N^{m+1}} \left[\frac{1}{6} m^{[-2]} X_2^{m-2} X_4 + \frac{1}{9} m^{[-3]} X_2^{m-3} X_3^2 - \frac{1}{2} m^{[-2]} X_2^m \right] \\ &+ \frac{1}{N^{m+2}} \left[\frac{1}{96} m^{[-3]} X_2^{m-3} X_6 + \frac{1}{45} m^{[-4]} X_2^{m-4} X_3 X_5 + \frac{1}{72} m^{[-4]} X_2^{m-4} X_4^2 \right. \\ &+ \frac{1}{54} m^{[-5]} X_2^{m-5} X_3^2 X_4 - \frac{m-1}{12} m^{[-3]} X_2^{m-2} X_4 + \frac{1}{486} m^{[-6]} X_2^{m-6} X_3^4 \\ &\left. \left. - \frac{(m-1)(m-2)}{18} m^{[-3]} X_2^{m-3} X_3^2 + \frac{3m-1}{24} m^{[-3]} X_2^m \right] + \dots \right\} \end{aligned} \right\} \quad (14).$$

$$\begin{aligned}
& E(\mu'_r - \mu_r)^{2m+1} = 1 \cdot 3 \cdot 5 \dots (2m+1) \left\{ \frac{1}{N^{m+1}} \frac{1}{3} m X_2^{m-1} X_3 \right. \\
& + \frac{1}{N^{m+2}} \left[\frac{1}{3 \cdot 0} m^{[-2]} X_2^{m-2} X_5 + \frac{1}{1 \cdot 8} m^{[-3]} X_2^{m-3} X_3 X_4 \right. \\
& \quad \left. + \frac{1}{8 \cdot 1} m^{[-4]} X_2^{m-4} X_3^3 - \frac{m}{6} m^{[-2]} X_2^{m-1} X_3 \right] \\
& + \frac{1}{N^{m+3}} \left[\frac{1}{6 \cdot 3 \cdot 0} m^{[-3]} X_2^{m-3} X_7 + \frac{1}{2 \cdot 7 \cdot 0} m^{[-4]} X_2^{m-4} X_3 X_6 \right. \\
& \quad \left. + \frac{1}{1 \cdot 8 \cdot 0} m^{[-4]} X_2^{m-4} X_4 X_5 \right. \\
& + \frac{1}{1 \cdot 3 \cdot 5} m^{[-5]} X_2^{m-5} X_3^2 X_5 - \frac{m-1}{60} m^{[-2]} X_2^{m-2} X_5 + \frac{1}{1 \cdot 0 \cdot 8} m^{[-5]} X_2^{m-4} X_3 X_4^2 \\
& + \frac{1}{8 \cdot 1} m^{[-6]} X_2^{m-6} X_3^3 X_4 - \frac{(m-1)(m-2)}{36} m^{[-3]} X_2^{m-3} X_3 X_4 \\
& \quad \left. + \frac{1}{7 \cdot 2 \cdot 9 \cdot 0} m^{[-7]} X_2^{m-7} X_3^5 \right. \\
& \left. - \frac{(m-1)(m-2)(m-4)}{162} m^{[-3]} X_2^{m-4} X_3^3 + \frac{m(3m-1)}{72} X_2^{m-1} X_3 \right] + \dots \Big\} \quad (15).
\end{aligned}$$

Hence :

$$\begin{aligned}
& E(\mu'_r - \mu_r)^2 = \frac{1}{N} [\mu_{2r} - \mu_r^2] \\
& E(\mu'_r - \mu_r)^3 = \frac{1}{N^2} [\mu_{3r} - 3\mu_{2r}\mu_r + 2\mu_r^3] \\
& E(\mu'_r - \mu_r)^4 = \frac{3}{N^2} [\mu_{2r} - \mu_r^2]^2 + \frac{1}{N^3} [\mu_{4r} - 4\mu_{3r}\mu_r - 3\mu_{2r}^2 + 12\mu_{2r}\mu_r^2 - 6\mu_r^4] \quad (16).
\end{aligned}$$

III

(1) Noting that

$$\begin{aligned}
E\mu'_{r_1}\mu'_{r_2} &= \frac{1}{N^2} E \left\{ \sum_{i=1}^N (x_i - m_1)^{r_1+r_2} + \sum_{i=1}^N \sum_{j \neq i} (x_i - m_1)^{r_1} (x_j - m_1)^{r_2} \right\} \\
&= \frac{1}{N^2} \{ N\mu_{r_1+r_2} + N^{[-2]} \mu_{r_1} \mu_{r_2} \} = \mu_{r_1} \mu_{r_2} + \frac{1}{N} [\mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2}],
\end{aligned}$$

we find :

$$E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2}) = E\mu'_{r_1}\mu'_{r_2} - \mu_{r_1}\mu_{r_2} = \frac{1}{N} [\mu_{r_1+r_2} - \mu_{r_1}\mu_{r_2}].$$

Similarly we find :

$$\begin{aligned}
E\mu'_{r_1}\mu'_{r_2}\mu'_{r_3} &= \frac{1}{N^3} \{ N\mu_{r_1+r_2+r_3} + N^{[-2]} [\mu_{r_1+r_2}\mu_{r_3} + \mu_{r_1+r_3}\mu_{r_2} + \mu_{r_2+r_3}\mu_{r_1}] \\
& \quad + N^{[-3]} \mu_{r_1}\mu_{r_2}\mu_{r_3} \} \\
&= \mu_{r_1}\mu_{r_2}\mu_{r_3} + \frac{1}{N} [\mu_{r_1+r_2}\mu_{r_3} + \mu_{r_1+r_3}\mu_{r_2} + \mu_{r_2+r_3}\mu_{r_1} - 3\mu_{r_1}\mu_{r_2}\mu_{r_3}] \\
& \quad + \frac{1}{N^2} [\mu_{r_1+r_2+r_3} - (\mu_{r_1+r_2}\mu_{r_3} + \mu_{r_1+r_3}\mu_{r_2} + \mu_{r_2+r_3}\mu_{r_1}) + 2\mu_{r_1}\mu_{r_2}\mu_{r_3}],
\end{aligned}$$

and

$$\begin{aligned}
& E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2})(\mu'_{r_3} - \mu_{r_3}) \\
&= \frac{1}{N^2} [\mu_{r_1+r_2+r_3} - \mu_{r_1+r_2} \mu_{r_3} - \mu_{r_1+r_3} \mu_{r_2} - \mu_{r_2+r_3} \mu_{r_1} + 2\mu_{r_1} \mu_{r_2} \mu_{r_3}]. \\
& E\mu'_{r_1} \mu'_{r_2} \mu'_{r_3} \mu'_{r_4} = \frac{1}{N^4} \{ N\mu_{r_1+r_2+r_3+r_4} \\
&+ N^{[-2]} [\mu_{r_1+r_2+r_3} \mu_{r_4} + \mu_{r_1+r_2+r_4} \mu_{r_3} + \mu_{r_1+r_3+r_4} \mu_{r_2} + \mu_{r_2+r_3+r_4} \mu_{r_1}] \\
&+ N^{[-2]} [\mu_{r_1+r_2} \mu_{r_3+r_4} + \mu_{r_1+r_3} \mu_{r_2+r_4} + \mu_{r_1+r_4} \mu_{r_2+r_3}] \\
&+ N^{[-3]} [\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} + \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} + \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} \\
&+ \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} + \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}] \\
&+ N^{[-4]} \mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4} \} \\
&= \mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4} + \frac{1}{N} [\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} + \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} + \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} \\
&+ \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} + \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}] \\
&+ \frac{1}{N^2} [\mu_{r_1+r_2+r_3} \mu_{r_4} + \mu_{r_1+r_2+r_4} \mu_{r_3} + \mu_{r_1+r_3+r_4} \mu_{r_2} + \mu_{r_2+r_3+r_4} \mu_{r_1} + \mu_{r_1+r_2} \mu_{r_3+r_4} \\
&+ \mu_{r_1+r_3} \mu_{r_2+r_4} + \mu_{r_1+r_4} \mu_{r_2+r_3} - 3(\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} + \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} \\
&+ \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} + \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} + \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}) + 11\mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4}] \\
&+ \frac{1}{N^3} [\mu_{r_1+r_2+r_3+r_4} - (\mu_{r_1+r_2+r_3} \mu_{r_4} + \mu_{r_1+r_2+r_4} \mu_{r_3} + \mu_{r_1+r_3+r_4} \mu_{r_2} + \mu_{r_2+r_3+r_4} \mu_{r_1} \\
&+ \mu_{r_1+r_2} \mu_{r_3+r_4} + \mu_{r_1+r_3} \mu_{r_2+r_4} + \mu_{r_1+r_4} \mu_{r_2+r_3}) + 2(\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} \\
&+ \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} + \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} + \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} + \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}) - 6\mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4}],
\end{aligned}$$

and

$$\begin{aligned}
& E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2})(\mu'_{r_3} - \mu_{r_3})(\mu'_{r_4} - \mu_{r_4}) \\
&= \frac{1}{N^2} \{ (\mu_{r_1+r_2} \mu_{r_3+r_4} + \mu_{r_1+r_3} \mu_{r_2+r_4} + \mu_{r_1+r_4} \mu_{r_2+r_3}) - (\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} \\
&+ \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} + \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} + \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} + \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}) + 3\mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4} \} \\
&+ \frac{1}{N^3} \{ \mu_{r_1+r_2+r_3+r_4} - (\mu_{r_1+r_2+r_3} \mu_{r_4} + \mu_{r_1+r_2+r_4} \mu_{r_3} + \mu_{r_1+r_3+r_4} \mu_{r_2} + \mu_{r_2+r_3+r_4} \mu_{r_1}) \\
&- (\mu_{r_1+r_2} \mu_{r_3+r_4} + \mu_{r_1+r_3} \mu_{r_2+r_4} + \mu_{r_1+r_4} \mu_{r_2+r_3}) \\
&+ 2(\mu_{r_1+r_2} \mu_{r_3} \mu_{r_4} + \mu_{r_1+r_3} \mu_{r_2} \mu_{r_4} + \mu_{r_1+r_4} \mu_{r_2} \mu_{r_3} + \mu_{r_2+r_3} \mu_{r_1} \mu_{r_4} + \mu_{r_2+r_4} \mu_{r_1} \mu_{r_3} \\
&+ \mu_{r_3+r_4} \mu_{r_1} \mu_{r_2}) - 6\mu_{r_1} \mu_{r_2} \mu_{r_3} \mu_{r_4} \}.
\end{aligned}$$

The coefficient of $1/N^2$ in

$$E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2})(\mu'_{r_3} - \mu_{r_3})(\mu'_{r_4} - \mu_{r_4})$$

can be expressed in the form

$$\begin{aligned} & [\mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2}] [\mu_{r_3+r_4} - \mu_{r_3} \mu_{r_4}] + [\mu_{r_1+r_3} - \mu_{r_1} \mu_{r_3}] [\mu_{r_2+r_4} - \mu_{r_2} \mu_{r_4}] \\ & + [\mu_{r_1+r_4} - \mu_{r_1} \mu_{r_4}] [\mu_{r_2+r_3} - \mu_{r_2} \mu_{r_3}] \\ & = [E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2})] \cdot [E(\mu'_{r_3} - \mu_{r_3})(\mu'_{r_4} - \mu_{r_4})] \\ & + [E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_3} - \mu_{r_3})] \cdot [E(\mu'_{r_2} - \mu_{r_2})(\mu'_{r_4} - \mu_{r_4})] \\ & + [E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_4} - \mu_{r_4})] \cdot [E(\mu'_{r_2} - \mu_{r_2})(\mu'_{r_3} - \mu_{r_3})]^*. \end{aligned}$$

(2) In the general case, putting

$$\begin{aligned} \mu_{r_1} \mu_{r_2} \dots \mu_{r_i} &= \Pi_{(i)}, \\ \mu'_{r_1} \mu'_{r_2} \dots \mu'_{r_i} &= \Pi'_{(i)}, \\ r_1 + r_2 + \dots + r_i &= r, \end{aligned}$$

and agreeing to denote

$$\sum_{f_1=1}^{i-h+1} \sum_{f_2=f_1+1}^{i-h+2} \sum_{f_3=f_2+1}^{i-h+3} \dots \sum_{f_h=f_{h-1}+1}^i \text{ by } \sum_{f, (h)},$$

we find:

$$\begin{aligned} E\Pi'_{(i)} &= \frac{1}{N^i} \sum_{j=1}^i N^{[-j]} H_{i,j}^{(i)} = \frac{1}{N^i} \sum_{j=0}^{i-1} N^{[-(i-j)]} H_{i,i-j}^{(i)} = H_{i,i}^{(i)} \\ &+ \sum_{j=1}^{i-1} \frac{1}{N^j} \sum_{k=0}^j (-1)^k \beta_{i-j+k,k} H_{i,i-j+k}^{(i)}. \end{aligned}$$

Here,

$$H_{i,i}^{(i)} = \Pi_{(i)},$$

and $H_{i,i-j}^{(i)}$ may, when $2j \leq i$, be written in the form

$$H_{i,i-j}^{(i)} = \sum_{l=1}^j \sum_{f, (j+l)} K_{j,l}^{(i)} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{j+l}}}},$$

and, when $2j > i$, in the form

$$H_{i,i-j}^{(i)} = \sum_{l=1}^{i-j} \sum_{f, (j+l)} K_{j,l}^{(i)} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{j+l}}}},$$

where

$$K_{j,i}^{(i)} = \mu_{r_{f_1}+r_{f_2}+\dots+r_{f_{j+1}}},$$

and $K_{j,l}^{(i)}$ denotes the sum of all products of l factors of type $\mu_{h_1} \mu_{h_2} \dots \mu_{h_l}$ possible, subject to the condition that $h_1, h_2, \dots, h_l$ appear in their correct order, in sums of not less than two terms taken from the numbers $r_{f_1}, r_{f_2}, \dots, r_{f_{j+l}}$, and that the number of summands composing h_k is not greater than the number composing h_{k+1} . We have in this manner:

$$\begin{aligned} H_{i,i-1}^{(i)} &= \sum_{f_1=1}^{i-1} \sum_{f_2=f_1+1}^i \mu_{r_{f_1}+r_{f_2}} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}}}, \\ H_{i,i-2}^{(i)} &= \sum_{f_1=1}^{i-2} \sum_{f_2=f_1+1}^{i-1} \sum_{f_3=f_2+1}^i \mu_{r_{f_1}+r_{f_2}+r_{f_3}} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \mu_{r_{f_3}}} \\ &+ \sum_{f_1=1}^{i-3} \sum_{f_2=f_1+1}^{i-2} \sum_{f_3=f_2+1}^{i-1} \sum_{f_4=f_3+1}^i [\mu_{r_{f_1}+r_{f_2}} \mu_{r_{f_3}+r_{f_4}} + \mu_{r_{f_1}+r_{f_3}} \mu_{r_{f_2}+r_{f_4}} \\ &\quad + \mu_{r_{f_1}+r_{f_4}} \mu_{r_{f_2}+r_{f_3}}] \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \mu_{r_{f_3}} \mu_{r_{f_4}}}, \end{aligned}$$

* Cf. Soper, "On the Probable Error of the Correlation Coefficient to a Second Approximation," p. 97 (*Biometrika*, Vol. IX).

$$\begin{aligned}
H_{i, i-3}^{(i)} &= \sum_{f, (4)} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \mu_{r_{f_3}} \mu_{r_{f_4}}} \mu_{r_{f_1}+r_{f_2}+r_{f_3}+r_{f_4}} \\
&+ \sum_{f, (5)} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \mu_{r_{f_3}} \mu_{r_{f_4}} \mu_{r_{f_5}}} [\mu_{r_{f_1}+r_{f_2}} \mu_{r_{f_3}+r_{f_4}+r_{f_5}} + \mu_{r_{f_1}+r_{f_3}} \mu_{r_{f_2}+r_{f_4}+r_{f_5}} + \dots] \\
&+ \sum_{f, (6)} \frac{\Pi_{(i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \mu_{r_{f_3}} \mu_{r_{f_4}} \mu_{r_{f_5}} \mu_{r_{f_6}}} [\mu_{r_{f_1}+r_{f_2}} \mu_{r_{f_3}+r_{f_4}} \mu_{r_{f_5}+r_{f_6}} + \mu_{r_{f_1}+r_{f_3}} \mu_{r_{f_2}+r_{f_4}} \mu_{r_{f_5}+r_{f_6}} + \dots],
\end{aligned}$$

and, on the other hand

$$\begin{aligned}
H_{i, 1}^{(i)} &= \mu_{r_1+r_2+\dots+r_i} = \mu_r, \\
H_{2i, 2}^{(2i)} &= \sum_{j=1}^{2i} \mu_{r_j} \mu_{r-r_j} + \sum_{j_1=1}^{2i-1} \sum_{j_2=j_1+1}^{2i} \mu_{r_{j_1}+r_{j_2}} \mu_{r-[r_{j_1}+r_{j_2}]} + \dots \\
&+ \sum_{j_1=1}^{2i-k+1} \sum_{j_2=j_1+1}^{2i-k+2} \dots \sum_{j_k=j_{k-1}+1}^{2i} \mu_{r_{j_1}+r_{j_2}+\dots+r_{j_k}} \mu_{r-[r_{j_1}+r_{j_2}+\dots+r_{j_k}]} + \dots \\
&+ \sum_{j_1=1}^{i+1} \sum_{j_2=j_1+1}^{i+2} \dots \sum_{j_i=j_{i-1}+i}^{2i} \mu_{r_{j_1}+r_{j_2}+\dots+r_{j_i}} \mu_{r-[r_{j_1}+r_{j_2}+\dots+r_{j_i}]}, \\
H_{2i+1, 2}^{(2i+1)} &= \sum_{j=1}^{2i+1} \mu_{r_j} \mu_{r-r_j} + \sum_{j_1=1}^{2i} \sum_{j_2=j_1+1}^{2i+1} \mu_{r_{j_1}+r_{j_2}} \mu_{r-[r_{j_1}+r_{j_2}]} + \dots \\
&+ \sum_{j_1=1}^{i+2} \sum_{j_2=j_1+1}^{i+3} \dots \sum_{j_i=j_{i-1}+1}^{2i+1} \mu_{r_{j_1}+r_{j_2}+\dots+r_{j_i}} \mu_{r-[r_{j_1}+r_{j_2}+\dots+r_{j_i}]}.
\end{aligned}$$

In the case in which $r_1 = r_2 = r_3 = \dots = r_i$, the coefficients $H_{i, j}^{(i)}$ become $R_{(i, j)}$ and we get again formulae (2), (4) and (5) of the present chapter.

(3) Let us agree to denote

$$E(\mu'_{r_1} - \mu_{r_1})(\mu'_{r_2} - \mu_{r_2}) \dots (\mu'_{r_i} - \mu_{r_i}) \text{ by } E_{(i)} d\mu.$$

We have :

$$\begin{aligned}
E_{(i)} d\mu &= E\Pi'_{(i)} - \sum_{j, (1)} \mu_{r_{j_1}} E\{\Pi'_{(i)}/\mu'_{r_{j_1}}\} + \sum_{j, (2)} \mu_{r_{j_1}} \mu_{r_{j_2}} E\{\Pi'_{(i)}/\mu'_{r_{j_1}} \mu'_{r_{j_2}}\} - \dots \\
&+ (-1)^h \sum_{j, (h)} \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_h}} E\{\Pi'_{(i)}/\mu'_{r_{j_1}} \mu'_{r_{j_2}} \dots \mu'_{r_{j_h}}\} + \dots \\
&+ (-1)^{i-2} \sum_{j, (i-2)} \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_{i-2}}} E\{\Pi'_{(i)}/\mu'_{r_{j_1}} \mu'_{r_{j_2}} \dots \mu'_{r_{j_{i-2}}}\} \\
&+ (-1)^{i-1} (i-1) \Pi_{(i)}.
\end{aligned}$$

Using here the values found above for $E\Pi'_{(i)}$, $E\{\Pi'_{(i)}/\mu'_{r_j}\}$, and so on, and denoting by $K_l^{(r_{f_1}, r_{f_2}, \dots, r_{f_{l-k+1}})}$ the sum of all products of l factors of type $\mu_{h_1} \mu_{h_2} \dots \mu_{h_l}$ possible, subject to the conditions that $h_1, h_2, \dots, h_l$ appear in order, that the sums contain not less than two summands chosen from the numbers $r_1, r_2, \dots, r_{f_{l-k+1}}$, and that the number of summands in h_k is not less than in h_{k+1} , we find that the coefficient of $1/N^t$ in the development of $E_{(i)} d\mu$ equals :

$$\begin{aligned}
& (-1)^t \Pi_{(2i)} \sum_{h=0}^{2i-t-1} (-1)^h C_{2i}^h \beta_{2i-h,t} + \sum_{k=0}^{t-1} (-1)^k \beta_{2i-t+k,k} \\
& \quad \times \sum_{l=1}^{t-k \text{ (or } 2i-t+k)} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\
& - \sum_{j, (1)} \mu_{r_{j_1}} \sum_{k=0}^{t-1} (-1)^k \beta_{2i-t+k-1,k} \\
& \quad \times \sum_{l=1}^{t-k \text{ (or } 2i-1-t+k)} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)} / \mu_{r_{j_1}}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\
& + \sum_{j, (2)} \mu_{r_{j_1}} \mu_{r_{j_2}} \sum_{k=0}^{t-1} (-1)^k \beta_{2i-t+k-2,k} \\
& \quad \times \sum_{l=1}^{t-k \text{ (or } 2i-2-t+k)} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)} / \mu_{r_{j_1}} \mu_{r_{j_2}}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) + \dots \\
& + (-1)^k \sum_{j, (h)} \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_h}} \sum_{k=0}^{t-1} (-1)^k \beta_{2i-t+k-h,k} \\
& \quad \times \sum_{l=1}^{t-k \text{ (or } 2i-h-t+k)} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)} / \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_h}}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) + \dots \\
& + (-1)^{2i-t-1} \sum_{j, (2i-t-1)} \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_{2i-t-1}}} \sum_{k=0}^{t-1} (-1)^k \beta_{k+1,k} \\
& \quad \times \sum_{l=1}^{t-k \text{ (or } k+1)} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)} / \mu_{r_{j_1}} \mu_{r_{j_2}} \dots \mu_{r_{j_{2i-t-1}}}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}})
\end{aligned} \quad \dots (17),$$

where the summation for l extends to all positive integer values of l from 1 to the least of the upper limits.

By the aid of simple transformations the coefficient of $1/N^t$ in the development of $E_{(2i)}$ can be expressed in the form:

$$\begin{aligned}
& E (-1)^t \Pi_{(2i)} \sum_{h=0}^{2i-t-1} (-1)^h C_{2i}^h \beta_{2i-h,t} \\
& + \sum_{k=\text{Ent.}(t/2)}^{t-1} (-1)^k \sum_{l=1}^{t-k} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\
& \quad \times \sum_{h=0}^{2i-t-1} (-1)^h C_{2i-t+k-l}^h \beta_{2i-t+k-h,k} \\
& + \sum_{k=0}^{\text{Ent.}(t/2)-1} (-1)^k \sum_{l=1}^{k+1} \sum_{f, (t-k+l)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\
& \quad \times \sum_{h=0}^{2i-t-1} (-1)^h C_{2i-t+k-l}^h \beta_{2i-t+k-h,k} \\
& + \sum_{k=0}^{\text{Ent.}(t/2)-1} (-1)^k \sum_{j=0}^{t-2k-2} \sum_{f, (2t-2k-j)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2t-2k-j}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{2t-2k-j}}) \\
& \quad \times \sum_{h=0}^{2i-2t+2k+j} (-1)^h C_{2i-2t+2k+j}^h \beta_{2i-t+k-h,k}
\end{aligned} \quad \dots (18).$$

Noticing that (cf. Introduction, II, 3)

$$\begin{aligned} \sum_{h=0}^{r-k-1} (-1)^h C_m^h \beta_{r-h,k} &= 0 \quad \text{when } m > 2k, \\ \sum_{h=0}^{r-k-1} (-1)^h C_{2k}^h \beta_{r-h,k} &= 1.3.5 \dots (2k-1), \\ \sum_{h=0}^{r-k-1} (-1)^h C_{2k-n}^h \beta_{r-h,k} &= \sum_{g=0}^n B_{k,g} C_{r-2k+n}^{n-g}, \end{aligned}$$

where $B_{k,g}$ has the value given in the Introduction (12), (13) and (14), we see that the coefficient of $1/N^t$ in the development of $E_{(2i)} d\mu$ is zero, if $t < i$. The coefficient of $1/N^i$ equals

$$\begin{aligned} &(-1)^i \Pi_{(2i)} 1.3.5 \dots (2i-1) + \sum_{k=1}^{i-1} (-1)^k 1.3.5 \dots (2k-1) \\ &\quad \times \sum_{f, (2i-2k)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2i-2k}}}} M_{2,2,\dots,2}^{(i-k)} + M_{2,2,\dots,2}^{(i)} \dots (19), \end{aligned}$$

where by $M_{2,2,\dots,2}^{(i-k)}$ is denoted the sum of all the different products of type $\mu_{h_1} \mu_{h_2} \dots \mu_{h_{i-k}}$ possible, subject to the condition that $h_1, h_2, \dots, h_{i-k}$ appear as sums of pairs of the numbers $r_{f_1}, r_{f_2}, \dots, r_{f_{2i-2k}}$.

When $r_1 = r_2 = \dots = r_{2i}$, the aggregate of terms denoted by $M_{2,2,\dots,2}^{(i-k)}$ reduces to

$$(2i-2k-1)(2i-2k-3) \dots 5.3.1 \mu_{2r}^{i-k},$$

and

$$\begin{aligned} 1.3.5 \dots (2k-1) &\sum_{f, (2i-2k)} \frac{\Pi_{(2i)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2i-2k}}}} M_{2,2,\dots,2}^{(i-k)} \\ &= 1.3.5 \dots (2k-1) C_{2i}^{2i-2k} \mu_r^{2k} (2i-2k-1)(2i-2k-3) \dots 5.3.1 \mu_{2r}^{i-k} \\ &= 1.3.5 \dots (2i-1) C_i^{i-k} \mu_r^{2k} \mu_{2r}^{i-k}. \end{aligned}$$

The coefficient of $1/N^i$ consequently becomes (cf. (14) above):

$$1.3.5 \dots (2i-1) (\mu_{2r} - \mu_r^2)^i.$$

Similarly, we find that the coefficient of $1/N^t$ in the development of $E_{(2i+1)} d\mu$ reduces to

$$\begin{aligned} &(-1)^t \Pi_{(2i+1)} \sum_{h=0}^{2i-t} (-1)^h C_{2i+1}^h \beta_{2i+1-h,t} \\ &+ \sum_{k=\text{Ent.}(t/2)}^{t-1} (-1)^k \sum_{l=1}^{t-k} \frac{\Pi_{(2i+1)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\ &\quad \times \sum_{h=0}^{2i-t} (-1)^h C_{2i+1-t+k-l}^h \beta_{2i+1-t+k-l,h,k} \\ &+ \sum_{k=0}^{\text{Ent.}(t/2)-1} (-1)^k \sum_{l=1}^k \sum_{f, (t-k+l)} \frac{\Pi_{(2i+1)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{t-k+l}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{t-k+l}}) \\ &\quad \times \sum_{h=0}^{2i-t} (-1)^h C_{2i+1-t+k-l}^h \beta_{2i+1-t+k-l,h,k} \\ &+ \sum_{k=0}^{\text{Ent.}(t/2)-1} (-1)^k \sum_{j=0}^{t-2k-1} \sum_{f, (2t-2k-j)} \frac{\Pi_{(2i+1)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2t-2k-j}}}} K_l(r_{f_1}, r_{f_2}, \dots, r_{f_{2t-2k-j}}) \\ &\quad \times \sum_{h=0}^{2i-2t+2k+j+1} (-1)^h C_{2i-2t+2k+j+1}^h \beta_{2i-t+k-h+1,k} \end{aligned} \quad \dots (20).$$

When $t \leq i$ the coefficient of $1/N^t$ in the development of $E_{(2i+1)} d\mu$ consequently reduces to zero. When $t = i + 1$, noting that

$$\begin{aligned} \sum_{h=0}^{i-1} (-1)^h C_{2k}^h \beta_{i+k-h, k} &= \sum_{h=0}^{2k} (-1)^h C_{2k}^h \beta_{i+k-h, k} = 1.3.5 \dots (2k-1), \\ \sum_{h=0}^{i-1} (-1)^h C_{2k-1}^h \beta_{i+k-h, k} &= \sum_{g=0}^1 B_{k,g} C_{i-k+1}^{1-g} = 1.3.5 \dots (2k-1) \left\{ (i-k+1) + \frac{2}{3}(k-1) \right\} \\ &= \sum_{h=0}^{2k-1} (-1)^h C_{2k-1}^h \beta_{i+k-h, k}, \end{aligned}$$

we find :

$$\begin{aligned} &(-1)^{i+1} 1.3.5 \dots (2i+1) \frac{2}{3} i \Pi_{(2i+1)} \\ &+ \sum_{k=1}^i (-1)^k 1.3.5 \dots (2k-1) \left\{ \sum_{f, (2i-2k+1)} \frac{\Pi_{(2i+1)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2i-2k+1}}}} M_{2, 2, \dots, 2, 3}^{(i-k)} \right. \\ &+ \left[(i-k+1) + \frac{2}{3}(k-1) \right] \sum_{f, (2i-2k+2)} \frac{\Pi_{(2i+1)}}{\mu_{r_{f_1}} \mu_{r_{f_2}} \dots \mu_{r_{f_{2i-2k+2}}}} M_{2, 2, \dots, 2, 2}^{(i-k+1)} \left. \right\} \\ &+ M_{2, 2, \dots, 2, 3}^{(i)} \end{aligned} \quad (21).$$

where $M_{2, 2, \dots, 2, 2}^{(i-k+1)}$ has the value given above, and $M_{2, 2, \dots, 2, 3}^{(i-k)}$ denotes an analogous sum of terms of type $\mu_{h_1} \mu_{h_2} \dots \mu_{h_{i-k}}$ with the single difference that in the aggregate h_{i-k} there occur not two but three of the numbers $r_{f_1}, r_{f_2}, \dots, r_{f_{2i-2k+1}}$, while the remaining $2i-2k-2$ numbers of this series are distributed in pairs among $h_1, h_2, \dots, h_{i-k-1}$.

In the case $r_1 = r_2 = \dots = r_{2i+1}$, the coefficient of $1/N^{i+1}$ in the development of $E_{(2i+1)} d\mu$ becomes :

$$\begin{aligned} &(-1)^{i+1} 1.3.5 \dots (2i+1) \frac{2}{3} i \mu_r^{2i+1} + (-1)^i 1.3.5 \dots (2i-1) \left[1 + \frac{2}{3}(i-1) \right] C_{2i+1}^2 \mu_r^{2i-1} \mu_{2r} \\ &+ \sum_{k=1}^{i-1} (-1)^k 1.3.5 \dots (2k-1) \left\{ C_{2i+1}^{2i-2k+1} \mu_r^{2k} \mu_{2r}^{i-k-1} \mu_{3r}^3 C_{2i-2k+1}^3 1.3.5 \dots (2i-2k-3) \right. \\ &+ \left[(i-k+1) + \frac{2}{3}(k-1) \right] C_{2i+1}^{2i-2k+2} \mu_r^{2k-1} \mu_{2r}^{i-k+1} 1.3.5 \dots (2i-2k+1) \left. \right\} \\ &+ \mu_{2r}^{i-1} \mu_{3r}^3 C_{2i+1}^3 1.3.5 \dots (2i-3) \\ &= 1.3.5 \dots (2i+1) \frac{2}{3} [\mu_{2r} - \mu_r^2]^{i-1} \cdot [\mu_{3r} - 3\mu_{2r} \mu_r + 2\mu_r^3]. \end{aligned}$$

(Cf. (15) above.)

(To be continued.)

MISCELLANEA.

I. Preliminary Note on the Association of Steadiness and Rapidity of Hand with Artistic Capacity.

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(1) This preliminary note is based on observations made by Miss M. Dalglish at the request of Professor K. Pearson. As a teacher of drawing in a large school, she had a long experience of the amount of artistic imagination and artistic craft in her pupils, and was able to obtain appreciations of their other abilities. The categories she supplied for about 60 pupils were: (a) their ages*, (b) the number of years during which they had learnt drawing†, (c) their artistic or non-artistic capacity, the former being subdivided into imagination and craft, (d) their mathematical, and (e) their musical ability. The steadiness and rapidity of hand were to be tested by the well-known "maze"-problem. Three mazes were prepared, I, II, III, of varying degree of tortuosity. The nature of the problem was explained to the pupils. They were to enter the maze at *A* and leave at *Q*, a continuous pencil track being drawn from the point of entry to the point of exit. The performance was to be considered the more excellent the fewer the occasions on which the pencil track touched the boundaries of the maze path—such touching being termed a "bump." The ideal pencil track would keep steadily in the mid-path and parallel to its borders. No distinction however in this preliminary experiment was made between a non-bumping wavy line and an ideal track. The efficiency due to keeping clear of the boundaries was simply determined by the number of bumps. The minimum number of bumps of any girl in any one maze was one and the maximum 72. Further, the performance was to be considered the more satisfactory the greater the celerity with which the track was completed. The minimum time (taken with a stop-watch) of any pupil in any one of the three mazes was 18 secs. and the maximum time practically 3 mins. Contrary to what might by some be anticipated, there was not a high negative correlation between the number of bumps and the time taken. Although on the one hand an over-hasty temperament might lead to many bumps, on the other a certain celerity tends to straightness of path while hesitation leads to bumping. These points will be more easily grasped from the correlation results provided below.

(2) A question arises as to the relative difficulty of the three mazes measured in time and number of bumps, the average values are:

	Maze I	Maze II	Maze III
Average number of minutes taken	2.002 $\pm$.043	1.208 $\pm$.031	1.391 $\pm$.037
Average number of bumps made	20.68 $\pm$ 1.297	15.39 $\pm$.927	31.82 $\pm$ 1.392

These numbers, however, can hardly be taken as measuring the absolute difficulty of the three mazes, for (i) they are not of equal length, and (ii) they have not the same number of changes of direction. Approximately the following hold:

	Maze I	Maze II	Maze III
Length of mid-path	1025 mm.	700 mm.	730 mm.
Number of changes of direction	128	94	84

* Their mean ages was 14.43 years with a standard deviation of 2.07 years, the actual range being from 10 to 18 years.

† The mean number of years during which the pupil had learnt drawing was 3.84 with a standard deviation of 2.20, the actual range being from 1 to 8 years.

and thus we have:

	Maze I	Maze II	Maze III
Number of cms. described per minute ...	51·20	57·95	52·48
Number of bumps per ten changes of direction	1·616	1·637	3·788

Thus judged by time whether absolutely or relatively to its length Maze II was the easiest, Maze I the hardest. But against this must be set the fact that Maze I was taken first, and probably the pupils in this case proceeded with more caution. Absolutely and relatively to the number of changes of direction Maze III was the hardest. Maze I does not seem to have been harder than Maze II, if we judge its mean number of bumps relative to the number of changes of direction.

Our total numbers being few it seemed at first desirable in order to reduce the probable errors of our results to treat each trial as an independent event and thus reach a total of over 150 cases. This possibility is, however, excluded by the difference in difficulty of the mazes; pooling would have produced spurious correlation. We were thus compelled to work out correlations for each maze, or to pool the total achievement of each pupil. We have sometimes adopted one and sometimes the other method.

To obtain a single general standard of efficiency in maze description we have taken as a combined measure the inverse product of the time taken and the number of bumps made. We shall speak of this as the "inverse product" simply. It receives some sort of justification, when we note that the factors are not highly correlated and further when we note that we desire a measure which shall *increase* with efficiency.

The fundamental problem we had in view is the following: To what extent are steadiness and rapidity of hand as exhibited in maze-tracing the result of training? to what extent are they innate?

Before proceeding to the discussion of this problem we may note the variability in period of time and in number of bumps for the three mazes.

	Maze I		Maze II		Maze III	
	S.D.	C. of V.	S.D.	C. of V.	S.D.	C. of V.
Time in maze...	·479 ± ·031	23·91 ± 1·61	·340 ± ·022	28·15 ± 1·93	·410 ± ·026	29·48 ± 2·04
No. of bumps...	14·39 ± ·92	69·59 ± 6·22	10·29 ± ·66	66·85 ± 5·86	15·44 ± ·98	48·54 ± 3·75

These results seem to indicate a conformity with the general law, that the harder the test the greater is the scatter*, i.e. the weak fail more conspicuously and the able succeed more markedly. This is a law manifested in most stiff competitive examinations, or again in the difficulty of making marked distinctions in the case of easy papers. We are speaking here of the scatter or variability as measured *absolutely* by the standard deviation. It is noteworthy that the *relative* variability (or the variability as percentage of the mean value) as measured by the coefficient of variation appears in the case of the bumps to be less in the case of the harder maze. We are thus driven to the conclusion that the emphasis of the difference between the ineffectual and effectual in a given task while increasing with the stiffness of the task does not increase proportionally to that stiffness, but probably at some lesser rate.

(3) The first problem to be answered is: How far is steadiness of hand an individual characteristic at all? Will the same individual do well in one maze and badly in another? The answer to this problem lies in the pupils' correlation in efficiency in performances in different mazes. Now whether the characteristic be acquired by training or be innate we should anticipate a change with age. Most innate characteristics grow stronger or weaker with age, and this must be taken into account. The following table gives the chief age correlations:

* The high variabilities in the case of Maze I are we think due to the manner in which different individuals attempted a novel task.

TABLE I.
Correlations with Age.

Characters	Maze I	Maze II	Maze III
No. of bumps and age	- .587 ± .059	- .532 ± .065	- .549 ± .063
Time taken and age	- .241 ± .085	+ .003 ± .090	+ .152 ± .088
Inverse product and age	+ .402 ± .076	+ .317 ± .081	+ .467 ± .071
No. of bumps and age for constant time taken	- .607 ± .057	- .539 ± .064	- .542 ± .064
Time taken and age for No. of bumps constant	- .336 ± .080	- .101 ± .089	- .112 ± .089

This table shows at once a very considerable relationship between age and the number of bumps made: the steadiness of hand increases with age. On the other hand in the case of Mazes II and III no relationship between age and time taken was demonstrated; in Maze I, however, there was possibly a slight relationship between time taken and age, of the opposite sense, however, to the insignificant values in Mazes II and III, i.e. the lower the age the longer the time taken; this is not improbably due to the novelty factor involved in Maze I. On the whole we may reasonably conclude that the relation between time taken and age is not important. Confirmation of this arises in the case of the inverse product measure of efficiency and age. The efficiency increases with age, but because it includes time is not so marked as in the factor of bumps alone. The two remaining correlations indicate what happens if we eliminate respectively the influence of time taken and number of bumps. We hardly improve the relation between the number of bumps and age, if we make the time taken constant. On the other hand we get one significant but small correlation and two insignificant correlations, but all three are now of the same sign if we measure the relation between time taken and age for constant number of bumps. It is thus possible that there is a very slight relation between time taken and age—rapidity slightly increasing with age for a given degree of steadiness of hand. This leads us to the direct problem of the relationship between rapidity and steadiness of hand.

TABLE II.
Correlations of Rapidity and Steadiness.

	Maze I	Maze II	Maze III
No. of bumps and time taken	- .052 ± .090	- .165 ± .088	- .432 ± .073
No. of bumps and time taken for constant age	- .246 ± .085	- .193 ± .087	- .422 ± .074
No. of bumps and time taken for time learnt constant	- .070 ± .090	- .164 ± .088	- .444 ± .072

Without regard to age, it is only in the case of Maze III that we can assert that the number of bumps increases inversely with the time taken. Allowing for age the associations are more marked, but by no means as intense as we had anticipated. Further they seem to be dependent on the difficulty of the maze—i.e. the harder the maze the closer the relationship. *A priori* one might imagine that a slow transit would escape bumps—it is so, but not in a very emphatic manner. We suggest that a certain degree of rapidity is really helpful in avoiding bumps; it keeps a straight course in the straighter parts of the mazes, while it is rapidity at the angles which is calculated to produce bumps. There are probably therefore two factors at work.

We can now turn to the question of individuality in maze-tracing. We find the following correlations:

TABLE III.
Individuality in Maze-Description.

Correlations						Mazes I and II	Mazes I and III	Mazes II and III
No. of bumps	...	...	...	...	...	+ ·754 ± ·039	+ ·761 ± ·029	+ ·876 ± ·021
Time taken	...	...	...	...	...	+ ·594 ± ·058	+ ·436 ± ·073	+ ·807 ± ·030
Inverse product	...	...	...	...	...	+ ·703 ± ·046	+ ·661 ± ·051	+ ·695 ± ·047
No. of bumps, age constant	...	...	...	...	...	+ ·644 ± ·053	+ ·648 ± ·052	+ ·826 ± ·029
Time taken, age constant	...	...	...	...	...	+ ·613 ± ·056	+ ·493 ± ·068	+ ·816 ± ·030
Inverse product, age constant	...	...	...	...	...	+ ·663 ± ·051	+ ·585 ± ·059	+ ·652 ± ·052

These are very noteworthy correlations and it is evident that there is a very marked degree of individuality in maze performances, whether we judge steadiness, rapidity or the combination of both involved in the inverse product measure of efficiency. These high correlations it is true are lessened, but still large, if we allow for age*. The existence of this marked individuality brings us then to our main problem. Are steadiness and rapidity of hand—which to an appreciable extent increase during childhood—products of training, i.e. of environment, or innate characters developing with age?

(4) How far does the length of time during which drawing has been learnt influence the rapidity and steadiness of hand in maze-tracing? The correlations are provided in the accompanying table.

TABLE IV.
Influence of Time Learnt on Steadiness and Rapidity of Hand.

Correlations						Maze I	Maze II	Maze III
No. of bumps and time learnt	...	...	...	...	...	- ·216 ± ·086	- ·286 ± ·083	- ·209 ± ·086
Time taken and time learnt	...	...	...	...	...	- ·077 ± ·090	+ ·029 ± ·090	- ·010 ± ·090
Inverse product and time learnt	...	...	...	...	...	+ ·151 ± ·088	+ ·272 ± ·084	+ ·287 ± ·083
No. of bumps and time learnt for constant age	...	...	...	...	...	+ ·132 ± ·089	- ·008 ± ·090	+ ·089 ± ·089
Time taken and time learnt for constant age	...	...	...	...	...	+ ·053 ± ·090	+ ·032 ± ·090	- ·102 ± ·089
Inverse product and time learnt for constant age	...	...	...	...	...	- ·066 ± ·090	+ ·137 ± ·088	+ ·067 ± ·090

Thus while there is no sensible correlation between rapidity of hand and time during which drawing has been learnt, the small amount of correlation between steadiness of hand and time learnt disappears when we take these correlations for constant age. As far as this material is concerned, we see that steadiness and rapidity of hand are not the result of drawing practice, but are probably innate characteristics developing with age. This result is so important that it needs of course independent verification, but if true its suggestiveness is great. *For crafts in which these characteristics are essential, they can better be obtained by selection than by training.*

We have endeavoured to throw further light on this point by approaching the subject from other standpoints. The correlation between age and time learnt is + ·5051 ± ·0671, and it may be argued that time learnt in the early years of life is not of very great importance, and that this possibly accounts for the correlation being rather low. We accordingly confined our attention to the 33 children who did not learn drawing before 10 years of age. The correlation of age and time learnt now rises to + ·8555 ± ·0315†. We then dealt only with steadiness of hand, and took as

* The correlations of times taken are *increased* not lessened, but we have already drawn attention to some irregularity in the relationship of age and time taken.

† This is only very slightly *lowered* if we confine our attention to children who make the same total number of bumps, i.e. age and time learnt for constant steadiness of hand is + ·8254 ± ·0374.

our measure the inverse of the number of bumps made on the three mazes added together. We found steadiness of hand (i.e. inverse of total bumps) and time learnt now had the significant correlation of $+ \cdot4030 \pm \cdot0755$, while age and steadiness of hand gave a correlation of

$$+ \cdot4877 \pm \cdot0895.$$

We now corrected these last two correlations for age and time learnt respectively and found: steadiness of hand and time learnt for constant age $- \cdot0034 \pm \cdot1174$, age and steadiness of hand for time learnt constant $+ \cdot3015 \pm \cdot1067$. There is thus:

(a) no relation between steadiness of hand and time during which drawing has been learnt, if we correct for age;

(b) a definite relation between steadiness of hand and age, if we correct for length of time drawing has been learnt.

The corresponding correlations in the case of the whole population under discussion were:

$$\text{Steadiness of hand and time learnt for constant age} + \cdot0334 \pm \cdot0900,$$

$$\text{Age and steadiness of hand for time learnt constant} + \cdot3731 \pm \cdot0776,$$

in sensible agreement with those for the special population who had not begun drawing before 10 years, although in the latter case the correlation of age and time learnt was $+ \cdot8555$ as against $+ \cdot5051$ for the general population.

As far as these results go they confirm the view that steadiness of hand is an innate character developing with age, but having little association with training in drawing.

(5) We now turn to "craft" and "imagination" as factors in drawing ability. If these be correlated with efficiency in maze-tracing, it will not necessarily follow that efficiency in maze-tracing is associated with effective drawing training, as apart from length of training. Possibly craft and imagination in drawing are themselves in the first place innate characters, developing no doubt with age, but not necessarily intimately associated with time during which training has been given.

In dealing with "imagination" and "craft" the method of "biserial- r " was adopted. Poor craft contains the classes "minus," very bad and bad, and good craft, the classes medium, good and very good. Poor imagination contains the classes minus, very bad, bad and medium, and good imagination the classes good and very good*.

We have first to note the influences of age and time learnt on craft and imagination.

TABLE V.

Influence of Time Learnt and Age on Craft and Imagination.

Character pair	Value of correlation
Good imagination and age	$- \cdot479 \pm \cdot094$
Good imagination and time learnt	$+ \cdot033 \pm \cdot090$
Good craft and age	$- \cdot096 \pm \cdot112$
Good craft and time learnt	$+ \cdot166 \pm \cdot114$
Good imagination and age for constant time learnt ...	$- \cdot575 \pm [\cdot060]^\dagger$
Good imagination and time learnt for constant age ...	$+ \cdot363 \pm [\cdot078]$
Good craft and age for constant time learnt	$- \cdot217 \pm [\cdot086]$
Good craft and time learnt for constant age	$+ \cdot261 \pm [\cdot083]$

Now the absolute correlations are extremely interesting, there is no relation between time learnt and either imagination or craft; these factors of drawing capacity appear like steadiness and

* The choice of series was made solely with a view to obtaining not too small frequency in the smaller series.

† Probable errors of these partial correlations are given as rough estimates for they are calculated on the basis of all the component correlations having been found by the product-moment method.

rapidity of hand to be innate. The influence of age on craft is not significant, but age seems to weaken imagination, i.e. the younger children are more imaginative in their drawing work. If we turn to the partial correlations we see, however, the bearing of these results. For a constant age imagination is moderately influenced by the time learnt; but for a constant amount of training it depreciates more markedly with age. The result is that there is no apparent relation of imagination to training. The diminution of the innate character with age is really more influential than its growth with training. Again for constant age there is a very moderate influence of training on craftsmanship, but for a constant time learnt craft diminishes with age*. The result again is that innate change with age masters training, and unless training is persistent, good craft will lessen, so that the correlation with age is either negative or insensible. The small influence of time learnt on these factors of drawing efficiency is remarkable, and it is highly suggestive to see that in certain characters training may only suffice to prevent deterioration, and does not provide a marked expansion of efficiency.

We may now turn to the influence of imagination and craft on the steadiness and rapidity of hand exhibited in maze-tracing.

TABLE VI.

Influence of Craft and Imagination on Steadiness and Rapidity of Hand.

Characters	Maze I	Maze II	Maze III
Good imagination and no. of bumps ...	+ .239 ± .109	+ .187 ± .109	+ .252 ± .109
Good imagination and time taken ...	+ .005 ± .115	+ .029 ± .114	+ .167 ± .112
Good craft and no. of bumps ...	- .163 ± .113	- .109 ± .114	- .039 ± .115
Good craft and time taken ...	+ .500 ± .093	+ .104 ± .114	- .308 ± .107
Good imagination and no. of bumps for constant age	- .059 ± [.090]†	- .092 ± [.089]	- .014 ± [.090]
Good imagination and time taken for constant age	- .130 ± [.089]	+ .035 ± [.090]	+ .277 ± [.083]
Good craft and no. of bumps for constant age	- .272 ± [.084]	- .189 ± [.087]	- .110 ± [.089]
Good craft and time taken for constant age	+ .494 ± [.068]	+ .104 ± [.089]	- .298 ± [.082]

Now these results are of much interest and suggestive for further inquiry as soon as we are able to deal with much larger numbers. In the correlation uncorrected for age there would appear to be slight relation between good imagination and a large number of bumps, but it is only because the younger students are more imaginative. Corrected for age the correlations are all reversed, but are seen to be of no significance. Good imagination and time taken cannot be considered to have significant relationship either before or after correction for age. Thus the imaginative factor in drawing skill is not sensibly associated with rapidity or steadiness of hand.

With regard to craft there do appear to be significant associations, but they are clearly changing with growth of experience in maze-tracing. Uncorrected for age, there is no really significant association between good craft and steadiness of hand although the constancy of sign is to be noted. After correction for age, it would appear probable that a small association exists—good craft having the steadier hand. But the relationship appears to be weakening with experience and is hardly significant in the third maze. The same change makes itself manifest in the time taken. Those with good craft took the longer time in the first maze and there is quite a

* Miss Dalglish reports that she judged of the craft capacity of her pupils quite apart from their age or technical ability. Thus given two children of 9 and 16 years of age whom she had rated with the same grade of craft capacity, the elder child would (if teachable) be doing the better drawing work, having had as a rule longer training. But this increased technical power was not regarded in the craft grading.

† See second footnote, p. 174.

sensible degree of correlation; in the second maze the correlation is insensible, while in the third it has become *negative*. In other words in the third maze good craft has begun to tell. It would require far more material and prolonged experiment to be certain how far it is the hesitation of the good craftsman over a novel task (Maze I) or the greater difficulty of the third maze which has told in the favour of good craft in that case. All we can assert is that within the small range of our experiment the slight relation of good craft to steadiness of hand appears to decrease, while the relationship of good craft to rapidity of hand is only beginning to develop in the third experiment, and is then not of any substantial intensity. We have already seen that there is only a small association of good craft and time learnt, about enough to allow for the deterioration of craft with age. Hence the slight relationship suggested between good craft and rapidity of hand is not necessarily an argument in favour of such rapidity arising from training, it may well be the result of an association of the innate characters.

(6) The remark at the end of the last section leads us directly to the problem of whether other qualities than those of draughtsmanship can be directly associated with steadiness and rapidity of hand. *A priori* we think there is much to be said for both mathematical and musical capacity being innate*. The former except in the case of geometrical drawing gives small training for the hand, but it does enable the owner of the capacity to realise more or less vividly a conception of the desired perfect maze description. On the other hand music not only gives much finger practice, but in the case of special ability probably signifies an inherited flexibility of hand. In our division of the material we have made only two classes—those of the students who possessed marked ability in music and in mathematics were separated into small classes from the remainder—the mediocre, the non-mathematical and the non-musical. We then applied the biserial method.

TABLE VII.

Association of Mathematical and Musical Capacity with Steadiness and Rapidity of Hand.

Characteristics	Maze I	Maze II	Maze III
Mathematical capacity and no. of bumps	− .112 ± .139	− .216 ± .136	− .136 ± .138
Musical capacity and no. of bumps ...	− .091 ± .170	− .042 ± .170	− .088 ± .170
Mathematical capacity and time taken ...	− .608 ± .106	− .390 ± .126	− .214 ± .135
Musical capacity and time taken ...	− .303 ± .162	− .233 ± .165	− .029 ± .170
Mathematical capacity and no. of bumps for constant age	− .012 ± [·090]	− .148 ± [·088]	− .049 ± [·090]
Musical capacity and no. of bumps for constant age	− .042 ± [·090]	+ .012 ± [·090]	− .042 ± [·090]
Mathematical capacity and time taken for constant age	− .592 ± [·059]	− .396 ± [·076]	− .237 ± [·085]
Musical capacity and time taken for constant age	− .290 ± [·083]	− .234 ± [·085]	− .046 ± [·090]

Now none of the correlations of mathematical or musical capacity with steadiness of hand are in themselves significant, but as all six of them are of the same sign, we may possibly assert a slender absolute relation between both and steadiness of hand above the average. On the other hand the rapidity of hand shows at first definite and in the case of mathematics marked relationship at first with both mathematical and musical capacity. But this relationship seems rapidly to

* The correlation of mathematical capacity with age was + .175 ± .137 and of musical capacity with age + .098 ± .169, which are satisfactory as showing that the teachers really judged *capacity* and not *knowledge*; they are as far as they go also some evidence for mathematical and musical capacity also being innate characteristics. Direct evidence for the hereditary character of musical capacity may be found of course in the pedigree of the Bach family. It is less demonstrated in the case of mathematics, but the Gregories might be cited, and possibly one or two recent instances will occur to those familiar with the Cambridge Tripos Lists.

diminish with experience of maze-tracing. It is probable that the mathematicians had at first an advantage which was not maintained. It may be suspected that they realised better at the outset what was required; an appreciation also of time taken may have been a factor at the outset with the musicians. But both these advantages seem to diminish as the non-mathematical and non-musical gain experience. The above remarks refer to the total correlations; when corrected for age we see that the conclusions for steadiness of hand are confirmed; even for constant age it has no sensible relationship to either mathematical or musical capacity. Rapidity of hand does show relationship with mathematics in a more marked, and with music in a less marked degree when we correct for age. But again both the associations are lessening with experience; for the third maze the superiority of the good mathematicians is only very moderate and the superiority of the good musicians has become insensible. There can be little doubt that the more marked superiority for Maze I was due to a better appreciation of what was needed in a novel task.

(7) *Conclusions.* The material dealt with is admittedly slender, and was analysed only as a step towards more elaborate returns; it was made in order to determine what additional experiments should be tried. Our conclusions are therefore suggestive rather than dogmatic. We have not been able to associate in a marked degree rapidity and steadiness of hand as exhibited in maze-tracing with any training; we more than suspect them to be innate characteristics*. Good craft, mathematical ability and musical capacity seem to some not very marked extent associated with rapidity of hand, but it is noteworthy that even in these cases the advantage was rather an initial one and tended to weaken with experience. An apparently noteworthy point, which is well worth confirmation or contradiction, is that continued training may only just suffice to maintain a grade of efficiency, which deteriorates with age. It would be of much interest to demonstrate that training in some cases does not create or even develop a faculty, but maintains it at the higher range of efficiency which belongs to an earlier age. It is possible that the teacher cannot develop imagination in the later stages of youthful growth, but may be able to preserve the greater imagination of the child by proper training. Certain faculties may be most intense at certain stages of growth. If education seizes upon them at this age and maintains their then intensity, we may be apt to overlook their history, and suppose them created by the educational process. The point is worth a direct and more intensive investigation. Here it is only a suggestion.

I have to thank Professor Pearson for his assistance during the preparation of this paper.

II. Sur les moments de la fonction de corrélation normale de n variables†.

PAR SVERKER BERGSTRÖM, STOCKHOLM.

1. La fonction de corrélation normale de n variables peut s'écrire d'après le théorème célèbre de M. Pearson‡

$$Z = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma_1 \sigma_2 \dots \sigma_n \sqrt{R}} e^{-\frac{1}{2R} \sum_{p=1}^{p=n} \sum_{q=1}^{q=n} \frac{R_{pq}}{\sigma_p \sigma_q} X_p X_q} \dots\dots\dots (1),$$

[* After thirty years' experience in teaching in a drawing office I think it safe to say that within a fortnight it is possible to assert of the bulk of freshmen engineers whether or no they will be good draughtsmen at the end of their two to three years' course. The power of rapid, steady, uniform bold work is there in germ or it is not. Knowledge of method and accuracy of result may be acquired, but only to a minor extent can anyone acquire that which distinguishes a good from a mediocre draughtsman. K. P.]

[† The present paper reached the editor later than the three other memoirs dealing with allied topics published in this part of the Journal, but the methods adopted are of sufficient interest to justify its appearance in association with those papers.]

‡ Voir K. Pearson, *Phil. Trans.* t. 187 (1896) et t. 200 (1903).

où

$$R = \begin{vmatrix} 1 & r_{12} & \dots & r_{1n} \\ r_{21} & 1 & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{n1} & r_{n2} & \dots & 1 \end{vmatrix} \equiv \|r_{pq}\| \dots \dots \dots (2),$$

et R_{pq} est le complément algébrique ou le mineur de l'élément dans la ligne $p^{\text{ième}}$ et la colonne $q^{\text{ième}}$. Les variables X_p sont mesurées à partir de ses valeurs moyennes et σ_p en sont les dispersions; r_{pq} désigne le coefficient de corrélation entre X_p et X_q , d'où

$$r_{pq} = r_{qp}.$$

$$r_{pp} = 1.$$

Le moment

$$\mu_{a_1 a_2 \dots a_p}$$

est, par définition,

$$\int \int \dots \int X_1^{a_1} X_2^{a_2} \dots X_p^{a_p} z dX_1 dX_2 \dots dX_p \dots \dots \dots (3).$$

C'est la valeur moyenne de

$$X_1^{a_1} X_2^{a_2} \dots X_p^{a_p}.$$

Pour simplifier l'écriture nous emploierons des *coordonnées normales*

$$x_p = \frac{X_p}{\sigma_p} \dots \dots \dots (4).$$

Introduisons, de plus, les notations

$$a_{pq} = \frac{R_{pq}}{R} \dots \dots \dots (5),$$

$$z_0 = \frac{1}{\sqrt{R}} \dots \dots \dots (6).$$

La fonction de corrélation devient

$$z = \frac{z_0}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum a_{pq} x_p x_q} \dots \dots \dots (7).$$

Notre but sera à calculer la valeur moyenne

$$M[x_1^{a_1} x_2^{a_2} \dots x_p^{a_p}].$$

2. Considérons l'intégrale

$$J = \int \int \dots \int x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} z dx_1 dx_2 \dots dx_n.$$

Il est facile de vérifier que cette intégrale est uniformément convergente, pourvu que

$$|a_{pq}| > k > 0,$$

 k étant un nombre fixe.En dérivant l'intégrale J par rapport à a_{pq}^* , on aura

$$J_1 = \int \int \dots \int x_1^{a_1} x_2^{a_2} \dots x_p^{a_{p+1}} \dots x_q^{a_{q+1}} \dots x_n^{a_n} z dx_1 dx_2 \dots dx_n \dots \dots \dots (8).$$

Donc, si J est une fonction comme de a_{pq} , on pourra obtenir, par une dérivation, J_1 . Il s'ensuit que notre question se ramène au calcul des dérivées par rapport aux coefficients a_{pq} de l'intégrale

$$I = \int \int \dots \int z dx_1 dx_2 \dots dx_n \dots \dots \dots (9),$$

dont la valeur est, d'après M. Pearson,

$$\frac{z_0}{\sqrt{D}} \dots \dots \dots (10),$$

où

$$D = \|a_{pq}\|.$$

* Il faut se rappeler que $a_{pq} = a_{qp}$.

3. Introduisons maintenant les notations

$$F'_{p_1 q_1} = \left(\frac{\partial}{\partial a_{p_1 q_1}} + \frac{\partial}{\partial a_{q_1 p_1}} \right) F \dots \dots \dots (11),$$

$$F''_{p_1 q_1, p_2 q_2} = \left(\frac{\partial}{\partial a_{p_1 q_1}} + \frac{\partial}{\partial a_{q_1 p_1}} \right) \left(\frac{\partial}{\partial a_{p_2 q_2}} + \frac{\partial}{\partial a_{q_2 p_2}} \right) F,$$

$$\dots \dots \dots F^{(k)}_{p_1 q_1, p_2 q_2, \dots, p_k q_k} = \prod_{i=1}^{i=k} \left(\frac{\partial}{\partial a_{p_i q_i}} + \frac{\partial}{\partial a_{q_i p_i}} \right) \dots \dots \dots (12).$$

Nous dirons que l'on calcule, par ce procédé, la *dérivée symétrique* de F . La définition s'étend au cas où $p = q$, pourvu que l'on convienne que

$$F'_{pp} = 2 \frac{\partial F}{\partial a_{pp}}.$$

Dérivons ainsi successivement (8) et (9).

$$I'_{p_1 q_1} = -M [x_{p_1} x_{q_1}] = -\frac{z_0}{2\sqrt{D^3}} D'_{p_1 q_1} \dots \dots \dots (13),$$

$$I''_{p_1 q_1, p_2 q_2} = M [x_{p_1 q_1} x_{p_2 q_2}] = \frac{3}{4} \frac{z_0}{\sqrt{D^5}} D'_{p_1 q_1} D'_{p_2 q_2} - \frac{z_0}{2\sqrt{D^3}} D''_{p_1 q_1, p_2 q_2} \dots \dots \dots (14).$$

On est conduit à poser, tout généralement,

$$M [x_{p_1 q_1} \dots x_{p_k q_k}] = \frac{z_0}{2^k \sqrt{D}} \{ (2k-1)!! \Sigma_k^k - (2k-3)!! 2 \Sigma_k^{k-1} + \dots \pm (2k-i+1)!! 2^{i-1} \Sigma_k^{k-i+1} \mp \dots \pm 2^{k-1} \Sigma_k^1 \} \dots \dots \dots (15),$$

où, pour abrégé,

$$(2p+1)!! = (2p+1)(2p-1) \dots 3 \cdot 1 \dots \dots \dots (16),$$

et où Σ_k^i désigne une somme de termes contenant i dérivées symétriques divisées par D et dont la somme des exposants symboliques est égale à k ; il y a dans Σ_k^i autant de tels termes que l'on puisse grouper en i groupes k éléments, en ne tenant compte de l'ordre des groupes égaux. Ce nombre est

$$g_k^i = \Sigma \frac{k!}{m_1! (a_1!)^{m_1} m_2! (a_2!)^{m_2} \dots} \dots \dots \dots (17),$$

où

$$\left. \begin{array}{l} a_1 \neq a_2 \neq \dots \\ m_1 a_1 + m_2 a_2 + \dots = k \\ m_1 + m_2 + \dots = i \end{array} \right\} \dots \dots \dots (18).$$

La formule (15) peut se vérifier par la démonstration par récurrence. Nous n'y insistons pas; elle ne présente aucune difficulté.

4. Il faut considérer une dérivée partielle dont se composent, en premier lieu, les dérivées symétriques et puis les sommes Σ_k^i .

La dérivée

$$\frac{\partial^k D}{\partial a_{p_1 q_1} \partial a_{p_2 q_2} \dots \partial a_{p_k q_k}}$$

doit être, évidemment, au signe près peut-être, égale au complément algébrique du terme

$$a_{p_1 q_1} a_{p_2 q_2} \dots a_{p_k q_k}.$$

Le signe s'obtient par l'évaluation du nombre des inversions dans $p_1, p_2, \dots, p_k$ —soit i —et dans $q_1, q_2, \dots, q_k$ —soit i' .

Or, en vertu du théorème bien connu sur la connexité entre un mineur d'un déterminant et celui de son réciproque, et à cause de l'identité

$$D = \frac{1}{R},$$

ce complément algébrique peut s'écrire

$$\frac{1}{R^{n-k}} R^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k} \cdot R^{n-k-1} = \frac{R^{q_1 q_2 \dots q_k}}{R^{p_1 p_2 \dots p_k}},$$

où le déterminant au numérateur est formé par les éléments communs aux lignes $p_1, p_2, \dots, p_k$ et aux colonnes $q_1, q_2, \dots, q_k$ du déterminant R (2).

Donc
$$\frac{\partial^k D}{\partial a_{p_1 q_1} \dots \partial a_{p_k q_k}} = (-1)^{i+i'} \frac{R^{q_1 q_2 \dots q_k}}{R^{p_1 p_2 \dots p_k}} \dots \dots \dots (19).$$

On pourra d'ailleurs éviter la considération du nombre des inversions en substituant à

$$R^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k}$$

un déterminant dont la diagonale principale consiste des éléments

$$r_{p_1 q_1}, r_{p_2 q_2}, \dots, r_{p_k q_k},$$

et que nous désignerons en échangeant la lettre R contre Q . Permutons, en effet, dans le déterminant Q des lignes ou des colonnes: on ne change que le signe. Les indices du déterminant R n'ayant pas d'inversion, on voit que

$$Q^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k} = (-1)^{i+i'} R^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k}.$$

On aura donc enfin
$$\frac{\partial^k D}{\partial a_{p_1 q_1} \dots \partial a_{p_k q_k}} = \frac{1}{R} \cdot Q^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k} \dots \dots \dots (20),$$

où

$$Q^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k} = \begin{vmatrix} r_{p_1 q_1} & r_{p_1 q_2} & \dots & r_{p_1 q_k} \\ r_{p_2 q_1} & r_{p_2 q_2} & \dots & r_{p_2 q_k} \\ \dots & \dots & \dots & \dots \\ r_{p_k q_1} & r_{p_k q_2} & \dots & r_{p_k q_k} \end{vmatrix} \dots \dots \dots (21).$$

Si deux des nombres p (ou des q) sont égaux, le déterminant (21) contiendra deux lignes (ou colonnes) égales: il s'évanouira; et, en effet, la dérivée seconde d'un déterminant par rapport à deux éléments appartenant à la même ligne (ou à la même colonne) est nulle.

La dérivée symétrique du $k^{\text{ième}}$ ordre consistera d'après (12) de 2^k dérivées simples, telles que (20). On les trouvera en faisant changer de place les p et les q à même indice de toutes les manières possibles. En désignant ces changements par S , on aura

$$D^{(k)}_{p_1 q_1, p_2 q_2 \dots p_k q_k} = \frac{1}{R} S(Q^{q_1 q_2 \dots q_k}_{p_1 p_2 \dots p_k}) \dots \dots \dots (22).$$

Il faut remarquer que le terme formé de la diagonale principale de (21) se reproduira une fois et une seule dans chaque dérivée simple; il y en aura 2^k dans la dérivée (22).

5. Revenons à l'expression (15). Elle va se simplifier. Tout d'abord

$$z_0 = \sqrt{D}.$$

Puis, considérons un terme dans Σ_k^i ; soit par exemple

$$\frac{D^{(n_1)}}{D} \cdot \frac{D^{(n_2)}}{D} \dots \frac{D^{(n_i)}}{D} \dots \dots \dots (23)$$

où

$$n_1 + n_2 + \dots + n_i = k.$$

D'après (22)
$$\frac{D^{(n)}}{D^{p_{a_1} q_{a_1}, p_{a_2} q_{a_2} \dots p_{a_n} q_{a_n}}} = S \left(Q^{q_{a_1} q_{a_2} \dots q_{a_n}}_{p_{a_1} p_{a_2} \dots p_{a_n}} \right).$$

C'est un polynôme en r , homogène et de l'ordre n .

Cela étant, on voit aisément que l'expression (23), homogène aussi, est de l'ordre

$$n_1 + n_2 + \dots + n_i = k.$$

Le terme

$$r_{p_1 q_1}, r_{p_2 q_2}, \dots, r_{p_k q_k} \dots \dots \dots (24)$$

y aura le coefficient

$$2^{n_1 + n_2 + \dots + n_k} = 2^k.$$

Donc le terme (24) obtiendra dans Σ_k^i le coefficient $2^k g_k^i$.

Généralement, en mettant dans l'expression (15), on voit qu'elle est une fonction homogène de l'ordre k en r .

La nature du problème exige que cette fonction est aussi symétrique.

Permutons, en effet, les indices du premier membre de (15); le résultat ne devant pas changer pour cela, le terme (24), y permutés les indices de la même manière, conserve le même coefficient.

6. Il ne nous faut donc considérer qu'un seul terme de (15), soit (24).

Ce coefficient est

$$(2k-1)!! g_k^k - (2k-3)!! 2g_k^{k-1} + (2k-5)!! 2^2 g_k^{k-2} - \dots \pm 2^{k-1} g_k^1 \\ \equiv \sum_{i=1}^{i=k} (-1)^{i-1} (2k-2i+1)!! 2^{i-1} g_k^{k-i+1} \dots \dots \dots (25).$$

Je dis que cette expression est égale à l'unité, quel que soit k.

Je vais employer le raisonnement par récurrence.

L'assertion étant évidente pour $k=1$, je la suppose démontrée pour k , et nous allons voir qu'elle subsistera encore pour $k+1$, c'est-à-dire que l'expression

$$\sum_{i=1}^{i=k+1} (-1)^{i-1} (2k-2i+3)!! 2^{i-1} g_{k+1}^{k-i+2} \dots \dots \dots (26)$$

sera égale à l'unité.

Envisageons d'abord les nombres g_n^i .

Cherchons g_{n+1}^i en supposant connu g_n^i .

On l'élément $(n+1)$ ième doit être placé dans l'un des i groupes formés de n éléments: il y en a

$$i g_n^i$$

éventualités.

On cet élément seul doit faire un groupe: les autres éléments formant alors $(i-1)$ groupes: on aura ainsi

$$g_n^{i-1}$$

éventualités.

D'où la formule

$$g_{n+1}^i = i g_n^i + g_n^{i-1} \dots \dots \dots (27).$$

En convenant que

$$g_n^{n+1} = 0, g_n^0 = 0,$$

cette formule subsiste encore pour

$$i = n + 1$$

et

$$i = 1.$$

L'expression (26) peut s'écrire maintenant, en vertu de (27),

$$\sum_{i=1}^{i=k+1} (-1)^{i-1} (2k-2i+3)!! 2^{i-1} \{ (k-i+2) g_k^{k-i+2} + g_k^{k-i+1} \} \\ = \sum_{i=2}^{i=k+1} (-1)^{i-1} (2k-2i+3)!! 2^{i-1} (k-i+2) g_k^{k-i+2} \\ + \sum_{i=1}^{i=k} (-1)^{i-1} (2k-2i+3)!! 2^{i-1} g_k^{k-i+1}.$$

Changeant dans la première somme i en $(i+1)$, on trouvera enfin pour (26)

$$\sum_{i=1}^{i=k} (-1)^{i-1} (2k-2i+1)!! 2^{i-1} \{ -2(k-i+1) + (2k-2i+3) \} g_k^{k-i+1}.$$

Ce n'est autre chose que (25); donc la somme (26) est bien égale à l'unité. C.Q.F.D.

$$M[x_1^{a_1} x_2^{a_2}] = \frac{a_1!}{(a_1 - a_2)!!} r_{12}^{a_2} + \frac{a_1!}{(a_1 - a_2 + 2)!!} \frac{a_2(a_2 - 1)}{2!!} r_{12}^{a_2 - 2} \\ + \frac{a_1!}{(a_1 - a_2 + 4)!!} \frac{a_2(a_2 - 1)(a_2 - 2)(a_2 - 3)}{4!!} r_{12}^{a_2 - 4} + \dots \dots \dots (32).$$

J'écris enfin les moments jusqu'au sixième ordre, en employant la notation

$$\beta_{a_1 a_2 \dots a_p} = M[x_1^{a_1} x_2^{a_2} \dots x_p^{a_p}].$$

$$\beta_2 = 1;$$

$$\beta_{11} = r_{12}.$$

$$\beta_4 = 3;$$

$$\beta_{31} = \beta_{13} = 3r_{12}, \beta_{22} = 2r_{12}^2 + 1;$$

$$\beta_{211} = 2r_{12}r_{13} + r_{23};$$

$$\beta_{1111} = r_{12}r_{34} + r_{13}r_{24} + r_{14}r_{23}.$$

$$\beta_6 = 15;$$

$$\beta_{51} = 15r_{12}, \beta_{42} = 12r_{12}^2 + 3, \beta_{33} = 6r_{12}^2 + 9r_{12};$$

$$\beta_{411} = 12r_{12}r_{13} + 3r_{23}, \beta_{321} = 6r_{12}^2r_{13} + 6r_{12}r_{23} + 3r_{13}, \beta_{222} = 8r_{12}r_{13}r_{23} + 2r_{12}^2 + 2r_{13}^2 + 2r_{23}^2 + 1;$$

$$\beta_{3111} = 6r_{12}r_{13}r_{14} + 3r_{12}r_{34} + 3r_{13}r_{24} + 3r_{14}r_{23}, \beta_{2211} = 2r_{12}^2r_{34} + 4r_{12}r_{13}r_{24} + 4r_{12}r_{14}r_{23} + 2r_{13}r_{14} \\ + 2r_{23}r_{24} + r_{34};$$

$$\beta_{21111} = 2r_{12}r_{13}r_{45} + 2r_{12}r_{14}r_{35} + 2r_{12}r_{15}r_{34} + 2r_{13}r_{14}r_{25} + 2r_{13}r_{15}r_{24} + 2r_{14}r_{15}r_{23} + r_{23}r_{45} + r_{24}r_{35} \\ + r_{25}r_{34};$$

$$\beta_{111111} = r_{12}r_{34}r_{56} + r_{13}r_{24}r_{56} + \dots (15 \text{ termes}).$$

Les moments jusqu'à l'ordre 4 à deux variables ont été calculés par M. Pearson et puis par M. Soper qui a donné une formule générale pour les moments à deux variables; *Biometrika*, t. IX, 1913, p. 101. (On doit remarquer cependant que sa formule (xxxii) est atteinte d'une erreur typographique.)

Le cas de trois variables a été traité par M. Wicksell dans "The general characteristics of the frequency function of stellar movements, etc." Lund 1915, p. 11.

Enfin M. Isserlis a déduit notre formule (28) pour le cas $2^k = 4$ ou de quatre variables dans *Biometrika*, t. XI, 1916, p. 189.

III. Formulae for determining the Mean Values of Products of Deviations of mixed Moment Coefficients in two to eight Variables in Samples taken from a limited Population.

By L. ISSERLIS, D.Sc.*

A. Let p_{12}, p_{34} denote the product moment coefficients, referred to a *fixed origin* in a sample of size n extracted out of a population of size N , there being *four* independent variables x_1, x_2, x_3, x_4 . The mean value of p_{12} in many samples is P_{12} , the corresponding product moment coefficient for the sampled population. Let $dp_{12} = p_{12} - P_{12}$, denote the deviation of the moment coefficient of the sample from its mean value, then

Mean value of $dp_{12} dp_{34}$ in many samples

$$= \frac{\chi}{n} (p_{1234} - P_{12}P_{34}),$$

where

$$\chi = \frac{N - n}{N - 1},$$

and p_{1234} is the product moment coefficient with respect to the four variables.

[* Dr Isserlis sent me a paper containing the results of the present note with others accompanied by proofs in the course of 1916. It has been impossible to publish that paper so far, but it is only fair to him, having regard to the fact that other investigators are now entering this field, to publish his formulae in association with the memoirs printed in this part of *Biometrika*. EDITOR.]

This result gives many particular cases if we identify two or more of the variables. For instance

$$\text{Mean value of } dp_1^2 dp_2^2 = (p_{1^2 2^2} - p_1^2 p_2^2) \frac{\chi}{n},$$

where p_1^2 is the second moment coefficient with regard to x , i.e. μ'_2 in the usual notation for one variable and so forth. Also

$$\text{Mean value of } (dp_1^2)^2 = \frac{\chi}{n} (p_1^4 - p_1^2^2).$$

B. Similarly if there are 6 independent variables $x_1, x_2, \dots, x_5, x_6$.

The mean value of $dp_{12} dp_{34} dp_{56}$

$$= \frac{\chi \chi'}{n^2} \{ p_{123456} - p_{1234} p_{56} - p_{3456} p_{12} - p_{1256} p_{34} + 2 p_{12} p_{34} p_{56} \},$$

where

$$\chi' = \frac{N - 2n}{N - 2}.$$

Particular cases are

$$\text{Mean value of } (dp_1^2)^3 = \frac{\chi \chi'}{n^2} (p_1^6 - 3 p_1^4 p_1^2 + 2 p_1^2^3),$$

or

$$\text{Mean value of } (d\mu'_2)^3 = \frac{\chi \chi'}{n^2} (\mu'^6_2 - 3 \mu'^4_2 \mu'^2_2 + 2 \mu'^2_2^3),$$

and Mean value of $dp_{12} dp_{23} dp_{31} = \frac{\chi \chi'}{n^2} [p_{1^2 2^2 3^2} - p_{12^2 3} p_{31} - p_{23^2 1} p_{12} - p_{31^2 2} p_{23} + 2 p_{12} p_{23} p_{31}]$.

C. Let there be 8 independent variables $x_1, x_2, \dots, x_7, x_8$.

The mean value of $dp_{12} dp_{34} dp_{56} dp_{78}$

$$\begin{aligned} &= \frac{\chi \phi}{n^2} [p_{1234} p_{5678} + p_{1256} p_{3478} + p_{1278} p_{3456} - p_{1234} p_{56} p_{78} - p_{5678} p_{12} p_{34} \\ &\quad - p_{1256} p_{34} p_{78} - p_{3478} p_{12} p_{56} - p_{1278} p_{34} p_{56} - p_{3456} p_{12} p_{78} + 3 p_{12} p_{34} p_{56} p_{78}] \\ &\quad + \frac{\chi \phi'}{n^3} [p_{12345678} + p_{1234} p_{5678} + p_{1256} p_{3478} + p_{1278} p_{3456} - p_{123456} p_{78} - p_{123478} p_{56} \\ &\quad - p_{125678} p_{34} - p_{345678} p_{12}]. \end{aligned}$$

where

$$\phi = 2 - 4\chi' + 3\chi'' + 4\chi'/n - 6\chi''/n,$$

$$\frac{\phi'}{n} = -1 + 3\chi' - 2\chi'' - 3\chi'/n + 4\chi''/n,$$

and

$$\chi'' = (N - 3n)/(N - 3).$$

When the sampled population is infinitely great

$$\phi' = \chi = \chi' = \chi'' = 1, \quad \phi = 1 - 2/n.$$

As a particular case,

$$\text{Mean value of } (dp_1^2)^4 = \frac{\chi \phi}{n^2} (3 p_1^4^2 - 6 p_1^4 p_1^2^2 + 3 p_1^2^4) + \frac{\chi \phi'}{n^3} (p_1^8 + 3 p_1^4^2 - 4 p_1^6 p_1^2),$$

or in the notation usual for a single variable,

$$\text{Mean value of } (d\mu'_2)^4 = \frac{\chi \phi}{n^2} (3 \mu'^4_2^2 - 6 \mu'^4_2 \mu'^2_2^2 + 3 \mu'^2_2^4) + \frac{\chi \phi'}{n^3} (\mu'^8_2 + 3 \mu'^4_2^2 - 4 \mu'^6_2 \mu'^2_2).$$

When the sampled population is normal the results of (A), (B) and (C) can be immediately expressed in terms of the correlation coefficients $r_{12}, r_{13}, \dots, r_{78}$, by means of the formulae established on p. 138 in the current issue of this journal.

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CHAPTER III

I

(1) Let us put

$$E \frac{1}{N} \sum_{i=1}^N [X_i - X_{(N)}]^r = \nu_{r, (N)} \dots \dots \dots (1).$$

We have

$$\nu_{1, (N)} = 0.$$

Noting that

$$E (X_i - X_{(N)})^r = E [X_j - X_{(N)}]^r,$$

we find

$$\nu_{r, (N)} = E [X_i - X_{(N)}]^r = E [X_1 - X_{(N)}]^r.$$

Replacing $X_1 - X_{(N)}$ by $(X_1 - m_1) - (X_{(N)} - m_1)$, we find

$$\nu_{r, (N)} = \mu_r + \sum_{j=1}^{r-1} (-1)^j C_r^j E (X_1 - m_1)^{r-j} (X_{(N)} - m_1)^j + (-1)^r \mu_{r, (N)}.$$

But $[X_{(N)} - m_1]^j = \frac{1}{N^j} [(X_1 - m_1) + (N-1)(X_{(N-1)} - m_1)]^j,$

where

$$X_{(N-1)} = \frac{1}{N-1} \sum_{i=2}^N X_i.$$

Hence

$$\left. \begin{aligned} \nu_{r, (N)} &= \mu_r + \sum_{j=1}^{r-1} \frac{(-1)^j}{N^j} C_r^j \sum_{h=0}^j C_j^h (N-1)^h \mu_{r-h} \mu_{h, (N-1)} + (-1)^r \mu_{r, (N)} \\ &= \mu_r + \sum_{j=1}^{r-1} \frac{(-1)^j}{N^j} C_r^j \mu_r + \sum_{j=2}^{r-2} \sum_{h=2}^j (-1)^j \frac{(N-1)^h}{N^j} C_r^j C_j^h \mu_{r-h} \mu_{h, (N-1)} \\ &\quad + (-1)^r \frac{1}{N^r} \left\{ \mu_r + \sum_{h=2}^{r-2} C_r^h (N-1)^h \mu_{r-h} \mu_{h, (N-1)} + (N-1)^r \mu_{r, (N-1)} \right\} \\ &= \left(\frac{N-1}{N} \right)^r \left\{ \mu_r + \sum_{h=2}^{r-2} (-1)^h C_r^h \mu_{r-h} \mu_{h, (N-1)} + (-1)^r \mu_{r, (N-1)} \right\} \\ &\quad = \left(\frac{N-1}{N} \right)^r \sum_{h=0}^r (-1)^h C_r^h \mu_{r-h} \mu_{h, (N-1)} \end{aligned} \right\} (2).$$

On the other hand, replacing $X_1 - X_{(N)}$ by

$$\frac{1}{N} [(N-1) X_1 - (N-1) X_{(N-1)}] = \frac{N-1}{N} [X_1 - X_{(N-1)}],$$

we find:

$$\nu_{r, (N)} = \left(\frac{N-1}{N} \right)^r \sum_{h=0}^r (-1)^h C_r^h m_{r-h} m_{h, (N-1)} \dots \dots \dots (3).$$



Replacing in (2) quantities of type $\mu_{h, (N-1)}$ by their values in terms of the quantities μ (cf. Chapter I, formulae (15), (17) and (18)) we obtain :

$$\nu_{2r, (N)} = \left(\frac{N-1}{N} \right)^{2r} \left\{ \mu_{2r} + \sum_{h=1}^{r-1} C_{2r}^{2h} \mu_{2r-2h} \sum_{i=0}^{h-1} \frac{1}{(N-1)^{h+i}} \sum_{j=0}^i (-1)^j \beta_{h-i+j, j} T_{2h, h-i+j} \right. \\ \left. - \sum_{h=1}^{r-2} C_{2r}^{2h+1} \mu_{2r-2h-1} \sum_{i=0}^{h-1} \frac{1}{(N-1)^{h+i+1}} \sum_{j=0}^i (-1)^j \beta_{h-i+j, j} T_{2h+1, h-i+j} \right. \\ \left. + \sum_{i=0}^{r-1} \frac{1}{(N-1)^{r+i}} \sum_{j=0}^i (-1)^j \beta_{r-i+j, j} T_{2r, r-i+j} \right\} \quad (4),$$

$$\nu_{2r+1, (N)} = \left(\frac{N-1}{N} \right)^{2r+1} \left\{ \mu_{2r+1} \right. \\ \left. + \sum_{h=1}^{r-1} C_{2r+1}^{2h} \mu_{2r+1-2h} \sum_{i=0}^{h-1} \frac{1}{(N-1)^{h+i}} \sum_{j=0}^i (-1)^j \beta_{h-i+j, j} T_{2h, h-i+j} \right. \\ \left. - \sum_{h=1}^{r-1} C_{2r+1}^{2h+1} \mu_{2r-2h} \sum_{i=0}^{h-1} \frac{1}{(N-1)^{h+i+1}} \sum_{j=0}^i (-1)^j \beta_{h-i+j, j} T_{2h+1, h-i+j} \right. \\ \left. - \sum_{i=0}^{r-1} \frac{1}{(N-1)^{r+i+1}} \sum_{j=0}^i (-1)^j \beta_{r-i+j, j} T_{2r+1, r-i+j} \right\} \quad \dots (5),$$

and, hence,

$$\nu_{r, (N)} = \mu_r - \frac{1}{N} [r\mu_r - \frac{1}{2}r^{[-2]} \mu_{r-2} \mu_2] \\ + \frac{1}{N^2} \left[\frac{1}{2}r^{[-2]} \mu_r - \frac{r-1}{2} r^{[-2]} \mu_{r-2} \mu_2 - \frac{1}{6}r^{[-3]} \mu_{r-3} \mu_3 + \frac{1}{8}r^{[-4]} \mu_{r-4} \mu_2^2 \right] + \dots \quad (6).$$

As N increases the ratio $\frac{\nu_{r, (N)}}{\mu_r}$ tends in this manner to the limit 1.

(2) Putting $r = 2, 3, 4, 5, 6$, in (2) we find without difficulty

$$\nu_{2, (N)} = \frac{N-1}{N} \mu_2 = \mu_2 - \frac{1}{N} \mu_2 \\ \nu_{3, (N)} = \frac{(N-1)(N-2)}{N^2} \mu_3 = \mu_3 - \frac{3}{N} \mu_3 + \frac{2}{N^2} \mu_3 \\ \nu_{4, (N)} = \mu_4 - \frac{2}{N} [2\mu_4 - 3\mu_2^2] + \frac{3}{N^2} [2\mu_4 - 5\mu_2^2] - \frac{3}{N^3} [\mu_4 - 3\mu_2^2]^* \\ \nu_{5, (N)} = \mu_5 - \frac{5}{N} [\mu_5 - 2\mu_3 \mu_2] + \frac{10}{N^2} [\mu_5 - 5\mu_3 \mu_2] - \frac{10}{N^3} [\mu_5 - 8\mu_3 \mu_2] \\ + \frac{4}{N^4} [\mu_5 - 10\mu_3 \mu_2] \\ \nu_{6, (N)} = \mu_6 - \frac{3}{N} [2\mu_6 - 5\mu_4 \mu_2] + \frac{5}{N^2} [3\mu_6 - 15\mu_4 \mu_2 - 4\mu_3^2 + 9\mu_2^3] \\ - \frac{5}{N^3} [4\mu_6 - 33\mu_4 \mu_2 - 16\mu_3^2 + 42\mu_2^3] + \frac{5}{N^4} [3\mu_6 - 36\mu_4 \mu_2^2 - 22\mu_3^2 + 63\mu_2^3] \\ - \frac{5}{N^5} [\mu_6 - 15\mu_4 \mu_2 - 10\mu_3^2 + 30\mu_2^3] \quad (7).$$

* For footnote see page 187.

(3) If the law of distribution of the values of X satisfies the condition :

$$\mu_{2i+1} = 0, \quad i = 0, 1, 2, \dots \infty,$$

then, as we saw above, $\mu_{2i+1, (N)} = 0, \quad i = 0, 1, 2, \dots \infty$ and, as appears from (5), $\nu_{2i+1, (N)} = 0, \quad i = 0, 1, 2, \dots \infty$. When the law of distribution of the values of X is Gaussian and $\mu_{2i} = 1.3.5 \dots (2i-1) \mu_2^i, \quad i = 1, 2, \dots \infty$, then, as we have seen $\mu_{2i, (N)} = 1.3.5 \dots (2i-1) \mu_2^i \frac{1}{N^i}$ and, consequently,

$$\left. \begin{aligned} \nu_{2i, (N)} &= \left(\frac{N-1}{N} \right)^{2i} \sum_{h=0}^i C_{2i}^{2h} \mu_{2i-2h} \mu_{2h, (N-1)} \\ &= \left(\frac{N-1}{N} \right)^{2i} \sum_{h=0}^i C_{2i}^{2h} \frac{1.3.5 \dots (2i-2h-1) 1.3.5 \dots (2h-1)}{(N-1)^h} \mu_2^i \\ &= \left(\frac{N-1}{N} \right)^{2i} 1.3.5 \dots (2i-1) \mu_2^i \sum_{h=0}^i C_i^h \frac{1}{(N-1)^h} \\ &= \left(\frac{N-1}{N} \right)^i \mu_{2i} = (N-1)^i \mu_{2i, (N)} \end{aligned} \right\} \quad (8).$$

II

(1) Let us put

$$\left. \begin{aligned} \nu'_{r, (N)} &= \frac{1}{N} \sum_{i=1}^N [X_i - X_{(N)}]^r \\ W_{r, (N)}^{(m)} &= E \left\{ \sum_{i=1}^N [X_i - X_{(N)}]^r \right\}^m \end{aligned} \right\} \dots\dots\dots (9).$$

We have :

$$\left. \begin{aligned} E\nu'_{r, (N)} &= \nu_{r, (N)} = \frac{1}{N} W_{r, (N)}^{(1)} \\ E[\nu'_{r, (N)}]^m &= \frac{1}{N^m} W_{r, (N)}^{(m)} \\ E[\nu'_{r, (N)} - \nu_{r, (N)}]^m &= \sum_{h=0}^m (-1)^h \frac{1}{N^{m-h}} C_m^h W_{r, (N)}^{(m-h)} \nu_{r, (N)}^h \end{aligned} \right\} \dots\dots\dots (10).$$

(2) When $m = 2$ we find :

$$\left. \begin{aligned} W_{r, (N)}^{(2)} &= E \left\{ \sum_{i=1}^N [X_i - X_{(N)}]^r \right\}^2 \\ &= E \sum_{i=1}^N [X_i - X_{(N)}]^{2r} + E \sum_{i=1}^N \sum_{j \neq i} [X_i - X_{(N)}]^r [X_j - X_{(N)}]^r \\ &= N\nu_{2r, (N)} + N(N-1) E[X_1 - X_{(N)}]^r [X_2 - X_{(N)}]^r \end{aligned} \right\} \dots\dots\dots (11).$$

* The quantity $\nu_{4, (N)}$ can also be written in the form

$$\nu_{4, (N)} = \frac{(N-1)(N-2)}{N^2} \mu_4 - \frac{(N-1)(N-3)}{N^3} (\mu_4 - 3\mu_2^2) + 3 \frac{(N-1)}{N^2} \mu_2^2.$$

R. Henderson ("Frequency Curves and Moments," *J. I. A.*, 1907), while giving correctly the values $\mu_{2, (N)}, \mu_{3, (N)}, \mu_{4, (N)}, \nu_{2, (N)}$ and $\nu_{3, (N)}$, erroneously gives (p. 435) :

$$\nu_{4, (N)} = \frac{(N-1)(N-2)}{N^2} \mu_4 - \frac{(N-1)(N-3)}{N^3} (\mu_4 - 3\mu_2^2).$$

The true value of $\nu_{4, (N)}$ exceeds the value obtained by Henderson by $\frac{3(N-1)}{N^2} \mu_2^2$.

Putting
$$X_{(N-2)} = \frac{1}{N-2} \sum_{i=3}^N X_i,$$

we have

$$\begin{aligned} [X_1 - X_{(N)}][X_2 - X_{(N)}] &= \frac{1}{N^2} \{[(N-1)(X_1 - m_1) - (X_2 - m_1) \\ &- (N-2)(X_{(N-2)} - m_1)][(N-1)(X_2 - m_1) - (X_1 - m_1) - (N-2)(X_{(N-2)} - m_1)]\} \\ &= f(X_1, X_2) - \left(\frac{N-2}{N}\right)^2 [(X_1 - m_1) + (X_2 - m_1)][X_{(N-2)} - m_1] + \left(\frac{N-2}{N}\right)^2 [X_{(N-2)} - m_1]^2, \end{aligned}$$

where

$$f(X_1, X_2) = [X_1 - m_1][X_2 - m_1] - \frac{N-1}{N^2} [(X_1 - m_1) + (X_2 - m_1)]^2.$$

Hence :

$$\begin{aligned} E[X_1 - X_{(N)}]^r [X_2 - X_{(N)}]^r &= E\{f(X_1, X_2)\}^r + \sum_{h=1}^r (-1)^h C_r^h E\{f(X_1, X_2)\}^{r-h} \left(\frac{N-2}{N}\right)^{2h} \\ &\quad \times \{[(X_1 - m_1) + (X_2 - m_1)][X_{(N-2)} - m_1] - [X_{(N-2)} - m_1]^2\}^h \\ &= E\{f(X_1, X_2)\}^r + r \left(\frac{N-2}{N}\right)^2 \mu_{2, (N-2)} E\{f(X_1, X_2)\}^{r-1} \\ &\quad + \sum_{h=2}^r \sum_{k=h}^{2h} (-1)^k C_r^h C_h^{k-h} \left(\frac{N-2}{N}\right)^{2h} \mu_{k, (N-2)} E\{f(X_1, X_2)\}^{r-h} [(X_1 - m_1) + (X_2 - m_1)]^{2h-k*}. \end{aligned}$$

But (see Chapter I (15) and (16))

$$\begin{aligned} \mu_{k, (N-2)} &= \sum_{i=0}^{\text{Ent.}(\frac{k}{2})-1} \frac{1}{(N-2)^{\text{Ent.}(\frac{k}{2})+i}} \sum_{j=0}^i (-1)^j \beta_{\text{Ent.}(\frac{k}{2})-i+j, i} T_{k, \text{Ent.}(\frac{k}{2})-i+j}, \\ E\{f(X_1, X_2)\}^s &= \sum_{f=0}^s \sum_{g=0}^{2f} (-1)^f \frac{(N-1)^f}{N^{2f}} C_s^f C_{2f}^g \mu_{s+f-g} \mu_{s-f+g}, \\ E\{f(X_1, X_2)\}^{s-h} [(X_1 - m_1) + (X_2 - m_1)]^{2h-k} &= \sum_{f=0}^{s-h} \sum_{g=0}^{2h+2f-k} (-1)^f C_{s-h}^f C_{2h+2f-k}^g \frac{(N-1)^f}{N^{2f}} \mu_{s+h+f-k-g} \mu_{s-h-f+g}. \end{aligned}$$

Hence :

$$\begin{aligned} N(N-1) E[X_1 - X_{(N)}]^r [X_2 - X_{(N)}]^r &= \sum_{f=0}^r \sum_{g=0}^{2f} (-1)^f \frac{(N-1)^{f+1}}{N^{2f-1}} C_r^f C_{2f}^g \mu_{r+f-g} \mu_{r-f+g} \\ &\quad + C_r^1 \mu_2 \sum_{f=0}^{r-1} \sum_{g=0}^{2f} (-1)^f \frac{(N-1)^{f+1} (N-2)}{N^{2f+1}} C_{r-1}^f C_{2f}^g \mu_{r-1+f-g} \mu_{r-1-f+g} \end{aligned}$$

* The development of $E\{f(X_1, X_2)\}^{r-h} [(X_1 - m_1) + (X_2 - m_1)]^{2h-k}$ in powers of N does not contain any terms of higher degree than N^0 while at the same time the development of $\mu_{k, (N-2)}$ contains no terms of higher degree than $\frac{1}{N^{\text{Ent.}(\frac{k+1}{2})}}$. To obtain $E[\nu'_{r, (N)} - \nu_{r, (N)}]^2$ correct to $1/N^t$ it is consequently sufficient to carry our calculations as far as $k=2t$.

$$\begin{aligned}
& + C_r^2 \mu_2 \sum_{f=0}^{r-2} \sum_{g=0}^{2f+2} (-1)^f C_{r-2}^f C_{2f+2}^g \frac{(N-1)^{f+1} (N-2)^3}{N^{2f+3}} \mu_{r+f-g} \mu_{r-2-f+g} \\
& + \sum_{k=3}^{2r} \sum_{i=0}^{\text{Ent.}(\frac{k}{2})-1} \sum_{j=0}^i \sum_{h=\text{Ent.}(\frac{k+1}{2})}^k \sum_{f=0}^{r-h} \sum_{g=0}^{2h+2f-k} (-1)^{k+f+j} C_r^h C_h^{k-h} C_{r-h}^f C_{2h+2f-k}^g \\
& \times \frac{(N-1)^{f+1} (N-2)^{2h-\text{Ent.}(\frac{k+1}{2})-i}}{N^{2f+2h-1}} \beta_{\text{Ent.}(\frac{k}{2})-i+j, j} T_{k, \text{Ent.}(\frac{k}{2})-i+j} \mu_{r+h+f-k-g} \mu_{r-h-f+g} \\
& = N^2 \mu_r^2 - N \{ \mu_r^2 + 2C_{r-1}^1 [\mu_{r+1} \mu_{r-1} + \mu_r^2] - C_r^1 \mu_{r-1}^2 \mu_2 - 2C_r^2 [\mu_r \mu_{r-2} + \mu_{r-1}^2] \mu_2 \} \\
& + \{ C_r^1 [4\mu_{r+1} \mu_{r-1} + 4\mu_r^2] + C_r^2 [2\mu_{r+2} \mu_{r-2} + 8\mu_{r+1} \mu_{r-1} + 6\mu_r^2] \\
& \quad - 3C_r^1 \mu_{r-1}^2 \mu_2 - 2C_r^1 C_{r-1}^1 [\mu_r \mu_{r-2} + \mu_{r-1}^2] \mu_2 \\
& \quad - 2C_r^2 C_{r-2}^1 [\mu_{r+1} \mu_{r-3} + 4\mu_r \mu_{r-2} + 3\mu_{r-1}^2] \mu_2 - 14C_r^2 \mu_r \mu_{r-2} \mu_2 \\
& \quad - 14C_r^2 \mu_{r-1}^2 \mu_2 - 4C_r^2 \mu_{r-1} \mu_{r-2} \mu_3 \\
& \quad - 2C_r^3 [\mu_r \mu_{r-3} + 3\mu_{r-1} \mu_{r-2}] \mu_3 + 3C_r^2 \mu_{r-2}^2 \mu_2^2 + 18C_r^3 [\mu_{r-1} \mu_{r-3} + \mu_{r-2}^2] \mu_2^2 \\
& \quad + 6C_r^4 [\mu_r \mu_{r-4} + 4\mu_{r-1} \mu_{r-3} + 3\mu_{r-2}^2] \mu_2^2 \} + \dots
\end{aligned}$$

Substituting in (11) and replacing $\nu_{2r, (N)}$ in (11) by its value in terms of the μ 's by formula (4), we find after reduction:

$$\begin{aligned}
W_{r, (N)}^{(2)} & = N^2 \mu_r^2 + N \{ \mu_{2r} - (2r+1) \mu_r^2 - 2r \mu_{r+1} \mu_{r-1} \\
& \quad + r^2 \mu_{r-1}^2 \mu_2 + r(r-1) \mu_r \mu_{r-2} \mu_2 \} \\
& - \left\{ 2r \mu_{2r} - r(2r-1) \mu_{2r-2} \mu_2 - 4r^2 \mu_{r+1} \mu_{r-1} - r(3r+1) \mu_r^2 - r(r-1) \mu_{r+2} \mu_{r-2} \right. \\
& + 3r^3 \mu_{r-1}^2 \mu_2 + r(r-1)(4r+1) \mu_r \mu_{r-2} \mu_2 + r^{[-3]} \mu_{r+1} \mu_{r-3} \mu_2 \\
& \quad \left. + r^2(r-1) \mu_{r-1} \mu_{r-2} \mu_3 \right. \\
& + \frac{r^{[-3]}}{3} \mu_r \mu_{r-3} \mu_3 - \frac{3}{4} r^2 (r-1)^2 \mu_{r-2}^2 \mu_2^2 - r^2 (r-1)(r-2) \mu_{r-1} \mu_{r-3} \mu_2^2 \\
& \left. - \frac{r^{[-4]}}{4} \mu_r \mu_{r-4} \mu_2^3 \right\} + \dots
\end{aligned} \quad (12),$$

and hence

$$\begin{aligned}
E[\nu'_{r, (N)} - \nu_{r, (N)}]^2 & = \frac{1}{N^2} W_{r, (N)}^{(2)} - \nu_{r, (N)}^2 \\
& = \frac{1}{N} \{ \mu_{2r} - \mu_r^2 - 2r \mu_{r+1} \mu_{r-1} + r^2 \mu_{r-1}^2 \mu_2 \} \\
& - \frac{1}{N^2} \left\{ 2r \mu_{2r} - r(2r-1) \mu_{2r-2} \mu_2 - r(r-1) \mu_{r+2} \mu_{r-2} - 4r^2 \mu_{r+1} \mu_{r-1} \right. \\
& \quad \left. + r^{[-3]} \mu_{r+1} \mu_{r-3} \mu_2 \right. \\
& - r(r+2) \mu_r^2 + 2(r+1)r(r-1) \mu_r \mu_{r-2} \mu_2 + 3r^3 \mu_{r-1}^2 \mu_2 + r^2(r-1) \mu_{r-1} \mu_{r-2} \mu_3 \\
& \quad \left. - r^2(r-1)(r-2) \mu_{r-1} \mu_{r-3} \mu_2^2 - \frac{r^2(r-1)^2}{2} \mu_{r-2}^2 \mu_2^2 \right\} + \dots
\end{aligned} \quad (13).$$

(3) When $n = 3$ and $n = 4$, we have similarly:

$$\begin{aligned} W_{r, (N)}^{(3)} &= N\nu_{3r, (N)} + 3N(N-1)E[X_1 - X_{(N)}]^{2r}[X_2 - X_{(N)}]^r \\ &\quad + N(N-1)(N-2)E[X_1 - X_{(N)}]^r[X_2 - X_{(N)}]^r[X_3 - X_{(N)}]^r, \\ W_{r, (N)}^{(4)} &= N\nu_{4r, (N)} + 4N(N-1)E[X_1 - X_{(N)}]^{3r}[X_2 - X_{(N)}]^r \\ &\quad + 3N(N-1)E[X_1 - X_{(N)}]^{2r}[X_2 - X_{(N)}]^{2r} \\ &\quad + 6N(N-1)(N-2)E[X_1 - X_{(N)}]^{2r}[X_2 - X_{(N)}]^r[X_3 - X_{(N)}]^r \\ &\quad + N(N-1)(N-2)(N-3)E[X_1 - X_{(N)}]^r[X_2 - X_{(N)}]^r[X_3 - X_{(N)}]^r[X_4 - X_{(N)}]^r. \end{aligned}$$

Determining $W_{r, (N)}^{(3)}$ and $W_{r, (N)}^{(4)}$ from these relations and substituting the values found in (10), we obtain $E[\nu'_{r, (N)} - \nu_{r, (N)}]^3$ and $E[\nu'_{r, (N)} - \nu_{r, (N)}]^4$. Their expression exhibits no special difficulties, but is so unwieldy that I do not give them here, contenting myself with the deduction of $E[\nu'_{r, (N)} - \nu_{r, (N)}]^3$ and $E[\nu'_{r, (N)} - \nu_{r, (N)}]^4$ which will be shown below (see Chapter IV, § III).

III

(1) Noting that

$$\sum_{i=1}^N [X_i - X_{(N)}]^2 = \sum_{i=1}^N (X_i - m_1)^2 - N[X_{(N)} - m_1]^2 = \sum_{i=1}^N (X_i - m_1)^2 - \frac{1}{N} \left[\sum_{i=1}^N (X_i - m_1) \right]^2,$$

and putting

$$\left. \begin{aligned} U_{2, (N)}^{(r, s)} &= E \left[\sum_{i=1}^N (X_i - m_1)^2 \right]^r \left[\sum_{i=1}^N (X_i - m_1) \right]^s \\ Z_{2, (N)}^{(r, s)} &= E \left[\sum_{i=1}^N (X_i - X_{(N)})^2 \right]^r \left[\sum_{i=1}^N (X_i - m_1) \right]^s \end{aligned} \right\} \dots\dots\dots (14),$$

we find

$$\left. \begin{aligned} W_{2, (N)}^{(m)} &= E \left[\sum_{i=1}^N (X_i - X_{(N)})^2 \right]^m \\ &= V_{2, (N)}^{(m)} + \sum_{h=1}^{m-1} (-1)^h \frac{1}{N^h} C_m^h U_{2, (N)}^{(m-h, 2h)} + (-1)^m N^m \mu_{2m, (N)} \\ V_{2, (N)}^{(m)} &= E \left[\sum_{i=1}^N (X_i - m_1)^2 \right]^m \\ &= W_{2, (N)}^{(m)} + \sum_{h=1}^{m-1} C_m^h \frac{1}{N^h} Z_{2, (N)}^{(m-h, 2h)} + N^m \mu_{2m, (N)} \end{aligned} \right\} \dots\dots\dots (15),$$

$$\text{or} \quad W_{2, (N)}^{(m)} = V_{2, (N)}^{(m)} - \sum_{h=1}^{m-1} C_m^h \frac{1}{N^h} Z_{2, (N)}^{(m-h, 2h)} - N^m \mu_{2m, (N)} \dots\dots\dots (16),$$

and, on the other hand,

$$\begin{aligned} U_{2, (N)}^{(r, s)} &= \sum_{h=0}^{r-1} C_r^h \frac{1}{N^h} Z_{2, (N)}^{(r-h, s+2h)} + N^{r+s} \mu_{2r+s, (N)}, \\ Z_{2, (N)}^{(r, s)} &= \sum_{h=0}^{r-1} (-1)^h C_r^h \frac{1}{N^h} U_{2, (N)}^{(r-h, s+2h)} + (-1)^r N^{r+s} \mu_{2r+s, (N)}, \end{aligned}$$

whence :

$$\sum_{h=1}^{r-1} C_r^h \frac{1}{N^h} [Z_{2,(N)}^{(r-h, s+2h)} + (-1)^h U_{2,(N)}^{(r-h, s+2h)}] + N^{r+s} \mu_{2r+s, (N)} [1 + (-1)^r] = 0 \dots (17).$$

(2) Noting that

$$\begin{aligned} U_{2,(N)}^{(r,s)} &= E \left[(X_1 - m_1)^2 + \sum_{i=2}^N (X_i - m_1)^2 \right]^r \left[(X_1 - m_1) + \sum_{i=2}^N (X_i - m_1) \right]^s \\ &= \mu_{2r+s} + \sum_{j=1}^{s-1} C_s^j \mu_{2r+s-j} (N-1)^j \mu_{j, (N-1)} + \mu_{2r} (N-1)^s \mu_{s, (N-1)} \\ &\quad + \sum_{h=1}^{r-1} C_r^h \mu_{2r+s-2h} V_{2, (N-1)}^{(h)} + \sum_{h=1}^{r-1} \sum_{j=1}^{s-1} C_r^h C_s^j \mu_{2r+s-2h-j} U_{2, (N-1)}^{(h,j)} \\ &\quad + \sum_{h=1}^{r-1} C_r^h \mu_{2r-2h} U_{2, (N-1)}^{(h,s)} + \mu_s V_{2, (N-1)}^{(r)} + \sum_{j=1}^{s-1} C_s^j \mu_{s-j} U_{2, (N-1)}^{(r,j)} + U_{2, (N-1)}^{(r,s)}, \end{aligned}$$

we find :

$$\begin{aligned} U_{2,(N)}^{(r,s)} - U_{2,(N-1)}^{(r,s)} &= \sum_{j=0}^s C_s^j \mu_{2r+s-j} (N-1)^j \mu_{j, (N-1)} + \sum_{h=1}^r C_r^h \mu_{2r+s-2h} V_{2, (N-1)}^{(h)} \\ &\quad + \sum_{h=1}^r \sum_{j=1}^{s-1} C_r^h C_s^j \mu_{2r+s-2h-j} U_{2, (N-1)}^{(h,j)} + \sum_{h=1}^{r-1} C_r^h \mu_{2r-2h} U_{2, (N-1)}^{(h,s)}, \end{aligned}$$

and hence :

$$\begin{aligned} U_{2,(N)}^{(r,s)} &= \left. \begin{aligned} &\sum_{j=0}^s C_s^j \mu_{2r+s-j} \sum_{f=1}^N (N-f)^j \mu_{j, (N-f)} + \sum_{h=1}^r C_r^h \mu_{2r+s-2h} \sum_{f=1}^N V_{2, (N-f)}^{(h)} \\ &+ \sum_{h=1}^{r-1} C_r^h \mu_{2r-2h} \sum_{f=1}^N U_{2, (N-f)}^{(h,s)} + \sum_{h=1}^r \sum_{j=1}^{s-1} C_r^h C_s^j \mu_{2r+s-2h-j} \sum_{f=1}^N U_{2, (N-f)}^{(h,j)} \end{aligned} \right\} (18). \end{aligned}$$

Putting here (see Chapter I (14) and Chapter II (10))

$$\begin{aligned} \sum_{f=1}^N (N-f)^j \mu_{j, (N-f)} &= \sum_{i=0}^{\text{Ent.} \left(\frac{j}{2} \right) - 1} N^{\left[\text{Ent.} \left(\frac{j}{2} \right) - i + 1 \right]} \frac{T_{j, \text{Ent.} \left(\frac{j}{2} \right) - i}}{\text{Ent.} \left(\frac{j}{2} \right) - i + 1}, \\ \sum_{f=1}^N V_{2, (N-f)}^{(h)} &= \sum_{i=1}^h \frac{N^{[-(i+1)]}}{i+1} D_{2, i}^{(h)}, \end{aligned}$$

and noting that

$$\begin{aligned} U_{2,(N)}^{(0,s)} &= N^s \mu_{s, (N)}, \\ U_{2,(N)}^{(h,0)} &= V_{2, (N)}^{(h)}, \\ U_{2,(N)}^{(1,s)} &= N \left\{ \mu_{s+2} + \sum_{h=2}^s C_s^h \mu_{s+2-h} (N-1)^h \mu_{h, (N-1)} \right\} \\ &= N \mu_{s+2} + \sum_{j=1}^{\text{Ent.} \left(\frac{s}{2} \right)} N^{[-(j+1)]} \sum_{h=2j}^s C_s^h \mu_{s+2-h} T_{h, j}, \end{aligned}$$

we find in consequence :

$$U_{2,(N)}^{(1,1)} = N\mu_3,$$

$$U_{2,(N)}^{(1,2)} = N\mu_4 + N^{[-2]} \mu_2^2,$$

$$U_{2,(N)}^{(1,3)} = N\mu_5 + 4N^{[-2]} \mu_3 \mu_2,$$

$$U_{2,(N)}^{(1,4)} = N\mu_6 + 7N^{[-2]} \mu_4 \mu_2 + 4N^{[-2]} \mu_3^2 + 3N^{[-3]} \mu_2^3,$$

$$U_{2,(N)}^{(1,6)} = N\mu_8 + N^{[-2]} [16\mu_6 \mu_2 + 26\mu_5 \mu_3 + 15\mu_4^2] \\ + N^{[-3]} [60\mu_4 \mu_2^2 + 70\mu_3^2 \mu_2] + 15N^{[-4]} \mu_2^4,$$

$$U_{2,(N)}^{(2,1)} = N\mu_5 + 2N^{[-2]} \mu_3 \mu_2,$$

$$U_{2,(N)}^{(2,2)} = N\mu_6 + 3N^{[-2]} \mu_4 \mu_2 + 2N^{[-2]} \mu_3^2 + N^{[-3]} \mu_2^3,$$

$$U_{2,(N)}^{(2,4)} = N\mu_8 + N^{[-2]} [8\mu_6 \mu_2 + 12\mu_5 \mu_3 + 7\mu_4^2] \\ + N^{[-3]} [16\mu_4 \mu_2^2 + 20\mu_3^2 \mu_2] + 3N^{[-4]} \mu_2^4,$$

$$U_{2,(N)}^{(3,2)} = N\mu_8 + N^{[-2]} [4\mu_6 \mu_2 + 6\mu_5 \mu_3 + 3\mu_4^2] \\ + N^{[-3]} [6\mu_4 \mu_2^2 + 6\mu_3^2 \mu_2] + N^{[-4]} \mu_2^4.$$

(3) Substituting the values we have found for $U_{2,(N)}^{(r,s)}$ in (15) we obtain

$$\left. \begin{aligned} W_{2,(N)}^{(1)} &= (N-1) \mu_2 \\ W_{2,(N)}^{(2)} &= \frac{N-1}{N} \{ (N-1) \mu_4 + (N^2 - 2N + 3) \mu_2^2 \} \\ &= N^2 \mu_2^2 + N [\mu_4 - 3\mu_2^2] - [2\mu_4 - 5\mu_2^2] + \frac{1}{N} [\mu_4 - 3\mu_2^2]^* \end{aligned} \right\} \dots(19),$$

$$\left. \begin{aligned} W_{2,(N)}^{(3)} &= \frac{N-1}{N^2} \{ (N-1)^2 \mu_6 + 3(N-1)(N^2 - 2N + 5) \mu_4 \mu_2 \\ &\quad - 2(3N^2 - 6N + 5) \mu_3^2 + (N-2)(N^3 - 3N^2 + 9N - 15) \mu_2^3 \} \\ &= N^3 \mu_2^3 + N^2 [3\mu_4 \mu_2^2 - 6\mu_2^3] + N [\mu_6 - 12\mu_4 \mu_2 - 6\mu_3^2 + 20\mu_2^3] \\ &\quad - [3\mu_6 - 30\mu_4 \mu_2 - 18\mu_3^2 + 48\mu_2^3] + \frac{1}{N} [3\mu_6 - 36\mu_4 \mu_2 - 22\mu_3^2 + 63\mu_2^3] \\ &\quad - \frac{1}{N^2} [\mu_6 - 15\mu_4 \mu_2 - 10\mu_3^2 + 30\mu_2^3] \end{aligned} \right\} (20),$$

* The value of $\frac{1}{N^2} W_{2,(N)}^{(2)}$ is given in Student's paper "The Probable Error of a Mean" (*Biometrika*, Vol. vi, p. 3), but unfortunately misprints have crept into the formula.

$$\begin{aligned}
W_{2, (N)}^{(4)} &= \frac{N-1}{N^3} \left\{ (N-1)^3 \mu_8 + 4(N-1)^2 (N^2 - 2N + 7) \mu_6 \mu_2 \right. \\
&\quad + (3N^4 - 12N^3 + 42N^2 - 60N + 35) \mu_4^2 \\
&\quad \quad \quad - (N-1)(24N^2 - 48N + 56) \mu_5 \mu_3 \\
&\quad + 6(N-2)(N^4 - 4N^3 + 16N^2 - 40N + 35) \mu_4 \mu_2^2 \\
&\quad \quad \quad - (N-2)(24N^3 - 120N^2 + 280N - 280) \mu_3^2 \mu_2 \\
&\quad + (N-2)(N-3)(N^4 - 4N^3 + 18N^2 - 60N + 105) \mu_2^4 \left. \right\} \\
&= N^4 \mu_2^4 + N^3 [6\mu_4 \mu_2^2 - 10\mu_2^4] \\
&\quad + N^2 [4\mu_6 \mu_2 + 3\mu_4^2 - 42\mu_4 \mu_2^2 - 24\mu_3^2 \mu_2 + 53\mu_2^4] \\
&\quad + N [\mu_8 - 20\mu_6 \mu_2 - 15\mu_4^2 - 24\mu_5 \mu_3 + 180\mu_4 \mu_2^2 + 192\mu_3^2 \mu_2 - 218\mu_2^4] \\
&\quad - [4\mu_8 - 64\mu_6 \mu_2 - 54\mu_4^2 - 96\mu_5 \mu_3 + 576\mu_4 \mu_2^2 + 688\mu_3^2 \mu_2 - 687\mu_2^4] \\
&\quad + \frac{1}{N} [6\mu_8 - 112\mu_6 \mu_2 - 102\mu_4^2 - 176\mu_5 \mu_3 \\
&\quad \quad \quad + 1122\mu_4 \mu_2^2 + 1360\mu_3^2 \mu_2 - 1398\mu_2^4] \\
&\quad - \frac{1}{N^2} [4\mu_8 - 92\mu_6 \mu_2 - 95\mu_4^2 - 160\mu_5 \mu_3 \\
&\quad \quad \quad + 1110\mu_4 \mu_2^2 + 1400\mu_3^2 \mu_2 - 1515\mu_2^4] \\
&\quad + \frac{1}{N^3} [\mu_8 - 28\mu_6 \mu_2 - 35\mu_4^2 - 56\mu_5 \mu_3 + 420\mu_4 \mu_2^2 + 560\mu_3^2 \mu_2 - 630\mu_2^4] \left. \right\} \quad (21).
\end{aligned}$$

In the case when the law of distribution of the variable X follows the Gauss-Laplace law :

$$\left. \begin{aligned}
W_{2, (N)}^{(1)} &= (N-1) \mu_2 = V_{2, (N-1)}^{(1)} \\
W_{2, (N)}^{(2)} &= (N-1)(N+1) \mu_2^2 = V_{2, (N-1)}^{(2)} \\
W_{2, (N)}^{(3)} &= (N-1)(N+1)(N+3) \mu_2^3 = V_{2, (N-1)}^{(3)} \\
W_{2, (N)}^{(4)} &= (N-1)(N+1)(N+3)(N+5) \mu_2^4 = V_{2, (N-1)}^{(4)*}
\end{aligned} \right\} \dots (22).$$

(4) Substituting the values found for $W_{2, (N)}^{(1)}$, $W_{2, (N)}^{(2)}$, $W_{2, (N)}^{(3)}$ and $W_{2, (N)}^{(4)}$ in (10), we find :

$$\left. \begin{aligned}
E[\nu'_{2, (N)} - \nu_{2, (N)}]^2 &= \frac{(N-1)^2}{N^3} \mu_4 - \frac{(N-1)(N-3)}{N^3} \mu_2^2 \\
&= \frac{1}{N} [\mu_4 - \mu_2^2] - \frac{2}{N^2} [\mu_4 - 2\mu_2^2] + \frac{1}{N^3} [\mu_4 - 3\mu_2^2]
\end{aligned} \right\} \dots (23),$$

* Cf. Student, "The Probable Error of a Mean" (*Biometrika*, Vol. VI).

$$\begin{aligned}
 E[\nu'_{2,(N)} - \nu_{2,(N)}]^3 &= \frac{(N-1)^3}{N^5} \mu_6 - \frac{3(N-1)^2(N-5)}{N^5} \mu_4 \mu_2 \\
 &\quad - \frac{2(N-1)(3N^2-6N+5)}{N^5} \mu_3^2 + \frac{2(N-1)(N^2-12N+15)}{N^5} \mu_2^3 \\
 &= \frac{1}{N^2} [\mu_6 - 3\mu_4 \mu_2 - 6\mu_3^2 + 2\mu_2^3] - \frac{1}{N^3} [3\mu_6 - 21\mu_4 \mu_2 - 18\mu_3^2 + 26\mu_2^3] \\
 &\quad + \frac{1}{N^4} [3\mu_6 - 33\mu_4 \mu_2 - 22\mu_3^2 + 54\mu_2^3] - \frac{1}{N^5} [\mu_6 - 15\mu_4 \mu_2 - 10\mu_3^2 + 30\mu_2^3]
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 E[\nu'_{2,(N)} - \nu_{2,(N)}]^4 &= \frac{3}{N^2} [\mu_4 - \mu_2^2]^2 \\
 &+ \frac{1}{N^3} [\mu_8 - 4\mu_6 \mu_2 - 15\mu_4^2 - 24\mu_5 \mu_3 + 48\mu_4 \mu_2^2 + 96\mu_3^2 \mu_2 - 30\mu_2^4] \\
 &- \frac{1}{N^4} [4\mu_8 - 40\mu_6 \mu_2 - 54\mu_4^2 - 96\mu_5 \mu_3 + 336\mu_4 \mu_2^2 + 528\mu_3^2 \mu_2 - 306\mu_2^4] \\
 &+ \frac{1}{N^5} [6\mu_8 - 96\mu_6 \mu_2 - 102\mu_4^2 - 176\mu_5 \mu_3 + 924\mu_4 \mu_2^2 + 1232\mu_3^2 \mu_2 - 1044\mu_2^4] \\
 &- \frac{1}{N^6} [4\mu_8 - 92\mu_6 \mu_2 - 95\mu_4^2 - 160\mu_5 \mu_3 + 1110\mu_4 \mu_2^2 + 1400\mu_3^2 \mu_2 - 1515\mu_2^4] \\
 &+ \frac{1}{N^7} [\mu_8 - 28\mu_6 \mu_2 - 35\mu_4^2 - 56\mu_5 \mu_3 + 420\mu_4 \mu_2^2 + 560\mu_3^2 \mu_2 - 630\mu_2^4]
 \end{aligned} \tag{25}$$

In the case when the law of distribution of values of X follows the Gauss-Laplace law :

$$E[\nu'_{2,(N)} - \nu_{2,(N)}]^2 = \frac{2(N-1)}{N^2} \mu_2^2,$$

$$E[\nu'_{2,(N)} - \nu_{2,(N)}]^3 = \frac{8(N-1)}{N^3} \mu_2^3,$$

$$E[\nu'_{2,(N)} - \nu_{2,(N)}]^4 = \frac{12(N-1)(N+3)}{N^4} \mu_2^4,$$

$$\frac{E[\nu'_{2,(N)} - \nu_{2,(N)}]^4}{\{E[\nu'_{2,(N)} - \nu_{2,(N)}]^2\}^2} = 3 \frac{N+3}{N-1}.$$

CHAPTER IV

I

(1) We may also follow another road, to deduce the formulae obtained above, a road nearer to the one usual in English literature.

Let us denote by n_j the number of times in N experiments the variable X takes the value ξ_j , one of its k possible values (cf. above, Chapter I, § I), and putting

$$p_j' = \frac{n_j}{N},$$

we have :

$$\begin{aligned}
 N &= \sum_{j=1}^k n_j, \\
 \sum_{j=1}^k p_j' &= 1, \\
 m_r &= \sum_{j=1}^k p_j \xi_j^r; \quad m_r' = \sum_{j=1}^k p_j' \xi_j^r, \\
 \mu_r &= \sum_{j=1}^k p_j [\xi_j - m_1]^r; \quad \mu_r' = \sum_{j=1}^k p_j' [\xi_j - m_1]^r, \\
 X_{(N)} &= \sum_{j=1}^k p_j' \xi_j = m_1'.
 \end{aligned}$$

But

$$\left. \begin{aligned}
 En_i &= Np_i \\
 En_i^2 &= Np_i + N^{[-2]} p_i^2 = N^2 p_i^2 + Np_i(1-p_i) \\
 En_i^3 &= Np_i + 3N^{[-2]} p_i^2 + N^{[-3]} p_i^3 \\
 &= N^3 p_i^3 + 3N^2 p_i^2(1-p_i) + Np_i(1-p_i)(1-2p_i) \\
 En_i^4 &= Np_i + 7N^{[-2]} p_i^2 + 6N^{[-3]} p_i^3 + N^{[-4]} p_i^4 \\
 &= N^4 p_i^4 + 6N^3 p_i^3(1-p_i) + N^2 p_i^2(1-p_i)(7-11p_i) \\
 &\quad + Np_i(1-p_i)(1-6p_i+6p_i^2)
 \end{aligned} \right\} (1),$$

and in general, as is not difficult to see*,

$$En_i^r = \sum_{f=1}^r N^{[-f]} \alpha_{r,f} p_i^f = \sum_{h=1}^r N^h p_i^h \sum_{f=0}^{r-h} (-1)^f \alpha_{r,h+f} \beta_{h+f,f} p_i^f \dots\dots(2).$$

Further, denoting by P_h the probability of n_i taking the value h , and by $E_{n_j}^{(h)}$, the conditional mathematical expectation of n_j on the assumption that n_i takes the value h , we find :

$$\begin{aligned}
 P_h &= C_N^h p_i^h (1-p_i)^{N-h}, \\
 E_{n_j}^{(h)} &= (N-h) \frac{p_j}{1-p_i}, \\
 En_i n_j &= \sum_{h=0}^N P_h h E_{n_j}^{(h)} = \sum_{h=0}^{N-1} (N-h) h C_N^h p_i^h (1-p_i)^{N-h-1} p_j = N(N-1) p_i p_j.
 \end{aligned}$$

Similarly we obtain :

$$\left. \begin{aligned}
 En_{i_1} n_{i_2} n_{i_3} &= N(N-1)(N-2) p_{i_1} p_{i_2} p_{i_3} \\
 En_{i_1} n_{i_2} \dots n_{i_k} &= N(N-1) \dots (N-k+1) p_{i_1} p_{i_2} \dots p_{i_k}
 \end{aligned} \right\} \dots\dots\dots(3),$$

* See my paper, "On the Mathematical Expectation of a Positive Integral Power of the Difference between the Frequency and the Probability of an Event," in the *Proceedings of the Petrograd Polytechnic Institute*.

$$\begin{aligned}
En_i^2 n_j &= N^{[-2]} p_i p_j + N^{[-3]} p_i^2 p_j \\
&= N^3 p_i^2 p_j + N^3 p_i p_j (1 - 3p_i) - N p_i p_j (1 - 2p_i) \\
En_i^2 n_j^2 &= N^{[-2]} p_i p_j + N^{[-3]} p_i p_j (p_i + p_j) + N^{[-4]} p_i^2 p_j^2 \\
&= N^4 p_i^2 p_j^2 + N^3 p_i p_j (p_i + p_j - 6p_i p_j) + N^2 p_i p_j (1 - 3p_i - 3p_j + 11p_i p_j) \\
&\quad - N p_i p_j (1 - 2p_i - 2p_j + 6p_i p_j) \\
En_i^3 n_j &= N^{[-2]} p_i p_j + 3N^{[-3]} p_i^2 p_j + N^{[-4]} p_i^3 p_j \\
&= N^4 p_i^3 p_j + 3N^3 p_i^2 p_j (1 - 2p_i) \\
&\quad + N^2 p_i p_j (1 - 9p_i + 11p_i^2) - N p_i p_j (1 - 6p_i + 6p_i^2) \\
En_i^2 n_j n_h &= N^{[-3]} p_i p_j p_h + N^{[-4]} p_i^2 p_j p_h \\
&= N^4 p_i^2 p_j p_h + N^3 p_i p_j p_h (1 - 6p_i) - N^2 p_i p_j p_h (3 - 11p_i) \\
&\quad + 2N p_i p_j p_h (1 - 3p_i)
\end{aligned} \tag{4}$$

$$En_i^{r_1} n_j^{r_2} = \sum_{h_1=1}^{r_1} \sum_{h_2=1}^{r_2} N^{[-(h_1+h_2)]} \alpha_{r_1, h_1} \alpha_{r_2, h_2} p_i^{h_1} p_j^{h_2} \dots \tag{5}$$

and in the general case :

$$En_{i_1}^{r_1} n_{i_2}^{r_2} \dots n_{i_k}^{r_k} = \sum_{h_1=1}^{r_1} \sum_{h_2=1}^{r_2} \dots \sum_{h_k=1}^{r_k} N^{[-(h_1+h_2+\dots+h_k)]} \alpha_{r_1, h_1} \alpha_{r_2, h_2} \dots \alpha_{r_k, h_k} p_{i_1}^{h_1} p_{i_2}^{h_2} \dots p_{i_k}^{h_k} \tag{6}$$

If the numbers $n_{i_1}, n_{i_2}, \dots, n_{i_k}$, referred to k series of independent experiments, then we should have :

$$\begin{aligned}
En_{i_1}^{r_1} n_{i_2}^{r_2} \dots n_{i_k}^{r_k} &= En_{i_1}^{r_1} En_{i_2}^{r_2} \dots En_{i_k}^{r_k} \\
&= \sum_{h_1=1}^{r_1} \sum_{h_2=1}^{r_2} \dots \sum_{h_k=1}^{r_k} N^{[-h_1]} N^{[-h_2]} \dots N^{[-h_k]} \alpha_{r_1, h_1} \alpha_{r_2, h_2} \dots \alpha_{r_k, h_k} p_{i_1}^{h_1} p_{i_2}^{h_2} \dots p_{i_k}^{h_k}.
\end{aligned}$$

(2) Passing from the mathematical expectations of the numbers of repetitions to the mathematical expectations of frequencies, we find :

$$\begin{aligned}
Ep_i' &= p_i \\
Ep_i'^2 &= p_i^2 + \frac{1}{N} p_i (1 - p_i) \\
Ep_i'^3 &= p_i^3 + \frac{3}{N} p_i^2 (1 - p_i) + \frac{1}{N^2} p_i (1 - p_i) (1 - 2p_i) \\
Ep_i'^4 &= p_i^4 + \frac{6}{N} p_i^3 (1 - 6p_i) + \frac{1}{N^2} p_i^2 (1 - p_i) (7 - 11p_i) \\
&\quad + \frac{1}{N^3} p_i (1 - p_i) (1 - 6p_i + 6p_i^2)
\end{aligned} \tag{7}$$

$$Ep_i'^r = \sum_{h=0}^{r-1} \frac{1}{N^h} p_i^{r-h} \sum_{f=0}^h (-1)^f \alpha_{r, r-h+f} \beta_{r-h+f, f} p_i^f \dots \tag{8}$$

$$\begin{aligned}
 Ep_i' p_j' &= p_i p_j - \frac{1}{N} p_i p_j \\
 Ep_i'^2 p_j' &= p_i^2 p_j + \frac{1}{N} p_i p_j (1 - 3p_i) - \frac{1}{N^2} p_i p_j (1 - 2p_i) \\
 Ep_i' p_j' p_h' &= p_i p_j p_h - \frac{3}{N} p_i p_j p_h + \frac{2}{N^2} p_i p_j p_h \\
 Ep_i'^3 p_j' &= p_i^3 p_j + \frac{3}{N} p_i^2 p_j (1 - 2p_i) + \frac{1}{N^2} p_i p_j (1 - 9p_i + 11p_i^2) \\
 &\quad - \frac{1}{N^3} p_i p_j (1 - 6p_i + 6p_i^2) \\
 Ep_i'^2 p_j'^2 &= p_i^2 p_j^2 + \frac{1}{N} p_i p_j (p_i + p_j - 6p_i p_j) + \frac{1}{N^2} \\
 &\quad \times p_i p_j (1 - 3p_i - 3p_j + 11p_i p_j) - \frac{1}{N^3} p_i p_j (1 - 2p_i - 2p_j + 6p_i p_j) \\
 Ep_i'^2 p_j' p_h' &= p_i^2 p_j p_h + \frac{1}{N} p_i p_j p_h (1 - 6p_i) - \frac{1}{N^2} p_i p_j p_h (3 - 11p_i) \\
 &\quad + \frac{2}{N^3} p_i p_j p_h (1 - 3p_i) \\
 Ep_i p_j p_h p_f &= p_i p_j p_h p_f - \frac{6}{N} p_i p_j p_h p_f + \frac{11}{N^2} p_i p_j p_h p_f - \frac{6}{N^3} p_i p_j p_h p_f
 \end{aligned} \tag{9}$$

and hence :

$$\begin{aligned}
 E(p_i' - p_i) &= 0 \\
 E(p_i' - p_i)^2 &= \frac{1}{N} p_i (1 - p_i) \\
 E(p_i' - p_i)^3 &= \frac{1}{N^2} p_i (1 - p_i) (1 - 2p_i) \\
 E(p_i' - p_i)^4 &= \frac{3}{N^2} p_i^2 (1 - p_i)^2 + \frac{1}{N^3} p_i (1 - p_i) (1 - 6p_i + 6p_i^2) \\
 E(p_i' - p_i)(p_j' - p_j) &= -\frac{1}{N} p_i p_j \\
 E(p_i' - p_i)^2(p_j' - p_j) &= -\frac{1}{N^2} p_i p_j (1 - 2p_i) \\
 E(p_i' - p_i)(p_j' - p_j)(p_h' - p_h) &= \frac{2}{N^2} p_i p_j p_h \\
 E(p_i' - p_i)^3(p_j' - p_j) &= -\frac{3}{N^2} p_i^2 p_j (1 - p_i) - \frac{1}{N^3} p_i p_j (1 - 6p_i + 6p_i^2) \\
 E(p_i' - p_i)^2(p_j' - p_j)^2 &= \frac{1}{N^2} p_i p_j (1 - p_i - p_j + 3p_i p_j) \\
 &\quad - \frac{1}{N^3} p_i p_j (1 - 2p_i - 2p_j + 6p_i p_j)
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 E(p_i' - p_i)^2(p_j' - p_j)(p_h' - p_h) &= -\frac{1}{N^2} p_i p_j p_h (1 - 3p_i) \\
 &\quad + \frac{2}{N^3} p_i p_j p_h (1 - 3p_i) \\
 E(p_i' - p_i)(p_j' - p_j)(p_h' - p_h)(p_f' - p_f) &= \frac{3}{N^2} p_i p_j p_h p_f \\
 &\quad - \frac{6}{N^3} p_i p_j p_h p_f^*
 \end{aligned}$$

In the general case we find:

$$E(p_i' - p_i)^r = \sum_{f=\text{Ent.}(\frac{r+1}{2})}^{r-1} \frac{1}{N^f} \sum_{k=0}^f (-1)^k p_i^{r+k-f} \nabla_{(r)}^r \alpha_{r, r+k-f} \beta_{r+k-f, k} \dots (11),$$

and hence†

$$\begin{aligned}
 E(p_i' - p_i)^{2r} &= 1 \cdot 3 \cdot 5 \dots (2r-1) \left\{ \frac{1}{N^r} p_i^r (1-p_i)^r \right. \\
 &\quad + \frac{1}{N^{r+1}} p_i^{r-1} (1-p_i)^{r-1} \frac{r(r-1)}{18} [(2r-1) - 2p_i(1-p_i)(4r+1)] \\
 &\quad + \frac{1}{N^{r+2}} p_i^{r-2} (1-p_i)^{r-2} \frac{r(r-1)(r-2)}{9720} [(20r^3 - 60r^2 + 31r + 15) \\
 &\quad \quad - 4p_i(1-p_i)(40r^3 - 30r^2 - r + 3) \\
 &\quad \quad \left. + 4p_i^2(1-p_i)^2(80r^3 + 120r^2 + 7r - 21)] + \dots \right\} \quad (12),
 \end{aligned}$$

$$\begin{aligned}
 E(p_i' - p_i)^{2r+1} &= 1 \cdot 3 \cdot 5 \dots (2r+1) (1-2p_i) \left\{ \frac{1}{N^{r+1}} p_i^r (1-p_i)^r \frac{r}{3} \right. \\
 &\quad + \frac{1}{N^{r+2}} p_i^{r-1} (1-p_i)^{r-1} \frac{r(r-1)}{810} [(10r^2 - 5r - 3) - 2p_i(1-p_i)(20r^2 + 35r + 12)] \\
 &\quad + \frac{1}{N^{r+3}} p_i^{r-2} (1-p_i)^{r-2} \frac{r(r-1)(r-2)}{204120} [(28r^4 - 84r^3 - 7r^2 + 147r - 54) \\
 &\quad \quad - 4p_i(1-p_i)(56r^4 + 42r^3 - 77r^2 - 42r + 9) \\
 &\quad \quad \left. + 4p_i^2(1-p_i)^2(112r^4 + 504r^3 + 665r^2 + 189r - 72)] + \dots \right\} \quad (13),
 \end{aligned}$$

* The expressions for $E(p_i' - p_i)^4$, $E(p_i' - p_i)^3(p_j' - p_j)$, and so on, show how dangerous it is to reject, without due qualification, terms containing $\frac{1}{N}$ to higher powers. Depending on the magnitude of p_i , $E(p_i' - p_i)^4$ may be either greater or less than $\frac{3}{N^2} p_i^2 (1-p_i)^2$, according as $p_i(1-p_i) \leq \frac{1}{6}$; when p_i is very small—of order $1/N$ —the term rejected and the term retained are of the same order.

† Cf. my paper, previously cited, "On the Mathematical Expectation of a Positive Integral Power of the Difference between the Frequency and the Probability of an Event." Both formulae may easily be obtained directly from formulae (22) and (23) of Chapter I.

$$\left. \begin{aligned}
 E(p_i' - p_i)^5 &= (1 - 2p_i) \left\{ \frac{10}{N^3} p_i^2 (1 - p_i)^2 + \frac{1}{N^4} p_i (1 - p_i) [1 - 12p_i + 12p_i^2] \right\} \\
 E(p_i' - p_i)^6 &= \frac{15}{N^3} p_i^3 (1 - p_i)^3 + \frac{5}{N^4} p_i^2 (1 - p_i)^2 [5 - 26p_i (1 - p_i)] \\
 &\quad + \frac{1}{N^5} p_i (1 - p_i) [1 - 30p_i (1 - p_i) + 120p_i^2 (1 - p_i)^2] \\
 E(p_i' - p_i)^7 &= (1 - 2p_i) \left\{ \frac{105}{N^4} p_i^3 (1 - p_i)^3 + \frac{1}{N^5} p_i^2 (1 - p_i)^2 [56 - 462p_i (1 - p_i)] \right. \\
 &\quad \left. + \frac{1}{N^6} p_i (1 - p_i) [1 - 60p_i (1 - p_i) + 360p_i^2 (1 - p_i)^2] \right\}
 \end{aligned} \right\} \quad (14).$$

Replacing $N^{-(h_1+h_2)}$ in (5) by $\sum_{l=0}^{h_1+h_2-1} (-1)^l \beta_{h_1+h_2, l} N^{h_1+h_2-l}$, we find after some transformations:

$$E p_i'^{r_1} p_j'^{r_2} = \sum_{f=0}^{r_1+r_2-1} \frac{1}{N^f} \sum_{h_1} \sum_{h_2} (-1)^{f-h_1-h_2} \alpha_{r_1, r_1-h_1} \alpha_{r_2, r_2-h_2} \beta_{r_1+r_2-h_1-h_2, f-h_1-h_2} p_i^{r_1-h_1} p_j^{r_2-h_2} \quad (15),$$

where the summation for h_1 extends to all positive integer values from 0 to the smaller of the numbers f and $r_1 - 1$, and the summation for h_2 to all integer values from 0 to the smaller of the numbers $r_2 - 1$ and $f - h_1$.

Substituting the values of $E p_i'^{r_1-h_1} p_j'^{r_2-h_2}$ in the development of

$$E(p_i' - p_i)^{r_1} (p_j' - p_j)^{r_2} = \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} (-1)^{l_1+l_2} C_{r_1}^{l_1} C_{r_2}^{l_2} p_i^{l_1} p_j^{l_2} E p_i'^{r_1-l_1} p_j'^{r_2-l_2},$$

we find after some rather tedious transformations:

$$\left. \begin{aligned}
 E(p_i' - p_i)^{r_1} (p_j' - p_j)^{r_2} &= \sum_{f=\text{Ent.} \left(\frac{r_1+r_2+1}{2} \right)}^{r_1+r_2-2} \frac{1}{N^f} \sum_{h_1=0}^{f \text{ (or } r_1-1)} \sum_{h_2=0}^{f-h_1 \text{ (or } r_2-1)} (-1)^{f-h_1-h_2} p_i^{r_1-h_1} p_j^{r_2-h_2} \nabla_{(r_1)}^{r_1} \nabla_{(r_2)}^{r_2} \alpha_{r_1, r_1-h_1} \alpha_{r_2, r_2-h_2} \\
 &\quad \beta_{r_1+r_2-h_1-h_2, f-h_1-h_2} \\
 &+ \frac{1}{N^{r_1+r_2-1}} \sum_{h_1=0}^{r_1-1} \sum_{h_2=0}^{r_2-1} (-1)^{r_1+r_2-h_1-h_2-1} \alpha_{r_1, r_1-h_1} \alpha_{r_2, r_2-h_2} \\
 &\quad \beta_{r_1+r_2-h_1-h_2, f-h_1-h_2} p_i^{r_1-h_1} p_j^{r_2-h_2}
 \end{aligned} \right\} \quad (16).$$

In the general case we have:

$$\left. \begin{aligned}
 E(p_{i_1}' - p_{i_1})^{r_1} (p_{i_2}' - p_{i_2})^{r_2} \dots (p_{i_k}' - p_{i_k})^{r_k} \\
 &= \sum_{f=\text{Ent.} \left(\frac{r_1+r_2+\dots+r_k+1}{2} \right)}^{r_1+r_2+\dots+r_k-2} \frac{1}{N^f} \sum_{h_1=0}^{f \text{ (or } r_1-1)} \sum_{h_2=0}^{f-h_1 \text{ (or } r_2-1)} \dots \sum_{h_k=0}^{f-h_1-h_2-\dots-h_{k-1} \text{ (or } r_k-1)} \\
 &\quad \times (-1)^{f-h_1-h_2-\dots-h_k} p_{i_1}^{r_1-h_1} p_{i_2}^{r_2-h_2} \dots p_{i_k}^{r_k-h_k} \nabla_{(r_1)}^{r_1} \nabla_{(r_2)}^{r_2} \dots \nabla_{(r_k)}^{r_k} \alpha_{r_1, r_1-h_1} \dots \\
 &\quad \alpha_{r_k, r_k-h_k} \beta_{r_1+\dots+r_k-h_1-\dots-h_k, f-h_1-\dots-h_k} \\
 &+ \frac{1}{N^{r_1+r_2+\dots+r_k-1}} \sum_{h_1=0}^{r_1-1} \sum_{h_2=0}^{r_2-1} \dots \sum_{h_k=0}^{r_k-1} (-1)^{r_1+\dots+r_k-h_1-\dots-h_k-1} p_{i_1}^{r_1-h_1} \dots p_{i_k}^{r_k-h_k} \\
 &\quad \times \alpha_{r_1, r_1-h_1} \dots \alpha_{r_k, r_k-h_k} \beta_{r_1+\dots+r_k-h_1-\dots-h_k, r_1+\dots+r_k-h_1-\dots-h_k}
 \end{aligned} \right\} \quad (17).$$

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If we agree to put $A_{0,0} = 1$ and $C_r^{-1} = 0$, then if $r_1 + r_2 + \dots + r_k = 2r$, the first term can be brought to the form (cf. Introduction (28)):

$$\frac{1}{N^r} \left\{ \sum_{h_1=0}^{r \text{ (or } r_1-1)} \sum_{h_2=0}^{r-h_1 \text{ (or } r_2-1)} \dots \sum_{h_k=0}^{r-h_1-h_2-\dots-h_{k-1} \text{ (or } r_k-1)} (-1)^{r-H} p_{i_1}^{r_1-h_1} p_{i_2}^{r_2-h_2} \dots p_{i_k}^{r_k-h_k} \right. \\ \left. \times C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1,0} A_{h_2,0} \dots A_{h_k,0} B_{r-H,0} \right\} \quad (18),$$

and when $r_1 + r_2 + \dots + r_k = 2r + 1$, to the form (cf. Introduction (29)):

$$\frac{1}{N^{r+1}} \left\{ \sum_{h_1=0}^{r+1 \text{ (or } r_1-1)} \sum_{h_2=0}^{r+1-h_1 \text{ (or } r_2-1)} \dots \sum_{h_k=0}^{r+1-h_1-h_2-\dots-h_{k-1} \text{ (or } r_k-1)} (-1)^{r+1-H} p_{i_1}^{r_1-h_1} \right. \\ \times p_{i_2}^{r_2-h_2} \dots p_{i_k}^{r_k-h_k} \left\{ H \cdot C_{r_1}^{2h_1} \dots C_{r_k}^{2h_k} A_{h_1,0} \dots A_{h_k,0} B_{r+1-H,0} \right. \\ + C_{r_1}^{2h_1} \dots C_{r_k}^{2h_k} A_{h_1,0} \dots A_{h_k,0} B_{r+1-H,1} + C_{r_1}^{2h_1-1} C_{r_2}^{2h_2} \dots C_{r_k}^{2h_k} A_{h_1,1} \\ \times A_{h_2,0} \dots A_{h_k,0} B_{r+1-H,0} + \dots \\ \left. + C_{r_1}^{2h_1} C_{r_2}^{2h_2} \dots C_{r_{k-1}}^{2h_{k-1}} C_{r_k}^{2h_k-1} A_{h_1,0} A_{h_2,0} \dots A_{h_{k-1},0} A_{h_k,1} B_{r+1-H,0} \right\} \left. \right\} \quad (19).$$

Noting that, in accordance with (3),

$$E p'_{i_1} p'_{i_2} \dots p'_{i_k} = \frac{1}{N^k} N^{[k]} p_{i_1} p_{i_2} \dots p_{i_k} = p_{i_1} p_{i_2} \dots p_{i_k} \sum_{h=0}^{k-1} (-1)^h \frac{1}{N^h} \beta_{k,h} \quad (20),$$

we find, on the other hand, easily:

$$E (p'_{i_1} - p_{i_1}) (p'_{i_2} - p_{i_2}) \dots (p'_{i_k} - p_{i_k}) \\ = p_{i_1} p_{i_2} \dots p_{i_k} \sum_{h=0}^{k-1} \frac{(-1)^h}{N^h} \nabla^k \beta_{k,h} = p_{i_1} p_{i_2} \dots p_{i_k} \sum_{h=0}^{k-1} \frac{(-1)^h}{N^h} B_{h,2h-k} \quad (21).$$

Hence:

$$\left. \begin{aligned} E (p'_{i_1} - p_{i_1}) (p'_{i_2} - p_{i_2}) \dots (p'_{i_5} - p_{i_5}) &= p_{i_1} p_{i_2} \dots p_{i_5} \left\{ -\frac{20}{N^3} + \frac{24}{N^4} \right\} \\ E (p'_{i_1} - p_{i_1}) \dots (p'_{i_6} - p_{i_6}) &= p_{i_1} \dots p_{i_6} \left\{ -\frac{15}{N^3} + \frac{130}{N^4} - \frac{120}{N^5} \right\} \\ E (p'_{i_1} - p_{i_1}) \dots (p'_{i_7} - p_{i_7}) &= p_{i_1} \dots p_{i_7} \left\{ \frac{210}{N^4} - \frac{924}{N^5} + \frac{720}{N^6} \right\} \end{aligned} \right\} \quad (22).$$

II

(1) Noting that $m_{r,(N)} = EX_{(N)}^r = E \left[\sum_{j=1}^k p_j' \xi_j \right]^r$,

we find

$$\begin{aligned} m_{2,(N)} &= E \left\{ \sum_{j=1}^k p_j'^2 \xi_j^2 + \sum_{j_1=1}^k \sum_{j_2 \neq j_1}^k p'_{j_1} p'_{j_2} \xi_{j_1} \xi_{j_2} \right\} \\ &= \sum_{j=1}^k \xi_j^2 E p_j'^2 + \sum_{j_1=1}^k \sum_{j_2 \neq j_1}^k \xi_{j_1} \xi_{j_2} E p'_{j_1} p'_{j_2} \\ &= \sum_{j=1}^k \xi_j^2 \left[p_j^2 + \frac{1}{N} p_j (1 - p_j) \right] + \sum_{j_1=1}^k \sum_{j_2 \neq j_1}^k \xi_{j_1} \xi_{j_2} \left[p_{j_1} p_{j_2} - \frac{1}{N} p_{j_1} p_{j_2} \right] \\ &= \left\{ \sum_{j=1}^k p_j \xi_j \right\}^2 + \frac{1}{N} \sum_{j=1}^k p_j \xi_j^2 - \frac{1}{N} \left\{ \sum_{j=1}^k p_j \xi_j \right\}^2 = m_1^2 + \frac{1}{N} [m_2 - m_1^2]. \end{aligned}$$

In a similar way, the other formulae deduced above may be obtained (Chapter I, § II).

(2) Noting that

$$m_r' - m_r = \sum_{j=1}^k (p_j' - p_j) \xi_j^r,$$

we find :

$$\left. \begin{aligned} & E(m_{r_1}' - m_{r_1})(m_{r_2}' - m_{r_2}) \\ &= E \left\{ \sum_{j=1}^k (p_j' - p_j)^2 \xi_j^{r_1+r_2} + \sum_{j_1=1}^k \sum_{j_2 \neq j_1}^k (p_{j_1}' - p_{j_1})(p_{j_2}' - p_{j_2}) \xi_{j_1}^{r_1} \xi_{j_2}^{r_2} \right\} \\ &= \sum_{j=1}^k \xi_j^{r_1+r_2} \left[\frac{1}{N} p_j(1-p_j) \right] + \sum_{j_1=1}^k \sum_{j_2 \neq j_1}^k \xi_{j_1}^{r_1} \xi_{j_2}^{r_2} \left[-\frac{1}{N} p_{j_1} p_{j_2} \right] \\ &= \frac{1}{N} \{m_{r_1+r_2} - m_{r_1} m_{r_2}\} \end{aligned} \right\} \quad (23).$$

Similarly we find :

$$\left. \begin{aligned} & E(m_{r_1}' - m_{r_1})(m_{r_2}' - m_{r_2})(m_{r_3}' - m_{r_3}) \\ &= \frac{1}{N^2} \{m_{r_1+r_2+r_3} - [m_{r_1+r_2} m_{r_3} + m_{r_1+r_3} m_{r_2} + m_{r_2+r_3} m_{r_1}] + 2m_{r_1} m_{r_2} m_{r_3}\} \end{aligned} \right\} \quad (24),$$

$$\left. \begin{aligned} & E(m_{r_1}' - m_{r_1})(m_{r_2}' - m_{r_2})(m_{r_3}' - m_{r_3})(m_{r_4}' - m_{r_4}) \\ &= \frac{N-2}{N^3} \{ [m_{r_1+r_2} - m_{r_1} m_{r_2}] [m_{r_3+r_4} - m_{r_3} m_{r_4}] \\ &\quad + [m_{r_1+r_3} - m_{r_1} m_{r_3}] [m_{r_2+r_4} - m_{r_2} m_{r_4}] \\ &\quad + [m_{r_1+r_4} - m_{r_1} m_{r_4}] [m_{r_2+r_3} - m_{r_2} m_{r_3}] \} \\ &\quad + \frac{1}{N^3} \{ m_{r_1+r_2+r_3+r_4} - [m_{r_1+r_2+r_3} m_{r_4} + m_{r_1+r_2+r_4} m_{r_3} \\ &\quad + m_{r_1+r_3+r_4} m_{r_2} + m_{r_2+r_3+r_4} m_{r_1}] \\ &\quad + m_{r_1+r_2} m_{r_3+r_4} + m_{r_1+r_3} m_{r_2+r_4} + m_{r_1+r_4} m_{r_2+r_3} \} \end{aligned} \right\} \quad (25),$$

and hence, or directly from the formulae of § I :

$$\left. \begin{aligned} & E(m_r' - m_r)^2 = \frac{1}{N} [m_{2r} - m_r^2] \\ & E(m_r' - m_r)^3 = \frac{1}{N^2} [m_{3r} - 3m_{2r} m_r + 2m_r^3] \\ & E(m_r' - m_r)^4 = \frac{3(N-2)}{N^2} [m_{2r} - m_r^2]^2 + \frac{1}{N^3} [m_{4r} - 4m_{3r} m_r + 3m_{2r}^2] \end{aligned} \right\} \quad (26).$$

III

(1) Denoting by ω' the difference $X_{(N)} - m_1$ and by $d\mu_r'$, the difference $\mu_r' - \mu_r$, we find

$$\left. \begin{aligned} \nu_r' &= \sum_{j=1}^k p_j' [\xi_j - X_{(N)}]^r = \sum_{j=1}^k p_j' \sum_{h=0}^r (-1)^h C_r^h (\xi_j - m_1)^{r-h} \omega'^h \\ &= \mu_r' - C_r^1 \omega' \mu_{r-1}' + C_r^2 \omega'^2 \mu_{r-2}' - C_r^3 \omega'^3 \mu_{r-3}' + \dots \\ &= \mu_r + d\mu_r' - C_r^1 \omega' \mu_{r-1} - C_r^1 \omega' d\mu_{r-1}' + C_r^2 \omega'^2 \mu_{r-2} \\ &\quad + C_r^2 \omega'^2 d\mu_{r-2}' - C_r^3 \omega'^3 \mu_{r-3} - C_r^3 \omega'^3 d\mu_{r-3}' + \dots \end{aligned} \right\} (27).$$

W. F. Sheppard, in his well-known investigation "On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation*," terminates the development at the third term, taking

$$\nu_r' = \mu_r + d\mu_r' - r\omega' \mu_{r-1} \dots \dots \dots (28).$$

Hence :

$$\begin{aligned} v_r &= E\nu_r' = \mu_r, \\ \nu_r' - \nu_r &= \nu_r' - \mu_r = d\mu_r' - r\mu_{r-1} \omega', \\ E(\nu_r' - \nu_r)^2 &= E(\nu_r' - \mu_r)^2 = E(d\mu_r')^2 - 2r\mu_{r-1} E\omega' d\mu_r' + r^2 \mu_{r-1}^2 E\omega'^2 \\ &= \frac{1}{N} (\mu_{2r} - \mu_r^2) - \frac{2}{N} r\mu_{r-1} \mu_{r+1} + \frac{1}{N} r^2 \mu_{r-1}^2 \mu_2 \dots \\ &= \frac{1}{N} \{ \mu_{2r} - \mu_r^2 - 2r\mu_{r+1} \mu_{r-1} + r^2 \mu_{r-1}^2 \mu_2 \}. \end{aligned}$$

We thus obtain, with full accuracy (cf. Chapter III (13)), the first term of the development of $E(\nu_r' - \nu_r)^2$ in powers of $\frac{1}{N}$. This is explained by the fact that the terms rejected by Sheppard in the formula (27) do not yield terms of order $\frac{1}{N}$ in $E(\nu_r' - \nu_r)^2$. Owing to the same circumstance, we also get accurately the first term in the development of $E(\nu_{r_1}' - \nu_{r_1})(\nu_{r_2}' - \nu_{r_2})$, starting with (28):

$$\left. \begin{aligned} E(\nu_{r_1}' - \nu_{r_1})(\nu_{r_2}' - \nu_{r_2}) &= E(\nu_{r_1}' - \mu_{r_1})(\nu_{r_2}' - \mu_{r_2}) \\ &= E[d\mu_{r_1}' - r_1 \mu_{r_1-1} \omega'] [d\mu_{r_2}' - r_2 \mu_{r_2-1} \omega'] \\ &= \frac{1}{N} \{ \mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2} - r_1 \mu_{r_1-1} \mu_{r_2+1} - r_2 \mu_{r_1+1} \mu_{r_2-1} + r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} \mu_2 \} \end{aligned} \right\} (29).$$

* *Phil. Trans. A*, Vol. 192.

But we cannot start with (28) in the calculation of the mathematical expectation of $(\nu_r' - \nu_r)^3$, just as we cannot obtain the further terms in the development of $E(\nu_r' - \nu_r)^2$ in powers of $1/N$. For these purposes Sheppard's method must be put into a slightly changed form: more than three terms must be retained in (27). In the calculation of the terms of order $1/N^2$, we must at the same time rely on the formulae of § III of the second Chapter and on the following relations easily deduced from them:

$$\left. \begin{aligned} E\omega' d\mu_r' &= \frac{1}{N} \mu_{r+1} \\ E\omega'^2 &= \frac{1}{N} \mu_2 \end{aligned} \right\} \dots\dots\dots (30),$$

$$\left. \begin{aligned} E\omega' d\mu_{r_1}' d\mu_{r_2}' &= \frac{1}{N^2} [\mu_{r_1+r_2+1} - \mu_{r_1+1} \mu_{r_2} - \mu_{r_2+1} \mu_{r_1}] \\ E\omega'^2 d\mu_r' &= \frac{1}{N^2} [\mu_{r+2} - \mu_r \mu_2] \\ E\omega'^3 &= \frac{1}{N^2} \mu_3 \end{aligned} \right\} \dots\dots\dots (31),$$

$$\left. \begin{aligned} E\omega' d\mu_{r_1}' d\mu_{r_2}' d\mu_{r_3}' &= \frac{1}{N^3} [\mu_{r_1+r_2+r_3+1} + \mu_{r_2+1} \mu_{r_1+r_3} + \mu_{r_3+1} \mu_{r_1+r_2} \\ &\quad - \mu_{r_1+1} \mu_{r_2} \mu_{r_3} - \mu_{r_1} \mu_{r_2+1} \mu_{r_3} - \mu_{r_1} \mu_{r_2} \mu_{r_3+1}] \\ &\quad + \frac{1}{N^3} [\mu_{r_1+r_2+r_3+1} - (\mu_{r_1+r_2+1} \mu_{r_3} + \mu_{r_1+r_3+1} \mu_{r_2} + \mu_{r_2+r_3+1} \mu_{r_1}) \\ &\quad - (\mu_{r_1+r_2} \mu_{r_3+1} + \mu_{r_1+r_3} \mu_{r_2+1} + \mu_{r_1+1} \mu_{r_2+r_3}) \\ &\quad + 2(\mu_{r_1+1} \mu_{r_2} \mu_{r_3} + \mu_{r_1} \mu_{r_2+1} \mu_{r_3} + \mu_{r_1} \mu_{r_2} \mu_{r_3+1})] \\ E\omega'^2 d\mu_{r_1}' d\mu_{r_2}' &= \frac{1}{N^2} [\mu_{r_1+r_2} \mu_2 + 2\mu_{r_1+1} \mu_{r_2+1} - \mu_{r_1} \mu_{r_2} \mu_2] \\ &\quad + \frac{1}{N^3} [\mu_{r_1+r_2+2} - (\mu_{r_1+2} \mu_{r_2} + \mu_{r_1} \mu_{r_2+2}) \\ &\quad - (\mu_{r_1+r_2} \mu_2 + 2\mu_{r_1+1} \mu_{r_2+1}) + 2\mu_{r_1} \mu_{r_2} \mu_2] \\ E\omega'^3 d\mu_r &= \frac{3}{N^2} \mu_{r+1} \mu_2 + \frac{1}{N^3} [\mu_{r+3} - \mu_r \mu_3 - 3\mu_{r+1} \mu_2] \\ E\omega'^4 &= \frac{3}{N^2} \mu_2^2 + \frac{1}{N^3} [\mu_4 - 3\mu_2^2] \end{aligned} \right\} (32).$$

(2) To get the exact value of terms of order $1/N^2$ in the development of $E(\nu_{r_1}' - \mu_{r_1})(\nu_{r_2}' - \mu_{r_2})$ in powers of $1/N$, it is necessary to start from

$$\begin{aligned} \nu_r' - \mu_r &= (d\mu_r' - r\mu_{r-1}\omega') - \left(r\omega' d\mu_{r-1}' - \frac{r(r-1)}{2} \mu_{r-2}\omega'^2 \right) \\ &\quad + \left(\frac{r(r-1)}{2} \omega'^2 d\mu_{r-2}' - \frac{r(r-1)(r-2)}{6} \mu_{r-3}\omega'^3 \right). \end{aligned}$$

After some simple transformations we find :

$$\begin{aligned}
 E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2}) = & \frac{1}{N} \{ \mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2} - r_1 \mu_{r_1-1} \mu_{r_2+1} - r_2 \mu_{r_1+1} \mu_{r_2-1} \\
 & + r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} \mu_2 \} \\
 + \frac{1}{N^2} \{ & -(r_1 + r_2) \mu_{r_1+r_2} + \frac{(r_1 + r_2)(r_1 + r_2 - 1)}{2} \mu_{r_1+r_2-2} \mu_2 \\
 & + \frac{r_2(r_2 - 1)}{2} \mu_{r_1+2} \mu_{r_2-2} + \frac{r_1(r_1 - 1)}{2} \mu_{r_1-2} \mu_{r_2+2} \\
 & + r_2(r_1 + r_2) \mu_{r_1+1} \mu_{r_2-1} + r_1(r_1 + r_2) \mu_{r_1-1} \mu_{r_2+1} - \frac{r_2^{[-3]}}{2} \mu_{r_1+1} \mu_{r_2-3} \mu_2 \\
 & - \frac{1}{2} r_1^{[-3]} \mu_{r_1-3} \mu_{r_2+1} \mu_2 + (r_1 + r_2 + 2r_1 r_2) \mu_{r_1} \mu_{r_2} \\
 & - \frac{r_2(r_2 - 1)}{2} (3r_1 + 2) \mu_{r_1} \mu_{r_2-2} \mu_2 - \frac{r_1(r_1 - 1)}{2} (3r_2 + 2) \mu_{r_1-2} \mu_{r_2} \mu_2 \\
 & - \frac{3}{2} r_1 r_2 (r_1 + r_2) \mu_{r_1-1} \mu_{r_2-1} \mu_2 - \frac{1}{2} r_1 r_2 (r_2 - 1) \mu_{r_1-1} \mu_{r_2-2} \mu_3 \\
 & - \frac{1}{2} r_1 (r_1 - 1) r_2 \mu_{r_1-2} \mu_{r_2-1} \mu_3 + \frac{1}{2} r_1 r_2^{[-3]} \mu_{r_1-1} \mu_{r_2-3} \mu_2^2 + \frac{1}{2} r_1^{[-3]} r_2 \mu_{r_1-3} \mu_{r_2-1} \mu_2^2 \\
 & + \frac{3}{4} r_1 (r_1 - 1) r_2 (r_2 - 1) \mu_{r_1-2} \mu_{r_2-2} \mu_2^2 \} + \dots \} \quad (33).
 \end{aligned}$$

Noting that

$$\begin{aligned}
 E(v'_{r_1} - v_{r_1})(v'_{r_2} - v_{r_2}) &= E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2}) - (v_{r_1} - \mu_{r_1})(v_{r_2} - \mu_{r_2}) \\
 &= E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2}) - \frac{1}{N^2} [r_1 r_2 \mu_{r_1} \mu_{r_2} - \frac{1}{2} r_1 r_2 (r_2 - 1) \mu_{r_1} \mu_{r_2-2} \mu_2 \\
 &\quad - \frac{1}{2} r_1 (r_1 - 1) r_2 \mu_{r_1-2} \mu_{r_2} \mu_2 + \frac{1}{4} r_1 (r_1 - 1) r_2 (r_2 - 1) \mu_{r_1-2} \mu_{r_2-2} \mu_2^2] - \dots,
 \end{aligned}$$

we find hence :

$$\begin{aligned}
 E(v'_{r_1} - v_{r_1})(v'_{r_2} - v_{r_2}) = & \frac{1}{N} \{ \mu_{r_1+r_2} - r_2 \mu_{r_1+1} \mu_{r_2-1} - r_1 \mu_{r_1-1} \mu_{r_2+1} \\
 & - \mu_{r_1} \mu_{r_2} + r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} \mu_2 \} \\
 + \frac{1}{N^2} \{ & -(r_1 + r_2) \mu_{r_1+r_2} + \frac{(r_1 + r_2)(r_1 + r_2 - 1)}{2} \mu_{r_1+r_2-2} \mu_2 \\
 & + \frac{1}{2} r_2 (r_2 - 1) \mu_{r_1+2} \mu_{r_2-2} + \frac{1}{2} r_1 (r_1 - 1) \mu_{r_1-2} \mu_{r_2+2} \\
 & + r_2 (r_1 + r_2) \mu_{r_1+1} \mu_{r_2-1} + r_1 (r_1 + r_2) \mu_{r_1-1} \mu_{r_2+1} - \frac{1}{2} r_2^{[-3]} \mu_{r_1+1} \mu_{r_2-3} \mu_2 \\
 & - \frac{1}{2} r_1^{[-3]} \mu_{r_1-3} \mu_{r_2+1} \mu_2 + (r_1 + r_2 + r_1 r_2) \mu_{r_1} \mu_{r_2} \\
 & - (r_1 + 1) r_2 (r_2 - 1) \mu_{r_1} \mu_{r_2-2} \mu_2 - r_1 (r_1 - 1) (r_2 + 1) \mu_{r_1-2} \mu_{r_2} \mu_2 \\
 & - \frac{3}{2} r_1 r_2 (r_1 + r_2) \mu_{r_1-1} \mu_{r_2-1} \mu_2 - \frac{1}{2} r_1 r_2 (r_2 - 1) \mu_{r_1-1} \mu_{r_2-2} \mu_3 \\
 & - \frac{1}{2} r_1 (r_1 - 1) r_2 \mu_{r_1-2} \mu_{r_2-1} \mu_3 + \frac{1}{2} r_1 r_2^{[-3]} \mu_{r_1-1} \mu_{r_2-3} \mu_2^2 + \frac{1}{2} r_1^{[-3]} r_2 \mu_{r_1-3} \mu_{r_2-1} \mu_2^2 \\
 & + \frac{1}{2} r_1 (r_1 - 1) r_2 (r_2 - 1) \mu_{r_1-2} \mu_{r_2-2} \mu_2^2 \} + \dots \} \quad (34).
 \end{aligned}$$

Putting $r_1 = r_2 = r$ in this, we find (cf. (13) of Chapter III):

$$E(v_r' - v_r)^2 = \frac{1}{N} \{ \mu_{2r} - 2r \mu_{r+1} \mu_{r-1} - \mu_r^2 + r^2 \mu_{r-1}^2 \mu_2 \} \\ + \frac{1}{N^2} \{ -2r \mu_{2r} + r(2r-1) \mu_{2r-2} \mu_2 + r(r-1) \mu_{r+2} \mu_{r-2} \\ + 4r^2 \mu_{r+1} \mu_{r-1} - r^{[-3]} \mu_{r+1} \mu_{r-3} \mu_2 \\ + r(r+2) \mu_r^2 - 2r(r^2-1) \mu_r \mu_{r-2} \mu_2 - 3r^3 \mu_{r-1}^2 \mu_2 - r^2(r-1) \mu_{r-1} \mu_{r-2} \mu_3 \\ + r^2(r-1)(r-2) \mu_{r-1} \mu_{r-3} \mu_2^2 + \frac{1}{2} r^2(r-1)^2 \mu_{r-2}^2 \mu_2^2 \} + \dots \quad (35).$$

(3) To find accurately the first term (of order $1/N^2$) in the development of

$$E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3}),$$

we must start with

$$v_r' - \mu_r = (d\mu_r' - r\mu_{r-1}\omega') - \left(r\omega' d\mu_{r-1}' - \frac{r(r-1)}{2} \mu_{r-2}\omega'^2 \right).$$

Using the relations (30), (31) and (32), we find without difficulty:

$$E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3}) \\ = \frac{1}{N^2} \left\{ \mu_{r_1+r_2+r_3} - [\mu_{r_1+r_2} \mu_{r_3} + \mu_{r_1+r_3} \mu_{r_2} + \mu_{r_1} \mu_{r_2+r_3}] + 2\mu_{r_1} \mu_{r_2} \mu_{r_3} \right. \\ - [r_1 \mu_{r_1-1} (\mu_{r_2+r_3+1} - \mu_{r_2+1} \mu_{r_3} - \mu_{r_2} \mu_{r_3+1}) + r_2 \mu_{r_2-1} (\mu_{r_1+r_3+1} - \mu_{r_1+1} \mu_{r_2} \\ - \mu_{r_1} \mu_{r_2+1}) + r_3 \mu_{r_3-1} (\mu_{r_1+r_2+1} - \mu_{r_1+1} \mu_{r_2} - \mu_{r_1} \mu_{r_2+1})] \\ + [r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} (\mu_{r_3+2} - \mu_{r_3} \mu_2) + r_1 r_3 \mu_{r_1-1} \mu_{r_3-1} (\mu_{r_2+2} - \mu_{r_2} \mu_2) \\ + r_2 r_3 \mu_{r_2-1} \mu_{r_3-1} (\mu_{r_1+2} - \mu_{r_1} \mu_2)] - r_1 r_2 r_3 \mu_{r_1-1} \mu_{r_2-1} \mu_{r_3-1} \mu_3 \\ - [r_3 (\mu_{r_1+1} \mu_{r_2+r_3-1} + \mu_{r_2+1} \mu_{r_1+r_3-1} + \mu_{r_3} \mu_{r_1+r_2} - \mu_{r_1+1} \mu_{r_2} \mu_{r_3-1} \\ - \mu_{r_1} \mu_{r_2+1} \mu_{r_3-1} - \mu_{r_1} \mu_{r_2} \mu_{r_3}) \\ + r_2 (\mu_{r_1+1} \mu_{r_2+r_3-1} + \mu_{r_3+1} \mu_{r_1+r_2-1} + \mu_{r_2} \mu_{r_1+r_3} - \mu_{r_1+1} \mu_{r_2-1} \mu_{r_3} \\ - \mu_{r_1} \mu_{r_2-1} \mu_{r_3+1} - \mu_{r_1} \mu_{r_2} \mu_{r_3}) \\ + r_1 (\mu_{r_3+1} \mu_{r_1+r_2-1} + \mu_{r_2+1} \mu_{r_1+r_3-1} + \mu_{r_1} \mu_{r_2+r_3} - \mu_{r_1-1} \mu_{r_2} \mu_{r_3+1} \\ - \mu_{r_1-1} \mu_{r_2+1} \mu_{r_3} - \mu_{r_1} \mu_{r_2} \mu_{r_3})] \\ + [r_1 r_3 \mu_{r_1-1} (\mu_{r_2+r_3-1} \mu_2 + 2\mu_{r_2+1} \mu_{r_3} - \mu_{r_2} \mu_{r_3-1} \mu_2) \\ + r_2 r_3 \mu_{r_2-1} (\mu_{r_1+r_3-1} \mu_2 + 2\mu_{r_1+1} \mu_{r_3} - \mu_{r_1} \mu_{r_3-1} \mu_2) \\ + r_1 r_2 \mu_{r_1-1} (\mu_{r_2+r_3-1} \mu_2 + 2\mu_{r_2} \mu_{r_3+1} - \mu_{r_2-1} \mu_{r_3} \mu_2) \\ + r_2 r_3 \mu_{r_2-1} (\mu_{r_1+r_2-1} \mu_2 + 2\mu_{r_1+1} \mu_{r_2} - \mu_{r_1} \mu_{r_2-1} \mu_2) \\ + r_1 r_2 \mu_{r_1-1} (\mu_{r_1+r_3-1} \mu_2 + 2\mu_{r_1} \mu_{r_3+1} - \mu_{r_1-1} \mu_{r_3} \mu_2) \\ + r_1 r_3 \mu_{r_3-1} (\mu_{r_1+r_2-1} \mu_2 + 2\mu_{r_1} \mu_{r_2+1} - \mu_{r_1-1} \mu_{r_2} \mu_2)] \\ - [3r_1 r_2 r_3 (\mu_{r_1-1} \mu_{r_2-1} \mu_{r_3} \mu_2 + \mu_{r_1-1} \mu_{r_2} \mu_{r_3-1} \mu_2 + \mu_{r_1} \mu_{r_2-1} \mu_{r_3-1} \mu_2)] \\ + [\frac{1}{2} r_3 (r_3-1) \mu_{r_3-2} (\mu_{r_1+r_2} \mu_2 + 2\mu_{r_1+1} \mu_{r_2+1} - \mu_{r_1} \mu_{r_2} \mu_2) \\ + \frac{1}{2} r_2 (r_2-1) \mu_{r_2-1} (\mu_{r_1+r_3} \mu_2 + 2\mu_{r_1+1} \mu_{r_3+1} - \mu_{r_1} \mu_{r_3} \mu_2) \\ + \frac{1}{2} r_1 (r_1-1) \mu_{r_1-1} (\mu_{r_2+r_3} \mu_2 + 2\mu_{r_2+1} \mu_{r_3+1} - \mu_{r_2} \mu_{r_3} \mu_2)] \quad (36).$$

$$\begin{aligned}
& -3 \left[\frac{1}{2} r_1 r_3 (r_3 - 1) \mu_{r_1-1} \mu_{r_2+1} \mu_{r_3-2} \mu_2 + \frac{1}{2} r_2 r_3 (r_3 - 1) \mu_{r_1+1} \mu_{r_2-1} \mu_{r_3-2} \mu_2 \right. \\
& \quad + \frac{1}{2} r_1 r_2 (r_2 - 1) \mu_{r_1-1} \mu_{r_2-2} \mu_{r_3+1} \mu_2 + \frac{1}{2} r_2 (r_2 - 1) r_3 \mu_{r_1+1} \mu_{r_2-2} \mu_{r_3-1} \mu_2 \\
& \quad + \frac{1}{2} r_1 (r_1 - 1) r_2 \mu_{r_1-2} \mu_{r_2-1} \mu_{r_3+1} \mu_2 + \frac{1}{2} r_1 (r_1 - 1) r_3 \mu_{r_1-2} \mu_{r_2+1} \mu_{r_3-1} \mu_2 \Big] \\
& + 3 \left[\frac{1}{2} r_1 r_2 r_3 (r_3 - 1) \mu_{r_1-1} \mu_{r_2-1} \mu_{r_3-2} \mu_2^2 + \frac{1}{2} r_1 r_2 (r_2 - 1) \right. \\
& \quad \times r_3 \mu_{r_1-1} \mu_{r_2-2} \mu_{r_3-1} \mu_2^2 + \frac{1}{2} r_1 (r_1 - 1) r_2 r_3 \mu_{r_1-2} \mu_{r_2-1} \mu_{r_3-1} \mu_2^2 \Big] \Big\} + \dots
\end{aligned}$$

Noting that

$$\begin{aligned}
& E(v'_{r_1} - v_{r_1})(v'_{r_2} - v_{r_2})(v'_{r_3} - v_{r_3}) = E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3}) \\
& \quad - \{(v_{r_1} - \mu_{r_1}) E(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3}) + (v_{r_2} - \mu_{r_2}) \\
& \quad \times E(v'_{r_1} - \mu_{r_1})(v'_{r_3} - \mu_{r_3}) + (v_{r_3} - \mu_{r_3}) E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})\} \\
& \quad + 2(v_{r_1} - \mu_{r_1})(v_{r_2} - \mu_{r_2})(v_{r_3} - \mu_{r_3}) \\
& = E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3}) + \frac{1}{N^2} \{ [r_1 \mu_{r_1} - \frac{1}{2} r_1 (r_1 - 1) \mu_{r_1-2} \mu_2] \\
& \quad \times [\mu_{r_2+r_3} - \mu_{r_2} \mu_{r_3} - r_2 \mu_{r_2-1} \mu_{r_3+1} - r_3 \mu_{r_2+1} \mu_{r_3-1} + r_2 r_3 \mu_{r_2-1} \mu_{r_3-1} \mu_2] \\
& \quad + [r_2 \mu_{r_2} - \frac{1}{2} r_2 (r_2 - 1) \mu_{r_2-2} \mu_2] [\mu_{r_1+r_3} - \mu_{r_1} \mu_{r_3} - r_1 \mu_{r_1-1} \mu_{r_3+1} \\
& \quad - r_3 \mu_{r_1+1} \mu_{r_3-1} + r_1 r_3 \mu_{r_1-1} \mu_{r_3-1} \mu_2] \\
& \quad + [r_3 \mu_{r_3} - \frac{1}{2} r_3 (r_3 - 1) \mu_{r_3-2} \mu_2] [\mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2} - r_1 \mu_{r_1-1} \mu_{r_2+1} \\
& \quad - r_2 \mu_{r_1+1} \mu_{r_2-1} + r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} \mu_2] \Big\} + \dots
\end{aligned} \tag{37}$$

we find hence the first term in the development of

$$E(v'_{r_1} - v_{r_1})(v'_{r_2} - v_{r_2})(v'_{r_3} - v_{r_3})$$

in powers of $1/N$.

Putting $r_1 = r_2 = r_3 = r$, we find:

$$\begin{aligned}
& E(v'_r - \mu_r)^3 = \frac{1}{N^2} \{ \mu_{3r} - 3r \mu_{2r+1} \mu_{r-1} - 3(r+1) \mu_{2r} \mu_r + \frac{3}{2} r(r-1) \mu_{2r} \mu_{r-2} \mu_2 \\
& \quad - 6r \mu_{2r-1} \mu_{r+1} + 6r^2 \mu_{2r-1} \mu_{r-1} \mu_2 + 3r^2 \mu_{r+2} \mu_{r-1}^2 + 12r(r+1) \mu_{r+1} \mu_r \mu_{r-1} \\
& \quad + 3r(r-1) \mu_{r+1}^2 \mu_{r-2} - 9r^2(r-1) \mu_{r+1} \mu_{r-1} \mu_{r-2} \mu_2 \\
& \quad + (3r+2) \mu_r^3 - \frac{3}{2} r(r-1) \mu_r^2 \mu_{r-2} \mu_2 \\
& \quad - 9r^2(r+1) \mu_r \mu_{r-1}^2 \mu_2 - r^3 \mu_{r-1}^3 \mu_3 + \frac{9}{2} r^3(r-1) \mu_{r-1}^2 \mu_{r-2} \mu_2^2 \Big\} + \dots
\end{aligned} \tag{38}$$

$$\begin{aligned}
& E(v'_r - v_r)^3 = E(v'_r - \mu_r)^3 + \frac{3}{N^2} [r \mu_r - \frac{1}{2} r(r-1) \mu_{r-2} \mu_2] \\
& \quad \times [\mu_{2r} - 2r \mu_{r+1} \mu_{r-1} - \mu_r^2 + r^2 \mu_{r-1}^2 \mu_2] + \dots \\
& = \frac{1}{N^2} \{ \mu_{3r} - 3r \mu_{2r+1} \mu_{r-1} - 3 \mu_{2r} \mu_r - 6r \mu_{2r-1} \mu_{r+1} \\
& \quad + 6r^2 \mu_{2r-1} \mu_{r-1} \mu_2 + 3r^2 \mu_{r+2} \mu_{r-1}^2 \\
& \quad + 6r(r+2) \mu_{r+1} \mu_r \mu_{r-1} + 3r(r-1) \mu_{r+1}^2 \mu_{r-2} \\
& \quad - 6r^2(r-1) \mu_{r+1} \mu_{r-1} \mu_{r-2} \mu_2 + 2 \mu_r^3 \\
& \quad - 3r^2(2r+3) \mu_r \mu_{r-1}^2 \mu_2 - r^3 \mu_{r-1}^3 \mu_3 + 3r^3(r-1) \mu_{r-1}^2 \mu_{r-2} \mu_2^2 \Big\} + \dots
\end{aligned} \tag{39}$$

(4) The first term (of order $1/N^2$) in the development of

$$E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4})$$

in powers of $1/N$ is obtained exactly from (24). We find:

$$\begin{aligned} & E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4}) \\ &= \frac{1}{N^2} \{ [\mu_{r_1+r_2} - \mu_{r_1} \mu_{r_2}] [\mu_{r_3+r_4} - \mu_{r_3} \mu_{r_4}] + [\mu_{r_1+r_3} - \mu_{r_1} \mu_{r_3}] [\mu_{r_2+r_4} - \mu_{r_2} \mu_{r_4}] \\ & \quad + [\mu_{r_1+r_4} - \mu_{r_1} \mu_{r_4}] [\mu_{r_2+r_3} - \mu_{r_2} \mu_{r_3}] \\ & \quad - [r_1 \mu_{r_1-1} (\mu_{r_2+1} \mu_{r_3+r_4} + \mu_{r_3+1} \mu_{r_2+r_4} + \mu_{r_1+1} \mu_{r_2+r_3} - \mu_{r_2+1} \mu_{r_3} \mu_{r_4} \\ & \quad - \mu_{r_2} \mu_{r_3+1} \mu_{r_4} - \mu_{r_2} \mu_{r_3} \mu_{r_4+1}) \\ & \quad + r_2 \mu_{r_2-1} (\mu_{r_1+1} \mu_{r_3+r_4} + \mu_{r_3+1} \mu_{r_1+r_4} + \mu_{r_4+1} \mu_{r_1+r_3} - \mu_{r_1+1} \mu_{r_3} \mu_{r_4} \\ & \quad - \mu_{r_1} \mu_{r_3+1} \mu_{r_4} - \mu_{r_1} \mu_{r_3} \mu_{r_4+1}) \\ & \quad + r_3 \mu_{r_3-1} (\mu_{r_1+1} \mu_{r_2+r_4} + \mu_{r_2+1} \mu_{r_1+r_4} + \mu_{r_4+1} \mu_{r_1+r_3} - \mu_{r_1+1} \mu_{r_2} \mu_{r_4} \\ & \quad - \mu_{r_1} \mu_{r_2+1} \mu_{r_4} - \mu_{r_1} \mu_{r_2} \mu_{r_4+1}) \\ & \quad + r_4 \mu_{r_4-1} (\mu_{r_1+1} \mu_{r_2+r_3} + \mu_{r_2+1} \mu_{r_1+r_3} + \mu_{r_3+1} \mu_{r_1+r_2} - \mu_{r_1+1} \mu_{r_2} \mu_{r_3} \\ & \quad - \mu_{r_1} \mu_{r_2+1} \mu_{r_3} - \mu_{r_1} \mu_{r_2} \mu_{r_3+1})] \\ & \quad + [r_1 r_2 \mu_{r_1-1} \mu_{r_2-1} (\mu_{r_3+r_4} \mu_2 + 2\mu_{r_3+1} \mu_{r_4+1} - \mu_{r_3} \mu_{r_4} \mu_2) \\ & \quad + r_1 r_3 \mu_{r_1-1} \mu_{r_3-1} (\mu_{r_2+r_4} \mu_2 + 2\mu_{r_2+1} \mu_{r_4+1} - \mu_{r_2} \mu_{r_4} \mu_2) \\ & \quad + r_1 r_4 \mu_{r_1-1} \mu_{r_4-1} (\mu_{r_2+r_3} \mu_2 + 2\mu_{r_2+1} \mu_{r_3+1} - \mu_{r_2} \mu_{r_3} \mu_2) \\ & \quad + r_2 r_3 \mu_{r_2-1} \mu_{r_3-1} (\mu_{r_1+r_2} \mu_2 + 2\mu_{r_1+1} \mu_{r_2+1} - \mu_{r_1} \mu_{r_2} \mu_2) \\ & \quad + r_2 r_4 \mu_{r_2-1} \mu_{r_4-1} (\mu_{r_1+r_3} \mu_2 + 2\mu_{r_1+1} \mu_{r_3+1} - \mu_{r_1} \mu_{r_3} \mu_2) \\ & \quad + r_3 r_4 \mu_{r_3-1} \mu_{r_4-1} (\mu_{r_1+r_2} \mu_2 + 2\mu_{r_1+1} \mu_{r_2+1} - \mu_{r_1} \mu_{r_2} \mu_2)] \\ & \quad - 3 [r_1 r_2 r_3 \mu_{r_1-1} \mu_{r_2-1} \mu_{r_3-1} \mu_{r_4+1} \mu_2 + r_1 r_2 r_4 \mu_{r_1-1} \mu_{r_2-1} \mu_{r_3+1} \mu_{r_4-1} \mu_2 \\ & \quad + r_1 r_3 r_4 \mu_{r_1-1} \mu_{r_2+1} \mu_{r_3-1} \mu_{r_4-1} \mu_2 + r_2 r_3 r_4 \mu_{r_1+1} \mu_{r_2-1} \mu_{r_3-1} \mu_{r_4-1} \mu_2] \\ & \quad + 3r_1 r_2 r_3 r_4 \mu_{r_1-1} \mu_{r_2-1} \mu_{r_3-1} \mu_{r_4-1} \mu_2^2 \} + \dots \end{aligned} \quad (40).$$

Noting that

$$\begin{aligned} & E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4}) \\ &= E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4}) \\ & \quad - \{(v_{r_1} - \mu_{r_1}) E(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_2} - \mu_{r_2}) E(v'_{r_1} - \mu_{r_1})(v'_{r_3} - \mu_{r_3})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_3} - \mu_{r_3}) E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_4} - \mu_{r_4}) E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})(v'_{r_3} - \mu_{r_3})\} \\ & \quad + \{(v_{r_1} - \mu_{r_1})(v_{r_3} - \mu_{r_3}) E(v'_{r_2} - \mu_{r_2})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_1} - \mu_{r_1})(v_{r_3} - \mu_{r_3}) E(v'_{r_2} - \mu_{r_2})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_2} - \mu_{r_2})(v_{r_3} - \mu_{r_3}) E(v'_{r_1} - \mu_{r_1})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_2} - \mu_{r_2})(v_{r_3} - \mu_{r_3}) E(v'_{r_1} - \mu_{r_1})(v'_{r_4} - \mu_{r_4}) \\ & \quad + (v_{r_3} - \mu_{r_3})(v_{r_4} - \mu_{r_4}) E(v'_{r_1} - \mu_{r_1})(v'_{r_2} - \mu_{r_2})\} \\ & \quad - 3(v_{r_1} - \mu_{r_1})(v_{r_2} - \mu_{r_2})(v_{r_3} - \mu_{r_3})(v_{r_4} - \mu_{r_4}), \end{aligned}$$

we see that the terms of order $1/N^2$ in the developments of

$$E(\nu'_{r_1} - \mu_{r_1})(\nu'_{r_2} - \mu_{r_2})(\nu'_{r_3} - \mu_{r_3})(\nu'_{r_4} - \mu_{r_4})$$

and

$$E(\nu'_{r_1} - \nu_{r_1})(\nu'_{r_2} - \nu_{r_2})(\nu'_{r_3} - \nu_{r_3})(\nu'_{r_4} - \nu_{r_4})$$

coincide.

Putting $r_1 = r_2 = r_3 = r_4 = r$, we find:

$$\left. \begin{aligned} E(\nu'_r - \nu_r)^4 &= \frac{1}{N^2} \{ 3[\mu_{2r} - \mu_r^2]^2 - 12r\mu_{r-1}[\mu_{2r}\mu_{r+1} - \mu_{r+1}\mu_r^2] \\ &+ 6r^2\mu_{r-1}^2[\mu_{2r}\mu_r + 2\mu_{r+1}^2 - \mu_r^2\mu_r] - 12r^3\mu_{r+1}\mu_{r-1}^2\mu_r + 3r^4\mu_{r-1}^4\mu_r^2 \} + \dots \\ &= \frac{3}{N^2} [\mu_{2r} - \mu_r^2 - 2r\mu_{r+1}\mu_{r-1} + r^2\mu_{r-1}^2\mu_r]^2 + \dots \end{aligned} \right\} \quad (41).$$

The same formula (41) gives also the first term (of order $1/N^2$) in the development of $E(\nu'_r - \mu_r)^4$.

(5) In the general case if we agree to denote

$$(\nu'_{r_1} - \nu_{r_1})(\nu'_{r_2} - \nu_{r_2}) \dots (\nu'_{r_i} - \nu_{r_i}) \text{ by } d^{(i)}\nu,$$

and

$$(\nu'_{r_1} - \mu_{r_1})(\nu'_{r_2} - \mu_{r_2}) \dots (\nu'_{r_i} - \mu_{r_i}) \text{ by } \delta^{(i)}\nu,$$

we have:

$$\left. \begin{aligned} Ed^{(i)}\nu &= E\delta^{(i)}\nu - \sum_{h=1}^i (\nu_{r_h} - \mu_{r_h}) E \frac{\delta^{(i)}\nu}{\nu'_{r_h} - \mu_{r_h}} \\ &+ \sum_{h_1=1}^{i-1} \sum_{h_2=h_1+1}^i (\nu_{r_{h_1}} - \mu_{r_{h_1}})(\nu_{r_{h_2}} - \mu_{r_{h_2}}) E \frac{\delta^{(i)}\nu}{(\nu'_{r_{h_1}} - \mu_{r_{h_1}})(\nu'_{r_{h_2}} - \mu_{r_{h_2}})} + \dots \end{aligned} \right\} \quad (42).$$

We see that, in the developments of $Ed^{(2i)}\nu$ and $E\delta^{(2i)}\nu$ in powers of $1/N$, the first terms (of order $1/N^i$) coincide, since

$$\sum_{h=1}^i (\nu_{r_h} - \mu_{r_h}) E \frac{\delta^{(2i)}\nu}{\nu'_{r_h} - \mu_{r_h}}$$

contains no terms of order $\frac{1}{N^i}$. On the contrary, in the developments of $Ed^{(2i+1)}\nu$ and $E\delta^{(2i+1)}\nu$ the first terms (of order $1/N^{i+1}$) are different, for

$$\sum_{h=1}^i (\nu_{r_h} - \mu_{r_h}) E \frac{\delta^{(2i+1)}\nu}{\nu'_{r_h} - \mu_{r_h}}$$

contains terms of order $\frac{1}{N^{i+1}}$.

Formulae (18) and (19) of Chapter II permit us to calculate $E\delta^{(i)}\nu$ and $Ed^{(i)}\nu$ in general, to an arbitrary degree of accuracy, in the same way as $Ed^{(2)}\nu$, $Ed^{(3)}\nu$, $Ed^{(4)}\nu$ were found above. The actual expression, however, is of so unwieldy a character that I shall limit myself to the calculation of the first term in the development of $E(\nu'_r - \nu_r)^{2i}$, coinciding with the first term in the development of $E(\nu'_r - \mu_r)^{2i}$.

In the calculation of the first term of the development of $E(\nu'_r - \mu_r)^{2i}$ we may take

$$\nu_r - \mu_r = d\mu'_r - r\mu_{r-1}\alpha',$$

and limit ourselves to the calculation of the terms of order $1/N^i$ in the development

$$E(d\mu_r' - r\mu_{r-1}\omega')^{2i} = \sum_{j=0}^{2i} (-1)^j C_{2i}^{j, r^j} \mu_{r-1}^j E(d\mu_r')^{2i-j} \omega'^j \dots\dots\dots(43).$$

In formula (19) of the second Chapter put $r_1 = r_2 = \dots = r_j = 1$, and

$$r_{j+1} = r_{j+2} = \dots = r_{2i} = r;$$

we find for $j=0$:

$$E(d\mu_r')^{2i} = \frac{1}{N^i} 1.3.5 \dots (2i-1) [\mu_{2r} - \mu_r^2]^i + \dots,$$

for $j=1$:

$$E(d\mu_r')^{2i-1} \omega' = \frac{1}{N^i} 1.3.5 \dots (2i-1) \mu_{r+1} [\mu_{2r} - \mu_r^2]^{i-1} + \dots\dots\dots(44).$$

When $j=2h$:

$$\begin{aligned} E(d\mu_r')^{2i-2h} \omega'^{2h} &= \frac{1}{N^i} \sum_{l=0}^{i-h} (-1)^{i-h-l} 1.3.5 \dots (2i-2h-2l-1) C_{2i-2h}^{2l} \mu_r^{2i-2h-2l} \sum_{f=0}^{l \text{ (or } h)} \\ &\times C_{2l}^{2f} [2h]^{[-2f]} 1.3.5 \dots (2l-2f-1).1.3.5 \dots (2h-2f-1) \mu_{2r}^{l-f} \mu_{r+1}^{2f} \mu_2^{h-f} \\ &= \frac{1.3.5 \dots (2h-1).1.3.5 \dots (2i-2h-1)}{N^i} \sum_{l=0}^{i-h} \sum_{f=0}^{l \text{ (or } h)} (-1)^{i-h-l} \\ &\times \frac{2^{2f} [i-h]^{[-l]} h^{[-f]} l^{[-f]} \mu_{2r}^{l-f} \mu_{r+1}^{2f} \mu_r^{2i-2h-2l} \mu_2^{h-f}}{(l-f)! (2f)!} \\ &= \frac{1.3.5 \dots (2h-1).1.3.5 \dots (2i-2h-1)}{N^i} \\ &\times \sum_{g=0}^{h \text{ (or } i-h)} \frac{2^{2g} \mu_{r+1}^{2g} h^{[-g]} [i-h]^{[-g]} \mu_2^{h-g}}{(2g)!} [\mu_{2r} - \mu_r^2]^{i-h-g} + \dots \end{aligned} \quad (45).$$

When $j=2h+1$:

$$\begin{aligned} E(d\mu_r')^{2i-2h-1} \omega'^{2h+1} &= \frac{1}{N^i} \sum_{l=0}^{i-h-1} (-1)^{i-h-1-l} 1.3.5 \dots (2i-2h-2l-3) C_{2i-2h-1}^{2l+1} \mu_r^{2i-2h-2l-2} \\ &\times \sum_{f=0}^{l \text{ (or } h)} C_{2l+1}^{2f+1} [2h+1]^{[-(2f+1)]} 1.3.5 \dots (2l-2f-1).1.3.5 \dots (2h-2f-1) \\ &\times \mu_{2r}^{l-f} \mu_{r+1}^{2f+1} \mu_2^{h-f} \\ &= \frac{1.3.5 \dots (2h+1).1.3.5 \dots (2i-2h-1)}{N^i} \sum_{l=0}^{i-h-1} \sum_{f=0}^{l \text{ (or } h)} (-1)^{i-h-1-l} \\ &\times \frac{2^{2f} [i-h-1]^{[-l]} h^{[-f]} l^{[-f]} \mu_{2r}^{l-f} \mu_{r+1}^{2f+1} \mu_r^{2i-2h-2l-2} \mu_2^{h-f}}{(l-f)! (2f+1)!} \\ &= \frac{1.3.5 \dots (2h+1).1.3.5 \dots (2i-2h-1)}{N^i} \\ &\times \sum_{g=0}^{h \text{ (or } i-h-1)} \frac{2^{2g} \mu_{r+1}^{2g+1} h^{[-g]} [i-h-1]^{[-g]} \mu_2^{h-g}}{(2g+1)!} [\mu_{2r} - \mu_r^2]^{i-h-1-g} + \dots \end{aligned} \quad (46).$$

Substituting in (43) we find, after suitable transformations:

$$E(\nu_r' - \nu_r)^{2i} = \frac{1 \cdot 3 \cdot 5 \dots (2i-1)}{N^i} [\mu_{2r} - \mu_r^2 + r^2 \mu_{r-1}^2 \mu_2 - 2r \mu_{r+1} \mu_{r-1}]^i + \dots \quad (47).$$

Noting that $\frac{E(\nu_r' - \nu_r)^{2i}}{[E(\nu_r' - \nu_r)^2]^i}$ tends with increasing N to the limit $1 \cdot 3 \cdot 5 \dots (2i-1)$, and the ratio $\frac{[E(\nu_r' - \nu_r)^{2i+1}]^2}{[E(\nu_r' - \nu_r)^2]^{2i+1}}$ tends with increasing N to the limit zero, we see that the law of distribution of the values of ν_r' tends with increasing N to the Gauss-Laplace law.

Comparing (47) with (14) of Chapter II, we see that $\frac{E(\nu_{2r}' - \nu_{2r})^{2i}}{E(\mu_{2r}' - \mu_{2r})^{2i}}$ tends with increasing N to the limit 1.

In the case in which the law of distribution of the values of the variable X follows the Gauss-Laplace law and $\mu_{2r+1} = 0$, for $r = 0, 1, 2, 3, \dots, \infty$, $\frac{E(\nu_{2r}' - \nu_{2r})^{2i}}{E(\mu_{2r}' - \mu_{2r})^{2i}}$ tends with increasing N to the limit 1, for every positive integer r . But $\frac{E(\nu_{2r+1}' - \nu_{2r+1})^{2i}}{E(\mu_{2r+1}' - \mu_{2r+1})^{2i}}$ tends even for a Gaussian distribution of the values of X to the limit different from unity:

$$1 - \frac{1 \cdot 3 \cdot 5 \dots (2r+1)}{(2r+3)(2r+5) \dots (4r+1)}.$$

Corrigenda to Part I, Biometrika, XII, pp. 140—169.

p. 142, Eqn (2) for $a_{k,i} N^{[-k]}$ read $a_{k,i} N^{[-i]}$.

p. 142, Eqn (4) for $\Delta^i 0^i$ read $\Delta^i 0^k$.

p. 147, l. 19 for g_{k+1} read g_{k-1} .

p. 151, last line Eqn (11) for $m_{r,(N)}$ read $m_{k,(N)}$.

p. 156, Eqn (27) for μ_{r-1} read μ_{r-i} .

p. 157, l. 6 for $\frac{1}{N} \sum_{i=1}^N (X_i - m_1)$ read $\frac{1}{N} \sum_{i=1}^N (X_i) - m_1$.

p. 157, footnote, for Proc. Imp. Acad. read Mém. Acad. and under Chebysheff refer to t. II, p. 478, especially of Russian edition of his works.

p. 160, l. 8 for $v_{r,(N)}^{(h)}$ read $v_{r,(N)}^{(l)}$.

p. 160, last Eqn of (11) for $D_{r,(m)}^{m-2}$ read $D_{r,m-2}^{(m)}$.

p. 162, throughout this section χ of author's MS. has been printed X .

p. 163, last line of Eqn (15) insert $m^{[-3]}$ after $\frac{m(3m-1)}{72}$, and in 6th line of Eqn (15) for $m^{[-2]}$

after $\frac{m-1}{60}$ read $m^{[-3]}$.

p. 167, 8th line from bottom of page for $E_{(2i)}$ read $E_{(2i)} d\mu$.

p. 167, 2nd line from bottom of page for $K_l^{(r_{f1}, r_{f2}, \dots, r_{f2l-2k-j})}$ read $K_{t-k-j}^{(r_{f1}, r_{f2}, \dots, r_{f2l-2k-j})}$.

p. 168, 2nd line from bottom of page for $K_l^{(r_{f1}, \dots)}$ read again $K_{t-k-j}^{(r_{f1}, \dots)}$.

AN EXPLANATION OF DEVIATIONS FROM POISSON'S LAW IN PRACTICE.

By "STUDENT."

In her paper on the Poisson Law of small numbers, *Biometrika*, x, p. 36 *et seq.* Miss Whitaker after a very interesting analysis of the various attempts which have been made to test Poisson's Law on actual statistics concludes that "A general interpretation based on a very simple conception seems needed for those demographic cases in which the law of small numbers appears far more often to correspond to a negative than to a positive binomial."

The following is an attempt to explore the general question of what effect various departures from the conditions which lead to Poisson's Law have on the resulting statistics, and especially which conditions lead to positive and which to negative binomials when the exponential might at first sight be expected.

Poisson's Law has been applied to the occurrence of different numbers of individuals in divisions of space or time: thus of yeast cells in squares of a haemacytometer, of deaths from the kick of a horse in Prussian Army Corps which may be taken as individuals occurring in divisions of space, or of suicides of children per year in Prussia which are individuals occurring in divisions of time. In such cases it has been asserted that if the chance of an individual being found in a given division is so small that when multiplied by the very large number of individuals the product is still a reasonably small number, then the frequency of divisions containing 0, 1, 2 ... r individuals will be given by the terms of the exponential

$$Ne^{-m} \left\{ 1 + m + \frac{m^2}{2} + \dots + \frac{m^r}{r} + \dots \right\},$$

where N is the number of divisions and m the mean number of individuals occurring in a division.

For the above to be true it is necessary

- (1) That the chance of falling in a division is the same for each individual.
- (2) That the chance of an individual falling in it is the same for each division.
- (3) That the fact that an individual has fallen in a division does not affect the chance of other individuals falling therein.

As to these three conditions (1) is seldom or never true. I propose to show that this is generally unimportant; unless the chances of some individuals falling in a particular division are relatively high the Poisson law holds; the tendency however is towards a positive binomial.

Next (2) is comparatively seldom true except in the case of artificial divisions. The result of this, as Pearson has shown, is that a negative binomial fits the results better than the exponential.

Lastly (3) is often untrue. It will be shown that if the presence of an individual makes another less likely to fall into a division the positive binomial, but if more likely, the negative binomial will fit the figures best.

We may start from the fact that if the chance of an event happening be q and of its not happening p , then the chances of its happening 0, 1, 2, etc. times in n trials are given by the terms of the expansion of $(p + q)^n$, viz.

$$p^n : np^{n-1}q : \frac{n(n-1)}{2} p^{n-2}q^2 : \text{etc.}$$

As the moment coefficients of this series about the zero end of the range are

$$\nu_1 = nq,$$

$$\nu_2 = npq + n^2q^2 \text{ whence } \mu_2 = npq,$$

the binomial is completely determined if we know ν_1 and μ_2 for

$$p = \frac{\mu_2}{\nu_1} q = 1 - p = 1 - \frac{\mu_2}{\nu_1} \text{ and } n = \frac{\nu_1}{q} = \frac{\nu_1^2}{\nu_1 - \mu_2},$$

and in particular the binomial is positive (i.e. n and q are positive) if $\frac{\mu_2}{\nu_1} < 1$ and negative if $\frac{\mu_2}{\nu_1} > 1$. In the particular case when $\frac{\mu_2}{\nu_1} = 1$ the binomial becomes the Poisson exponential.

It is therefore unnecessary to deal with higher moments than the second for the purpose in hand.

Let us first consider the result of each individual having a different chance of falling in a given division:—

Let the chances of n individuals falling in a given division be $q_1, q_2, q_3 \dots q_n$. The chances of their not doing so are therefore $(1 - q_1), (1 - q_2), (1 - q_3) \dots (1 - q_n)$, and the chances that 0, 1, 2 ... n of them will fall in that division are given by the various terms of the expansion of

$$\{(1 - q_1) + q_1\} \{(1 - q_2) + q_2\} \{(1 - q_3) + q_3\} (\dots) \{(1 - q_n) + q_n\},$$

i.e. by

$$\begin{aligned} & (1 - q_1)(1 - q_2) \dots (1 - q_n) + S \{q_1(1 - q_2)(\dots)(1 - q_n)\} \\ & + S \{q_1q_2(1 - q_3) \dots (1 - q_n)\} + \dots + S \{q_1q_2q_3 \dots q_r(1 - q_{r+1}) \dots (1 - q_n)\} + \dots \\ & + q_1q_2q_3 \dots q_n, \end{aligned}$$

the term $S \{q_1q_2q_3 \dots q_r(1 - q_{r+1}) \dots (1 - q_n)\}$ giving the chance that exactly r individuals will fall in the division.

The sum of the above series is clearly unity so that the 1st and 2nd moment coefficients about the zero end of the series are given by two series of which the r th terms are

$$rS\{q_1q_2 - q_r(1 - q_{r+1}) \dots (1 - q_n)\} \text{ and } r^2S\{q_1q_2 \dots q_r(1 - q_{r+1}) \dots (1 - q_n)\}$$

respectively.

These series may be summed by rearranging them in the ascending order of the q products thus:

$$\begin{aligned} S\{q_1(1 - q_2)(1 - q_3) \dots (1 - q_n)\} &= S(q_1) - 2S(q_1q_2) + \dots + (-1)^{r-1} r.S(q_1q_2 \dots q_r) + \dots \\ 2S\{q_1q_2(1 - q_3)(1 - q_4) \dots (1 - q_n)\} &= 2S(q_1q_2) + \dots + (-1)^{r-2} r(r-1)S(q_1q_2 \dots q_r) + \dots \\ &\dots \dots \dots \\ tS\{q_1q_2 \dots q_t(1 - q_{t+1}) \dots (1 - q_n)\} &= tS(q_1q_2 \dots q_t) + \dots \\ &\quad + (-1)^{r-t} \frac{r \cdot |r-1|}{|t-1| |r-t|} S(q_1q_2 \dots q_r) + \dots \\ &\dots \dots \dots \\ rS\{q_1q_2 \dots q_r(1 - q_{r+1}) \dots (1 - q_n)\} &= \dots \dots \dots r.S(q_1q_2 \dots q_r) + \dots \end{aligned}$$

Adding these we get on the left ν_1 and on the right $S(q_1)$ + a number of terms of the form $r(1 - 1)^{r-1} S(q_1q_2 \dots q_r)$ which accordingly vanish and we get

$$\nu_1 = S(q_1).$$

In a similar manner it can be shown that

$$\nu_2 = S(q_1) + 2S(q_1q_2),$$

and other moment coefficients about zero can be found in the same way, but we are not here concerned with them*.

If $\bar{q}$, $\bar{q}^2$ are the mean values of q and q^2 , obviously

$$\nu_1 = S(q_1) = n\bar{q} \dots \dots \dots (1),$$

and

$$\begin{aligned} \nu_2 = S(q_1) + 2S(q_1q_2) &= S(q_1) + \{S(q_1)^2 - S(q_1^2)\} \\ &= n\bar{q} + n^2\bar{q}^2 - n\bar{q}^2 \dots \dots \dots (2), \\ &= n\bar{q} + n^2\bar{q}^2 - n\bar{q}^2 - n\sigma_q^2 \dots \dots \dots (3), \end{aligned}$$

$$\begin{aligned} \therefore \mu_2 &= n\bar{q} - n\bar{q}^2 - n\sigma_q^2 \\ &= n\bar{q} \left(1 - \bar{q} - \frac{\sigma_q^2}{\bar{q}}\right) \dots \dots \dots (4). \end{aligned}$$

* The moment coefficients are :

$$\mu_2 = n\bar{p}\bar{q} - n_q\mu_2,$$

$$\mu_3 = n\bar{p}\bar{q}(\bar{p} - \bar{q}) - 3n(\bar{p} - \bar{q})_q\mu_2 + 2n_q\mu_3,$$

$$\mu_4 = n\bar{p}\bar{q}\{1 + 3(n-2)\bar{p}\bar{q}\} - n\{7 + 6(n-6)\bar{p}\bar{q}\}_q\mu_2 + 12n(\bar{p} - \bar{q})_q\mu_3 - 6n_q\mu_4 + 3n^2_q\mu_2^2,$$

where ${}_q\mu_2$ etc. are the moment coefficients of the q distribution and $\bar{p} = 1 - \bar{q}$.

If now the distribution of chances is to be represented by the binomial $(P + Q)^N$. Then

$$Q = 1 - \frac{\mu_2}{\nu_1} = 1 - \frac{n\bar{q}(1 - \bar{q} - \sigma_q^2/\bar{q})}{n\bar{q}} \\ = \bar{q} + \frac{\sigma_q^2}{\bar{q}} \dots\dots\dots(5).$$

Since the original q 's are the chances of events happening they are always positive so that the above expression must be positive and the binomial positive.

If now we introduce the Poisson condition that $\bar{q}$ though positive is negligibly small (5) becomes in general zero, for σ_q is usually of the same order as $\bar{q}$, and in that case Poisson's law holds in spite of the inequality of the original q 's. If however $\frac{\sigma_q^2}{\bar{q}}$ is appreciably greater than zero (as in the extreme case

$$q_1 = \frac{1}{2} q_2 = q_3 = \dots = q_n = 0 \text{ when } \frac{\sigma_q^2}{\bar{q}} = \frac{n-1}{2n} = \frac{1}{2} \Big),$$

the distribution of chances is to be represented by a positive binomial.

Next we have to consider the effect of disregarding condition (2), namely that the chance of an individual falling into it must be the same for each division.

Let us suppose then that the q 's are all different for each division so that $n\bar{q}$ is also different.

Then writing m for $n\bar{q}$ and $\bar{m}$, $\bar{m}^2$, $\overline{nq^2}$ for the means of m , m^2 and nq^2 taken over all the divisions.

$$\text{We get from (1)} \qquad \nu_1 = \bar{m} \dots\dots\dots(6),$$

$$\text{from (2)} \qquad \nu_2 = \bar{m} + \bar{m}^2 - \overline{nq^2} \\ = \bar{m} + \bar{m}^2 + \sigma_m^2 - \overline{nq^2} \dots\dots\dots(7),$$

$$\therefore \mu_2 = \bar{m} + \sigma_m^2 - \overline{nq^2}^* \dots\dots\dots(8).$$

As before if $(P + Q)^N$ is the best fitting binomial,

$$Q = 1 - \frac{\mu_2}{\nu_1} = \frac{\overline{nq^2} - \sigma_m^2}{m}.$$

Hence if $\sigma_m^2 > \overline{nq^2}$, which if there is any appreciable variation in m is probable, since as explained above nq^2 is generally negligible, a negative binomial will be found to fit better than the exponential.

Clearly condition (2) is usually not fulfilled in the vital and demographic statistics; divisions either of space or time are generally governed by different

* If we suppose that q does not vary with the individual but that nq ($=m$) varies with the division, the moment-coefficients of the m distribution being written ${}_m\mu$, then the moment-coefficients of the resulting distribution of divisions are as follows :

$$\mu_2 = \bar{m} + {}_m\mu_2, \\ \mu_3 = \bar{m} + 3{}_m\mu_2 + {}_m\mu_3, \\ \mu_4 = \bar{m} + 3\bar{m}^2 + (7 + 6\bar{m}){}_m\mu_2 + 6{}_m\mu_3 + {}_m\mu_4.$$

environments which will vary the chances of an individual falling into them, and so we may expect that as a rule negative binomials will occur in place of the exponential.

Finally, suppose that the presence of an individual in a division influences the chance of other individuals falling in that division.

Clearly it may do so either by way of increasing the chance or diminishing it.

If the chance be increased it is clear that we shall get for the same mean number of individuals per division a larger number of divisions containing high numbers of individuals and a larger number of zero divisions. In other words, for the same mean we shall get a larger Standard Deviation, so that μ_2/ν_1 will be greater than 1 and a negative binomial will fit better than the exponential. On the other hand, if the chance of other individuals is decreased by the presence of one already in a division μ_2/ν_1 will become less than unity and the best fitting binomial will be positive. The first of these two cases includes linking or clumping of events or bacteria, the second such a thing as the counting of large cells on a haemocytometer whose divisions are comparable in size with them.

We have now shown that a population which might be expected at first sight to follow Poisson's law

(1) Will do so if the only deviation from the ideal conditions is that the chances of different individuals falling into the same division are not equal, as long as these chances are all small.

(2) If in addition to this the chances of some individuals are large a positive binomial will fit the results better than the exponential.

(3) If the different divisions have different chances of containing individuals, as is usual, a negative binomial will fit the results better than the exponential, except in so far as (2) may interfere.

(4) If the presence of one individual in a division increases the chance of other individuals falling into that division, a negative binomial will fit best, but if it decreases the chance a positive binomial.

Generally speaking (3) is the operating deviation from Poisson's conditions and accordingly most statistics give negative binomials.

Finally I should like to point out that the object of my original paper (*Biometrika*, Vol. v) was to give the user of the haemocytometer a guide to the error which he may expect from its use, and that the net result was that the probable error of his count was $\cdot6745 \sqrt{N}$ where N was the total number counted* and that if N be a reasonably large number tables of the probability integral may be used, otherwise the exponential (or better still go on counting). This result is not affected by slight deviations from the Poisson law, any more than slight deviations from the normal law affect our use of the probability integral tables.

* *Biometrika*, Vol. v, p. 355. The probable error of mean is $\cdot6745 \sqrt{m/M}$ where m is the mean and M the number of unit areas counted. If in this we put $M=1$, then $m=N$ and the total count is $N \pm \cdot6745 \sqrt{N}$ as above.

THE CRITERION OF GOODNESS OF FIT OF PSYCHOPHYSICAL CURVES.

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(1) *Introduction.*

The object of the present paper is to inquire what is the proper method of examining psychophysical curves as to their goodness of fit. In psychophysics various mathematical processes are employed for fitting theoretical curves of "ogive" form* (known to psychologists as psychometric functions, but really error functions), to data of a certain kind, usually threshold measurements collected by the "Method of Right and Wrong Cases." The best known of these mathematical processes is the Müller "Constant Process," using the probability integral†. To make the material in which we are about to work understandable, it is necessary first to go into some detail as to the nature of the experiments which supply the data to be fitted, and as to the theories which have led to such mathematical curves being drawn through these data.

Most of the experiments in question have for their object the determination of the conditions of our experiences of equality and difference. For example, suppose we compare two weights, one of which is 100 grams, by lifting them in succession by the right hand with a number of experimental precautions, into which we need

* The term in this connexion is Galton's.

† G. T. Fechner, *Elemente der Psychophysik*, 1860; G. E. Müller, "Ueber die Maassbestimmungen des Ortsinnes der Haut mittels der Methode der richtigen und falschen Fälle," *Pflügers Archiv für die ges. Physiologie*, 1879, xix, pp. 191—235, especially par. 5 *et seq.*; G. E. Müller, *Die Gesichtspunkte und die Thatsachen der Psychophysischen Methodik*, Wiesbaden, 1904, par. 11; F. M. Urban, "Die Psychophysischen Massmethoden," *Archiv für die ges. Psychologie*, 1909, xv, p. 287; G. H. Thomson, "The Accuracy of the $\phi(\gamma)$ Process," *Brit. Journal of Psychol.*, 1914, vii, p. 46, and in various text-books, e.g. Titchener's *Experimental Psychology*, and W. Brown's *Essentials of Mental Measurement*.

not here enter. We wish to know under what conditions the unknown weight appears lighter than, equal to, or heavier than the standard weight. An important condition is of course the "actual" weight of the unknown weight, as measured in the usual manner. But this is by no means the only important condition. The order in which the weights are lifted (whether standard first or unknown first); the number of categories into which our judgments have to be classified; the order of succession of the several unknown weights, whether rising or falling or at random; the range over which the succession of unknown weights stretches, whether or no it contains any which are quite easily distinguished from the standard; all these and many other conditions are of great importance. Steps can however be taken to eliminate some of these factors, by means of judicious experimental precautions, and the attempt can be made to keep the others as constant as possible during a series of trials. The judgments which are given by the subjects then depend mainly on the difference between the standard stimulus and the variable stimulus; in the case of our example on the difference between the standard weight and the variable weight. Among other points of importance in the fitting of the curves is the possibility of deciding by means of the goodness of fit whether the experimental conditions have really been kept as constant as has been hoped, for lack of constancy in this respect will lead to heterogeneity which will show itself by the necessity of using a compound curve to obtain a good fit.

To fix ideas, it is desirable at this point to have an actual set of data to refer to. In some very carefully conducted experiments on weight-lifting, Professor F.M. Urban (*op. cit.*) found that, with one of his subjects, under certain experimental conditions, the standard weight being 100 grams, the following numbers of answers *heavier* were returned, out of 300 trials with each of the several unknown weights. It should be mentioned that the experimental method used involved that the unknown weights were presented to the subject in random sequence, accompanied each by the standard, so that the 300 trials referred to were not one after the other, but were separated from each other by trials with the other unknowns. Otherwise expectation and other psychological factors produce a considerable correlation between one judgment and the next, which is reduced to a minimum by Urban's procedure. Moreover, precautions against fatigue and several other factors were taken. For the details the reader is referred to Urban's memoir, with the warning that much of the mathematical part thereof is incorrect.

Grams s	Answers heavier	Proportion p
84	7 out of 300	·0233
88	8 out of 300	·0267
92	35 out of 300	·1167
96	107 out of 300	·3567
100	183 out of 300	·6100
104	265 out of 300	·8833
108	279 out of 300	·9300

It is to numbers such as these that the curves to be considered are fitted.

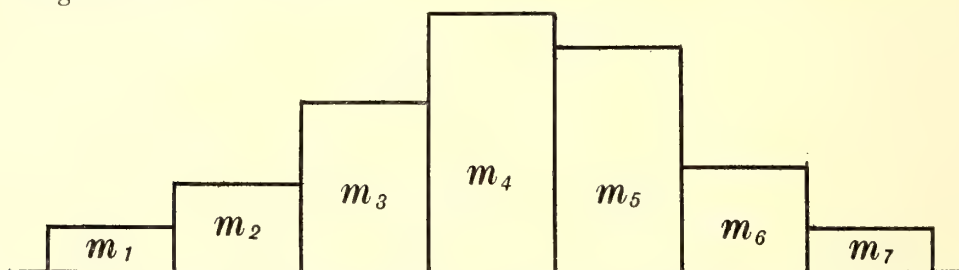
Any suitable curve which happened to occur to one might of course be employed. For example, a parabola of higher order can be used, and the curve $\tan^{-1} \theta$ has also been used. But clearly the whole experiment suggests that an error function of some sort is wanted, and as early as 1860 G. T. Fechner (*op. cit.*) suggested that such numbers formed the integral of a normal or Gaussian curve. One usual argument is somewhat as follows, using for clearness terms applying directly to the above example.

The existence of a hypothetical point is postulated, called the limen or threshold for the judgment *heavier*, such that above this point the subject always returns the answer *heavier*, and below it he always returns some other answer, *not heavier*. But this limen is supposed to be fluctuating from moment to moment, either really or apparently, owing to changes in the physical, physiological and psychological conditions of the experiment. If at one moment the answer *heavier* is returned, for the variable 96 grams, then at that moment the limen is below 96 grams. Later the answer *lighter*, or the answer *equal* may be returned for 96 grams, and at that moment the limen is above 96 grams. The values p in the above table will then be integrals of the distribution curve of this limen.

(2) *Peculiarities of Psychophysical Data from the Point of View of Curve Fitting.*

The problem of fitting a distribution curve integral to such data, say in the first place the probability integral, has certain peculiarities which differentiate it from many biometric curve-fitting problems.

Usually, when we are required to fit a normal curve, we are given the data in histogram form:



That is, a number M of direct measurements is made, and m_i are found to fall into a certain short range, m_2 into another adjacent range, and so on. To fit a Gauss curve requires the mean and the standard deviation, and these quantities can be directly found from such a histogram, Sheppard's adjustments being used if necessary.

Quantities analogous to our proportions p can be formed from such a biometric histogram, viz.:

$$\begin{aligned} p_1 &= m_1/M, \\ p_2 &= (m_1 + m_2)/M, \\ p_3 &= (m_1 + m_2 + m_3)/M, \\ &\dots\dots\dots \end{aligned}$$

and *vice versa*, quantities analogous to the m 's can be formed from the proportions p of the psychometric experiment, viz.:

$$\begin{aligned} m_1 &= p_1 M, \\ m_2 &= (p_2 - p_1) M, \\ m_3 &= (p_3 - p_2) M, \\ &\dots\dots\dots \end{aligned}$$

In the case of our example we should have:

Below 84 grams,	7 cases,	$m/M =$	·0233
84—88 " "	1 "	"	·0034
88—92 " "	27 "	"	·0900
92—96 " "	72 "	"	·2400
96—100 " "	76 "	"	·2533
100—104 " "	82 "	"	·2733
104—108 " "	14 "	"	·0467
Above 108 " "	21 "	"	·0700
Totals ...	300 "		1·0000

There are however important differences which make the analogy inexact from the curve-fitting point of view.

In the biometric histogram, if any one of the cells m_k is larger than it ought to be, then any other must have a tendency to be smaller than it ought to be. There is a strong negative correlation between the numbers in the cells, a correlation, that is, from trial to trial. In the psychometric pseudohistogram however, formed from the proportions p , this is otherwise, because the p 's are measured quite separately from one another.

In the biometric histogram the m 's, the numbers in each cell, are necessarily positive quantities. In the psychometric pseudohistogram they may be negative, if the p 's do not rise steadily. In the biometric histogram the actual range found in a trial is as a rule known, that is the points where p is zero and p is unity are known. In the psychometric case these points are as a rule not known, and there may be psychological reasons why extreme stimuli (such as would be required to find these points) should not be used. In our example we do not know whether the subject would have given no answers *heavier* at 80 grams, or whether at the other end he would have given only answers *heavier* at 112 grams.

When we do know these points, or can assume them, in the psychometric case, we can fit our probability integral by forming the pseudohistogram, and calculating the mean and the standard deviation as though it were a real histogram*. This has been suggested by more than one writer, in England by Professor C. Spearman†, who does not however point out the difficulty that it cannot as a rule be done, because the points $p = 0$ and $p = 1$ are not known.

* The actual arithmetical formation of the histogram is unnecessary if a summation method of finding moments is employed.

† *Brit. Journ. Psychol.*, 1908, II, p. 227.

In biometric language, the problem is to fit a normal curve to data for which the "tails" are undefined as to range, although their areas are known. This problem was solved by Müller (*op. cit.*) as follows:

(3) *The Constant Process.*

Call the stimuli $s_1, s_2, s_3, \dots s_n$,
and the proportions p $p_1, p_2, p_3, \dots p_n$,
then we have n equations

$$p - \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{h(s-S)} e^{-x^2} dx = 0 \dots \dots \dots (1),$$

to find the mean S and the precision h . We retain for the present this form of the integral as being more familiar to psychologists. The more modern form would have the standard deviation instead of the precision as the second unknown.

These equations are slightly inconsistent with one another. No pair of values S and h will exactly satisfy all n equations; instead of giving zero they leave small residuals v_i .

Müller assumed tacitly that these n equations if based on the same number of experiments each, are of equal importance or weight*. We shall allow this assumption to pass for the present but shall return to it later.

If we now make the usual assumptions of the Method of Least Squares, we can take as the *best* values of S and h those which make

$$\Sigma (v_i^2) \text{ a minimum,}$$

where the summation is over the n stimuli or n equations. The conditions that this should be so are

$$\left. \begin{aligned} \frac{d}{dh} \Sigma (v_i^2) &= 0 \text{ for constant } S \\ \frac{d}{dS} \Sigma (v_i^2) &= 0 \text{ for constant } h \end{aligned} \right\} \dots \dots \dots (2).$$

Unfortunately, the n equations however are very far from being simple and linear as in usual applications of Least Squares. To avoid this difficulty we look up in tables of the Probability Integral (which psychologists call Fechner's Fundamental Table) those n values of

$$\gamma = h(s - S) \dots \dots \dots (3)$$

which correspond exactly to our n values of p . These equations are not yet linear in S and h , but if we write

$$c = Sh \dots \dots \dots (4)$$

they become $\gamma - hs + c = 0 \dots \dots \dots (5),$

* There is unfortunately a possibility of ambiguity of language here as the word *weight* also occurs in the particular example we are using as illustration, where weights of 84 grams etc. are employed.

and are now linear in h and c . We have now n linear equations in which γ and s are known, h and c are required. If we insert any pair of values h and c into these n equations they will leave residuals v_2 . If we were now to proceed to make

$$\Sigma (v_2^2) \text{ a minimum,}$$

this would not effect our purpose. It is $\Sigma (v_1^2)$ we wish to make a minimum, not $\Sigma (v_2^2)$. If however we can find multipliers or weights M such that each

$$Mv_2^2 = v_1^2 \dots\dots\dots(6)$$

we can then make $\Sigma (Mv_2^2)$ a minimum.

That is, we can apply Least Squares to the equations (3), weighted with certain artificial weights M . The use of this device is Müller's particular credit in this connexion.

Clearly the residuals v_1 , which may be regarded as errors in p , are connected with the residuals v_2 , which may be regarded as errors in γ , by the equation

$$\frac{1}{\sqrt{\pi}} e^{-\gamma^2} v_2 = v_1$$

from equations (1) and (3). Therefore

$$M = e^{-2\gamma^2}/\pi.$$

Herein we can omit the π since it is only the relative values of the Müller weights which are of importance. These weights are given in most works on psychophysics, e.g. W. Brown, or Titchener, *op. cit.*

The condition that $\Sigma (v_1^2)$ should be a minimum has now become, that $\Sigma (Mv_2^2)$ should be a minimum. With this substitution, the Normal Equations (2) give

$$\left. \begin{aligned} [Ms^2]h - [Ms]c &= [Ms\gamma] \\ -[Ms]h + [M]c &= -[M\gamma] \end{aligned} \right\} \dots\dots\dots(7),$$

the square brackets being the sign of summation used by Gauss, and still persisting in psychophysics in this connexion. The summation here is over the n equations. Thence we have

$$\left. \begin{aligned} c &= \frac{[Ms][Ms\gamma] - [M\gamma][Ms^2]}{[M][Ms^2] - [Ms]^2} \\ h &= \frac{[M][Ms\gamma] - [Ms][M\gamma]}{[M][Ms^2] - [Ms]^2} \\ S = \frac{c}{h} &= \frac{[Ms][Ms\gamma] - [M\gamma][Ms^2]}{[M][Ms\gamma] - [M\gamma][Ms]} \end{aligned} \right\} \dots\dots\dots(8).$$

(4) *The Probability of a Certain Category of Judgment.*

The Constant Process remained in this form from 1879 to 1909. It is very much mixed up with the psychological *method* of experimenting and collecting the data, so that frequently the name "Method of Right and Wrong Cases," really the name of a certain method of collecting data, has been used to include this mathe-

matical process. To avoid this mental confusion I have elsewhere* suggested that the two words *Method* and *Process* should in psychophysics be consistently used in the way in which they are employed in the above sentence, viz. Method of collecting data, and Process of calculating. Frequently the Constant Process has been called the *phi-gamma* method, from the use of the name phi-gamma for the probability function.

In 1909, F. M. Urban (*op. cit.*) suggested alterations to the Müller weights, or rather suggested the necessity of another set of weights in addition†. These alterations arise from the notion of comparing the judgments *heavier* with the drawing of black balls from a bag containing black balls and white balls. The analogy is in detail as follows.

(1) From a bag containing black balls and white balls 300 drawings are made, one at a time, the ball being returned each time before the next drawing is made. 107 black balls are observed out of the 300.

(2) A subject on performing a certain experiment with weights sometimes gives the answer *heavier*, sometimes some other answer. On one occasion, when the weights were 100 grams standard and 96 grams unknown, this experiment was repeated 300 times, with due precautions against fatigue, etc., and the answer *heavier* was returned 107 times out of the 300.

Now if p is the observed proportion (here 107/300) of black balls in a bag, then the probable error of p is known to vary with $\sqrt{p(1-p)}_+^\dagger$. With the same sized sample, a result $p = .5$ has a larger probable error than a result $p = .8$, say. If anything similar holds, as the analogy suggests, for the psychometric experiment, then the n equations (1) or (5) are not equally reliable, even although based on the same number, 300, of experiments each. In addition to the weights M they need other weights

$$\frac{1}{4p(1-p)} = \frac{1}{4pq} \dots\dots\dots(9)$$

to allow for this new variation in reliability. The combined weights $M/4pq$ are known as Urban's weights, and a table of these is usually given in psychophysical textbooks alongside the ordinary Müller weights. Urban discusses the matter at some length in his already cited article, and a discussion will also be found in Wirth's *Psychophysik* (Leipzig, 1912) where on page 151 the actual scatter of various p 's is given in a diagram.

* *Brit. Journ. Psychol.* 1912, v, p. 203.

† There are many errors in the article of Urban's quoted. See my articles in the *Brit. Journ. Psychol.*, 1913, vi, p. 217, and 1914, vii, p. 44. But these errors, though making Urban's conclusions in that article invalid, do not touch the point here raised, in which I think Urban's suggestion marks an advance.

‡ Really the *true* values of p and $1-p$ should be used but this is the best we can do. And further, the expression, probable error, ceases to have an accurate meaning when p is too close to zero or unity and the distribution in consequence is very skew. But these refinements do not matter at this stage of our argument.

Replacing in equation (7) therefore the weights M by the new Urban weights P , Urban found in the present instance

$$S = 98.24 \text{ grams,}$$

$$h = 0.117995.$$

That is he represents the proportions p theoretically by using the hypothesis that the "psychometric function," as psychologists call it, is given by

$$p' = \frac{1}{\sqrt{\pi}} \int_{-x}^{0.117995(98.24-s)} e^{-x^2} dx \dots\dots\dots (10).$$

The theoretical values p' thus calculated are compared with the actual values p in this table.

Grams	p	p'	Difference x
84	.0233	.0088	+ .0145
88	.0267	.0438	- .0171
92	.1167	.1489	- .0322
96	.3567	.3544	+ .0023
100	.6100	.6155	- .0055
104	.8833	.8319	+ .0514
108	.9300	.9483	- .0183

The object of the present paper is to make clear the proper methods (*a*) of ascertaining, in all such cases, whether the theoretical numbers are a reasonable fit to the observed numbers, or not, and (*b*) of comparing the fits obtained by different hypotheses, that is by different error functions. The psychologist would express this by saying that he was comparing different psychometric functions. To the statistician the comparison is one of error functions, the natural procedure being to try first the normal curve, then members of Pearson's family of curves, then compound curves; the conclusion in the latter case being that the material was not homogeneous. This work I have as a matter of fact already carried out, and have come to that conclusion; but it is beyond the scope of the present paper, which hopes to interest psychologists in modern statistical methods, and statisticians in modern psychology.

(5) *Pearson's Criterion of Goodness of Fit.*

This problem, of comparing the goodness of fit of curves in psychophysics, although it has not as far as I am aware ever been correctly performed, is really very simple, and could be handled at once from first principles. For the sake however of showing the connexion with other work it is advisable to treat it as a special case of the application of Pearson's Criterion of Goodness of Fit*, which is in brief as follows.

* Karl Pearson, *Phil. Mag.*, July 1900 and April 1916.

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Let

$$x_1, x_2, x_3, \dots x_n,$$

be a system of deviations from the means of n variables whose standard deviations are

$$\sigma_1, \sigma_2, \sigma_3, \dots \sigma_n,$$

and intercorrelations

$$r_{12}, r_{13}, r_{23}, \dots r_{n-1, n}.$$

Then the frequency "surface" giving the frequency of occurrence of each possible combination of x 's is

$$z = z_0 e^{-\frac{1}{2}\chi^2} \dots\dots\dots(11),$$

where

$$\chi^2 = S_1 \left(\frac{R_{kk}}{R} \frac{x_k^2}{\sigma_k^2} \right) + 2S_2 \left(\frac{R_{kl}}{R} \frac{x_k x_l}{\sigma_k \sigma_l} \right) \dots\dots\dots(12).$$

Herein R is the determinant

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} & \dots & r_{1n} \\ r_{21} & 1 & r_{23} & \dots & r_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ r_{n1} & r_{n2} & r_{n3} & \dots & 1 \end{vmatrix} \dots\dots\dots(13),$$

and R_{kk}, R_{kl} , are the minors corresponding to r_{kk} and r_{kl} . S_1 is a sum over all k 's, and S_2 is a sum over all pairs kl other than $k = l$.

When χ^2 has been calculated, a probability P can be found, from Table XII in Pearson's *Tables for Statisticians*. This table is entered by $n' = (n + 1)$ and χ^2 , and gives values of

$$P = \frac{\int_{\chi}^{\infty} \chi^{n-1} e^{-\frac{1}{2}\chi^2} d\chi}{\int_0^{\infty} \chi^{n-1} e^{-\frac{1}{2}\chi^2} d\chi} \dots\dots\dots(14),$$

that is, P is the probability that a *random* sample of as bad a fit as the data, or worse, would be obtained from the theory which is being tested. The kind of data for which this criterion was first invented was data in real histogram form, of the kind called in earlier sections of this paper a biometric histogram. When the data are of this form, Pearson has shown that equation (12) reduces to the very simple form

$$\chi^2 = S \left(\frac{e^2}{m'} \right) \dots\dots\dots(15),$$

where m' is the theoretical value of m , and e is $m - m'$, and S indicates summation over all the cells of the histogram. Psychophysical data of the kind here considered, however, as has already been pointed out, are not really in histogram form. Although a histogram can be deduced from them, it is only by making certain assumptions, and the intercorrelations of the cells of this artificial histogram are different from the intercorrelations of a natural directly observed histogram.

It is not correct therefore to use equation (15) above. It is more accurate and withal exceedingly simple to apply equation (12) direct to the p 's. Since the latter are independent, all the intercorrelations r are zero. Therefore R is unity, R_{kk} is unity, and R_{kl} is zero. Equation (12) therefore becomes

$$\chi^2 = S \left(\frac{x^2}{\sigma^2} \right) \dots\dots\dots(16),$$

and as the distributions of each p will be binomial in form provided the experimental conditions remain constant enough we have

$$\sigma^2 = p'q'/\mu \dots\dots\dots(17)^*,$$

where μ = the number of experiments on which p is based, and $p' = 1 - q'$, so that

$$\chi^2 = S \left(\frac{\mu x^2}{p'q'} \right) \dots\dots\dots(18)^\dagger.$$

Herein the x 's are the differences between observed p 's and theoretical p 's. The probability P is then found as before.

* If we look upon the judgments *heavier*, as suggested in an earlier paragraph, as being comparable with drawing black balls out of a bag containing black balls and white balls in the proportion p' and $1 - p'$, then the probable error of p is $\cdot 67449 \sqrt{\frac{p'(1-p')}{\mu}}$, μ being the number of judgments of which $p\mu$ are of the category *heavier*.

For the chances of obtaining 0, 1, 2, ... $\mu - 1$, or μ black balls in a drawing of μ are given by the terms of

$$(p' + q')^\mu,$$

q' being $1 - p'$: that is, the chances of obtaining

$$p = \frac{0}{\mu}, \frac{1}{\mu}, \frac{2}{\mu}, \dots, \frac{\mu-1}{\mu}, \text{ or } \frac{\mu}{\mu}.$$

The s. d. of the above binomial is $\sqrt{\mu p'q'}$ and the s. d. of p therefore $\frac{1}{\mu} \sqrt{\mu p'q'} = \sqrt{\frac{p'q'}{\mu}}$.

† Compare Professor K. Pearson on "Goodness of Fit in Statistics and Physics," *Biometrika*, 1916, xi, pp. 239—261, especially p. 257.

We can check our equation (18) by treating the matter from first principles, and not as a special case included in Pearson's formulae. We have, from this point of view, n quantities p which are independently measured, and n quantities p' which are theoretically given. The variations from p' are binomial in form, that is, they are *approximately* Gaussian. The probability of an error

$$x_k = p_k - p_k'$$

is therefore

$$w_k = \frac{\sqrt{\mu}}{\sqrt{p'q'}\sqrt{2\pi}} e^{-\frac{\mu x^2}{2p'q'}} \dots\dots\dots(a).$$

The probability of the whole set of observed values $p_1, p_2, p_3, \dots p_n$ occurring is the product

$$z = w_1 w_2 w_3 \dots w_n \dots\dots\dots(b).$$

Write this

$$z = z_0 e^{-\frac{1}{2}\chi^2}.$$

Then

$$\chi^2 = S \left(\frac{\mu x^2}{p'q'} \right)$$

from equation (a).

(6) *Numerical Example.*

Let us apply these formulae to the example already cited. The calculations are carried out in the following table. The *theoretical* $p'q'$'s should be used, clearly, as denominators of the terms of χ^2 .

p'	q'	$p'q'$	x^2	$x^2/p'q'$
·0088	·9912	·00872	·00021025	·0241
·0438	·9562	·04198	·00029241	·0070
·1489	·8511	·12673	·00103684	·0082
·3544	·6456	·22880	·00000529	·0000
·6155	·3845	·23665	·00003025	·0001
·8319	·1681	·13983	·00264196	·0189
·9483	·0517	·04903	·00033489	·0069
—	—	—	—	·0652 = $S(x^2/p'q')$

The number of experiments was the same for each p , viz. 300, therefore

$$\chi^2 = S \left(\frac{\mu x^2}{p'q'} \right) = 300 \times \cdot 0652 = 19\cdot 56^*.$$

The Table XII in Pearson's *Tables* to find P has to be entered with χ^2 and $n' = (n + 1)$, where n is the number of *variates*, here the number of p 's, i.e. 7. We find there

$$n' = 8, \quad \chi^2 = 19, \quad \cdot 008187 = P, \quad \chi^2 = 20, \quad \cdot 005570 = P.$$

It is unnecessary, with data such as we are here handling, to interpolate elaborately. Clearly, for $\chi^2 = 19\cdot 56$, P is of the order

$$P = \cdot 007.$$

That is to say, in only seven cases in a thousand should we expect to get our present observed p 's from our theoretical p 's by random sampling. It is therefore not at all probable that the equation (1) truly represents the "psychometric function" for this subject and this reaction.

(7) *Urban's incorrect method of comparing Goodness of Fit.*

In the article from which the above example is taken, Professor Urban was *inter alia* desirous of comparing various hypotheses of the "psychometric function" among themselves. Those which he fully works out are (1) the above assumption that it is the integral of the normal probability curve, and (2) the assumption that it is an "arctan." curve ($\tan^{-1} \theta$). (It is needless to point out surely that the latter hypothesis is in itself most unlikely; however, we are here concerned with an empirical comparison of the two hypotheses, and it is important that the method should be correct since it will be necessary to compare other and more likely theories, as for example Pearson's curves.)

* Compare Appendix.

It can now be shown that the methods which Professor Urban employed in comparing these two hypotheses are incorrect and inadequate. What these inadequate methods are can best be shown by continuing the above example, which is taken at random from among Urban's material.

We have already found the squares x^2 of the differences between theory and observation in the case of the normal integral, or as Urban calls it the $\phi(\gamma)$ hypothesis. They are given in the table just above, and

$$S(x^2) = \cdot00455189.$$

We now proceed to form the analogous quantity in the case of the arctan. hypothesis.

Grams	Observed p	p'	x	x^2
84	·0233	·0795	— ·0562	·00315844
88	·0267	·1086	— ·0819	·00670761
92	·1167	·1682	— ·0515	·00265225
96	·3567	·3259	+ ·0308	·00094864
100	·6100	·6464	— ·0364	·00132496
104	·8833	·8222	+ ·0611	·00373321
108	·9300	·8872	+ ·0428	·00183184
—	—	—	—	·02035695 = $S(x^2)$

Urban now compares the $\phi(\gamma)$ hypothesis with the arctan. hypothesis by comparing

$$\cdot00455189 \text{ with } \cdot02035695,$$

and deciding that as the former is smaller, therefore the $\phi(\gamma)$ hypothesis is superior. This procedure is firstly inaccurate and secondly inadequate. It is inaccurate because not $S(x^2)$ but $S(x^2/p'q')$ should be compared, and it is inadequate because no idea is given whether the observed difference is significant or not.

The former point deserves a little more examination, because it is another form of an error which Urban was the first to correct, in this same article. In the form of the Constant Process as it left the hands of G. E. Müller, certain weights are to be used on the observation equations. These weights may be called Müller's weights. Urban pointed out, however, that they needed amendment, and published (*loc. cit.*) a table of weights to replace them. These weights differ from Müller's by the factor $1/4pq$, which arises in Urban's treatment from an application of what he calls Bernoulli's Theorem. It is these very Bernoulli weights, $1/pq$, which Urban himself has omitted in his above comparison of the $\phi(\gamma)$ and arctan. hypotheses.

In order to discuss the inadequacy of his comparison we need a measure of the probable error of the quantity P used above.

(8) *The Probable Error of χ^2 and P .*

We have

$$\chi^2 = \mu S \frac{(p - p')^2}{p'q'}, \quad (\text{from eqn. 18}).$$

If the accurate values of p' were known, the variation of χ^2 would be due entirely to the variations in the observed values p . In point of fact, of course, the p 's which are available are themselves functions of the p 's: but like Pearson in his 1914 article on the probable error of a coefficient of contingency*, and for the same reason and with I think the same justification, we shall assume that the p 's do not vary. Then the mean square deviation

$$\sigma_{\chi^2}^2 = \mu^2 S \left\{ \sigma_p^2 \left(\frac{2(p - p')}{p'q'} \right)^2 \right\} = \mu S \frac{4(p - p')^2}{p'q'} = 4\chi^2 \dots \dots \dots (19).$$

Therefore the probable error of χ^2 calculated in the way suitable for the Constant Process and other processes for fitting psychometric functions is

$$.6745\sigma_{\chi^2} = 1.349 \sqrt{\chi^2} \dots \dots \dots (20).$$

In the case above where $\chi^2 = 19.56$, its probable error is therefore about 5.9, so that we have

$$\chi^2 = 19.6 \pm 5.9.$$

We must next find χ^2 and its probable error for the arctan. hypothesis. The calculations are partly carried out above in finding $S(x^2)$. Completing them we obtain the following table:

p'	q'	$q'p'$	x^2	$x^2/p'q'$
.0795	.9205	.07318	.00315844	.0431
.1086	.8914	.09680	.00670761	.0693
.1682	.8318	.13991	.00265225	.0189
.3259	.6741	.21968	.00094864	.0043
.6464	.3536	.22856	.00132496	.0058
.8222	.1778	.14718	.00373321	.0254
.8872	.1128	.10006	.00183184	.0183
—	—	—	—	.1851 = $S(x^2/p'q')$

$$\chi^2 = \mu S \frac{x^2}{p'q'} = 55.53.$$

$$\text{Probable error of } \chi^2 = 1.349 \sqrt{\chi^2} = 10.0.$$

For arctan.,

$$\chi^2 = 55.53 \pm 10.0.$$

For $\phi(\gamma)$,

$$\chi^2 = 19.6 \pm 5.9.$$

$$\text{Difference} = 35.9 \pm 11.6,$$

where

$$11.6 = \sqrt{10.0^2 + 5.9^2}.$$

* *Biometrika*, Vol. x.

The difference is therefore three times its probable error and is just significant. The final conclusion is therefore that in this particular case the arctan. hypothesis is just significantly worse than the normal integral or $\phi(\gamma)$ hypothesis, but that the latter itself is very improbable. The P of the normal integral hypothesis it will be remembered was .007. The P of the arctan. hypothesis can be found from Table XII of Pearson's *Tables*. The entry has to be made with $n' = 7 + 1 = 8$, and $\chi^2 = 55.5$, and we find given $P = .000000$, i.e. it is less than .0000005, showing how very improbable the arctan. hypothesis is.

The probable error of P is discussed by Professor Pearson in the *Phil. Mag.* for April 1916 and he shows that the standard deviation

$$\sigma_P = \frac{1}{2} \sigma_{\chi^2} \{P_{n'} - P_{n'-2}\} \dots\dots\dots (21),$$

and using equation (19) we get

$$\sigma_P = (P_{n'} - P_{n'-2}) \chi \text{ in our case } \dots\dots\dots (22).$$

It must be borne in mind that $n' = n + 1$, where n = number of variates. In our case therefore, n' is one more than the number of stimuli. $P_{n'-2}$ is similarly obtained from Table XII of Pearson's *Tables* by entering with the column with heading one less than the number of stimuli. For the above $\phi(\gamma)$ hypothesis we have

$$\chi^2 = 19.6,$$

$$\text{Number of stimuli} = 7,$$

$$P \text{ or } P_8 = .007 \text{ approximately,}$$

$$P_6 = .002 \quad ,,$$

$$\sigma_P = (P_8 - P_6) \chi = .005 \sqrt{19.6} = .022,$$

$$\text{Probable error of } P = .6745 \sigma_P = .015.$$

Therefore for the $\phi(\gamma)$ hypothesis the criterion of goodness of fit is in this case

$$P = .007 \pm .015.$$

It is most improbable, therefore, that P is at all large, and the fit is significantly a bad one. The probable error of P for the arctan. hypothesis is too minute to be found from the table.

The calculations we have performed have been for Urban's Subject IV (*heavier answers*). For his other data similar calculations can be carried out. The arctan. hypothesis is usually worse than the normal integral, but not always significantly worse, and the normal integral itself is an atrociously bad fit to the data in most cases.

(9) *Summary of Rules for Testing and Comparing Goodness of Fit of Psychometric Curves.*

Let there be n stimuli, and let

$$p'_1, p'_2, p'_3 \dots p'_n$$

be the theoretical frequencies at these stimuli, and

$$p_1, p_2, p_3 \dots p_n$$

the observed values. Let

$$\mu_1, \mu_2, \mu_3 \dots \mu_n$$

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be the number of experiments at each stimulus. Calculate χ^2 , the sum of the quantities

$$\frac{\mu (p - p')^2}{p'q'} \quad (q' = 1 - p').$$

Then in Table XII of Pearson's *Tables**, in the column $n' = n + 1$ and the row χ^2 (interpolate if necessary) find the value of P , the probability of obtaining the observed p 's or a worse set, from the p 's by random sampling.

The probable error of χ^2 is here approximately 1.35χ and the probable error of P is approximately

$$.6745 (P_{n+1} - P_{n-1}) \chi,$$

where P_{n+1} is P itself, and P_{n-1} is the value found in Table XII by using the $(n - 1)$ th instead of the $(n + 1)$ th column.

APPENDIX.

What value will be obtained for χ^2 if, in the example used above (normal integral hypothesis), we were to proceed by first forming a histogram and then treating this histogram as though it were an ordinary directly observed one, i.e. using equation (15) above? The cells of the histogram will be occupied by the quantities $m = \delta p \times \mu$ (observation) or $m' = \delta p' \times \mu$ (theory) where δp is the change in p from one stimulus to the next and μ the number of observations at each stimulus, here the same throughout.

p	δp	p'	$\delta p'$	$\pm \frac{e}{\mu} = \delta p - \delta p'$	e^2/μ^2	$e^2/m'\mu$
.0233	.0233	.0088	.0088	.0145	.00021025	.0239
.0267	.0034	.0438	.0350	.0316	.00099856	.0285
.1167	.0900	.1489	.1051	.0151	.00022801	.0022
.3567	.2400	.3544	.2055	.0345	.00119025	.0058
.6100	.2533	.6155	.2611	.0078	.00006084	.0002
.8833	.2733	.8319	.2164	.0569	.00323741	.0150
.9300	.0467	.9483	.1164	.0697	.00485809	.0418
	.0700		.0517	.0183	.00033489	.0065
—	—	—	—	—	—	.1239 = $S(e^2/m'\mu)$

whence $\chi^2 = 300 \times .1239 = 37.17$, instead of the proper value 19.56. If the calculation is performed in this inaccurate way, therefore (by analogy with data which are *really* in histogram form), a very wrong idea of the closeness of fit would be obtained. The reason, as stated above, is that the correlations between the cells of the histogram derived from an ogive with independently measured p 's are not such as to lead to equation (15).

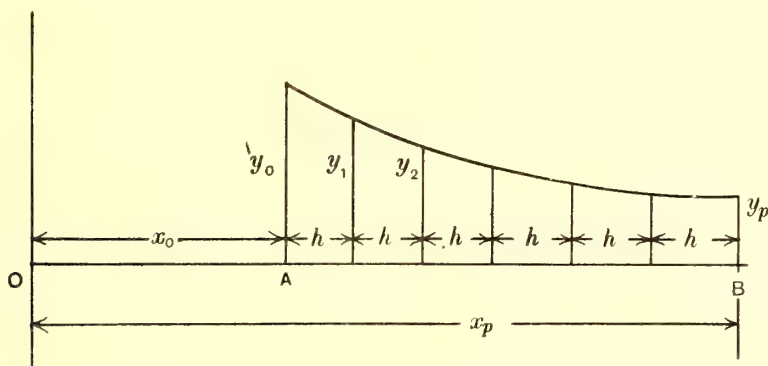
* *Tables for Statisticians and Biometricians*, Cambridge University Press, 1914.

ON CORRECTIONS FOR THE MOMENT-COEFFICIENTS OF LIMITED RANGE FREQUENCY DISTRIBUTIONS WHEN THERE ARE FINITE OR INFINITE ORDINATES AND ANY SLOPES AT THE TERMINALS OF THE RANGE.

BY ELEANOR PAIRMAN AND KARL PEARSON, F.R.S.

Part I. Non-Asymptotic Curves.

(1) We have in recent practice found the importance of full corrections for the moment-coefficients in the case of singly and doubly curtailed blocks of frequency such as are indicated in the accompanying figure. It has not been adequately



recognised that even the mean of such distributions is not correctly obtained by grouping at the midpoints of the subranges h , and merely finding the mean of these concentrated groups. Still less is this a correct process in the case of the higher moment-coefficients. The practical statisticians, aware possibly of the existence of "Sheppard's corrections," have been warned that they are only exact for the case of high contact, and regarding this have in their doubt neglected all corrections whatever. Now Sheppard's corrections are still valid when there is no high contact, and they should therefore always be used, but they form only part of the full correction* and may indeed merely amount to some 50 % of its value, although 75 % is a more usual average proportion, if the frequency block does not end in finite ordinates. We propose in the first part of this paper to deal with frequency

* In certain cases although part of the full correction they are in the wrong sense, and therefore if used alone would be worse than the raw moments.

blocks such as are indicated in the figure above, reserving for the second part the treatment of the corrections needful when the frequency curve asymptotes to the frequency axis, i.e. the cases of J- and U-shaped frequency distributions. The general treatment of non-asymptotic frequency blocks will follow the lines of pp. 282-8 of the paper: "On the systematic Fitting of Curves," contributed in 1902 by one of the present writers to the first volume of this *Journal*.

(2) The method there adopted started from the Euler-Maclaurin formula:

$$\int_{x_0}^{x_p} Z' dx = \left(\frac{1}{2} Z'_0 + Z'_1 + Z'_2 + \dots + Z'_{p-1} + \frac{1}{2} Z'_p \right) h - h \left[\frac{1}{12} h \frac{dZ'}{dx} - \frac{h^3}{720} \frac{d^3 Z'}{dx^3} \right. \\ \left. + \frac{h^5}{30240} \frac{d^5 Z'}{dx^5} - \frac{h^7}{1209600} \frac{d^7 Z'}{dx^7} + \frac{h^9}{47900160} \frac{d^9 Z'}{dx^9} - \frac{h^{11} B_{11}}{12!} \frac{d^{11} Z'}{dx^{11}} + \dots \right]_{x_0}^{x_p} \dots (i),$$

where Z' is any function of x and $Z'_0, Z'_1, Z'_2, \dots, Z'_p$ are the $p+1$ values of this function corresponding to p subranges taken from $x = x_0$ to $x = x_p$ of the range l . Clearly $ph = l = x_p - x_0$. $B_{11}, B_{13} \dots$ are the higher Bernoulli numbers. The first term on the right involving the $p+1$ values of the function Z' is the "chordal area"; the term between square brackets depends on the values of certain differential coefficients at the ends of the range, and these again depend on the form we assume for the frequency curve in the neighbourhood of the terminals. The value we are going to take for Z' is $x^s Z$, where Z is the integral $\int_x^{x_p} y dx$, or, y being the frequency ordinate, Z is the total frequency on the section $x_p - x$ of the range. In evaluating the limits we need not proceed beyond the ninth differential, for the 11th vanishes for $s=5$ with our assumptions for Z , and in our experience of actual frequency the ninth term as a rule contributes very little to the total correction. In order to obtain our results we must assume some form for Z at the terminals of the frequency block. Clearly at $x = x_0$, $Z = N$, i.e. the total frequency under consideration; at $x = x_p$, $Z = 0$. We shall assume Z given by high order parabolae in the neighbourhood of the terminals, i.e.

$$Z = N \left(1 + \frac{a_1 (x - x_0)}{1! h_0} + \frac{a_2 (x - x_0)^2}{2! h_0^2} + \frac{a_3 (x - x_0)^3}{3! h_0^3} + \frac{a_4 (x - x_0)^4}{4! h_0^4} + \frac{a_5 (x - x_0)^5}{5! h_0^5} \right)$$

in the neighbourhood of $x = x_0$, and

$$Z = N \left(\frac{b_1 (x_p - x)}{1! h_p} + \frac{b_2 (x_p - x)^2}{2! h_p^2} + \frac{b_3 (x_p - x)^3}{3! h_p^3} + \frac{b_4 (x_p - x)^4}{4! h_p^4} + \frac{b_5 (x_p - x)^5}{5! h_p^5} \right)$$

in the neighbourhood of $x = x_p$.

These lead at once to

$$\left(\frac{d^s Z}{dx^s} \right)_{x=x_0} = N a_s / h_0^s \quad \text{and} \quad \left(\frac{d^s Z}{dx^s} \right)_{x=x_p} = N (-1)^s b_s / h_p^s \dots \dots \dots (ii).$$

Exactly as in the earlier memoir we shall determine the a 's and b 's from five frequencies adjacent to the terminals of the range. In many cases, however,

e.g. deaths in infancy, disease incidence in infancy, wages, incomes or house-valuations, we have details for the ends of the range on different subranges to those for the bulk of the curve. These modified subranges will be termed h_0 and h_p , and when either of them is less than h , we shall get more accurate corrective terms if they are used instead of the frequencies on subranges h . At the same time it must be remembered that in calculating the value of the chordal area terms, sufficient of these h_0 and h_p subrange frequencies must be clubbed together to give sub-frequencies on ranges h .

Let the frequencies on the first five subranges h_0 or h from $x = x_0$ be Nn_1' , Nn_2' , Nn_3' , Nn_4' , Nn_5' , so that n_1' , n_2' , n_3' , n_4' , n_5' are proportional frequencies, then

$$N(1 - n_1') = N \left(1 + \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} \right),$$

or
$$-n_1' = \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!}.$$

Similarly
$$-n_1' - n_2' = \frac{a_1}{1!} 2 + \frac{a_2}{2!} 2^2 + \frac{a_3}{3!} 2^3 + \frac{a_4}{4!} 2^4 + \frac{a_5}{5!} 2^5,$$

$$-n_1' - n_2' - n_3' = \frac{a_1}{1!} 3 + \frac{a_2}{2!} 3^2 + \frac{a_3}{3!} 3^3 + \frac{a_4}{4!} 3^4 + \frac{a_5}{5!} 3^5,$$

$$-n_1' - n_2' - n_3' - n_4' = \frac{a_1}{1!} 4 + \frac{a_2}{2!} 4^2 + \frac{a_3}{3!} 4^3 + \frac{a_4}{4!} 4^4 + \frac{a_5}{5!} 4^5,$$

$$-n_1' - n_2' - n_3' - n_4' - n_5' = \frac{a_1}{1!} 5 + \frac{a_2}{2!} 5^2 + \frac{a_3}{3!} 5^3 + \frac{a_4}{4!} 5^4 + \frac{a_5}{5!} 5^5.$$

Solving these equations we find

$$\left. \begin{aligned} a_1 &= -\frac{1}{60} \{ 137n_1' - 163n_2' + 137n_3' - 63n_4' + 12n_5' \}, \\ a_2 &= -\frac{1}{12} \{ 45n_1' - 109n_2' + 105n_3' - 51n_4' + 10n_5' \}, \\ a_3 &= -\frac{1}{4} \{ 17n_1' - 54n_2' + 64n_3' - 34n_4' + 7n_5' \}, \\ a_4 &= \{ 3n_1' - 11n_2' + 15n_3' - 9n_4' + 2n_5' \}, \\ a_5 &= \{ n_1' - 4n_2' + 6n_3' - 4n_4' + n_5' \}. \end{aligned} \right\} \dots\dots\dots(\text{iii})$$

Similarly we find for the b coefficients

$$\left. \begin{aligned} b_1 &= +\frac{1}{60} \{ 137n_p' - 163n_{p-1}' + 137n_{p-2}' - 63n_{p-3}' + 12n_{p-4}' \}, \\ b_2 &= -\frac{1}{12} \{ 45n_p' - 109n_{p-1}' + 105n_{p-2}' - 51n_{p-3}' + 10n_{p-4}' \}, \\ b_3 &= +\frac{1}{4} \{ 17n_p' - 54n_{p-1}' + 64n_{p-2}' - 34n_{p-3}' + 7n_{p-4}' \}, \\ b_4 &= -\{ 3n_p' - 11n_{p-1}' + 15n_{p-2}' - 9n_{p-3}' + 2n_{p-4}' \}, \\ b_5 &= \{ n_p' - 4n_{p-1}' + 6n_{p-2}' - 4n_{p-3}' + n_{p-4}' \}, \end{aligned} \right\} \dots\dots\dots(\text{iv})$$

where Nn_{p-4}' , Nn_{p-3}' , Nn_{p-2}' , Nn_{p-1}' , Nn_p' are the five successive frequencies adjacent to the terminal $x = x_p$ of the subranges h_p or h as the case may be.

Since $dZ/dx = -y$, it follows that

$$\left. \begin{aligned} y_0 &= -\frac{Na_1}{h_0} = \frac{N}{60h_0} \{137n'_1 - 163n'_2 + 137n'_3 - 63n'_4 + 12n'_5\}, \\ y_p &= -\frac{Nb_1}{h_p} = \frac{N}{60h_p} \{137n'_p - 163n'_{p-1} + 137n'_{p-2} - 63n'_{p-3} + 12n'_{p-4}\}. \end{aligned} \right\} \dots(v)$$

These results enable us to determine *approximately* the terminal ordinates of the frequency distributions given by sub-frequencies, and to discover how nearly the frequency curve comes to zero at the terminals of the range. Similarly the smallness of the quantities a_2, a_3, a_4, a_5 and b_2, b_3, b_4, b_5 marks the character of the terminal contact. At the same time the reader must remember two points (i) that the terminal frequencies if small may be subject to large probable errors and (ii) that we have supposed $y=0$, when $x=x_0$ and $x=x_p$, the terminals of an *integer* number of subranges. It is extremely unlikely that the frequency curve would cut the variate axis *exactly* at such places. Hence on both counts, (i) and (ii), we must not anticipate in actual practice that a_1 and b_1 will vanish at $x=x_0$ and $x=x_p$ for non-abruptly terminating frequency, unless we know *a priori* the terminals of the range and have chosen our subranges to fit this knowledge.

(3) The next stage in our work must be to tabulate the values of

$$dZ'/dx, d^3Z'/dx^3, \dots d^9Z'/dx^9,$$

where $Z'=Zx^s$ at the two terminals of the range. We may do this for $s=0, 1, 2, 3, 4, 5$. The theorem of Leibnitz provides the needful expansions which are

$$\begin{aligned} \frac{dZ'}{dx} &= x^s \frac{dZ}{dx} + sx^{s-1}Z, \\ \frac{d^3Z'}{dx^3} &= x^s \frac{d^3Z}{dx^3} + 3sx^{s-1} \frac{d^2Z}{dx^2} + 3s(s-1)x^{s-2} \frac{dZ}{dx} + s(s-1)(s-2)x^{s-3}Z, \\ \frac{d^5Z'}{dx^5} &= x^s \frac{d^5Z}{dx^5} + 5sx^{s-1} \frac{d^4Z}{dx^4} + 10s(s-1)x^{s-2} \frac{d^3Z}{dx^3} + 10s(s-1)(s-2)x^{s-3} \frac{d^2Z}{dx^2} \\ &\quad + 5s(s-1)(s-2)(s-3)x^{s-4} \frac{dZ}{dx} + s(s-1)(s-2)(s-3)(s-4)x^{s-5}Z, \\ \frac{d^7Z'}{dx^7} &= 21s(s-1)x^{s-2} \frac{d^5Z}{dx^5} + 35s(s-1)(s-2)x^{s-3} \frac{d^4Z}{dx^4} + 35s(s-1)(s-2)(s-3)x^{s-4} \frac{d^3Z}{dx^3} \\ &\quad + 21s(s-1)(s-2)(s-3)(s-4)x^{s-5} \frac{d^2Z}{dx^2} \\ &\quad + 7s(s-1)(s-2)(s-3)(s-4)(s-5)x^{s-6} \frac{dZ}{dx} \\ &\quad + s(s-1)(s-2)(s-3)(s-4)(s-5)(s-6)x^{s-7}Z, \\ \frac{d^9Z'}{dx^9} &= 126s(s-1)(s-2)(s-3)x^{s-4} \frac{d^5Z}{dx^5} + 126s(s-1)(s-2)(s-3)(s-4)x^{s-5} \frac{d^3Z}{dx^3} \\ &\quad + 84s(s-1)(s-2)(s-3)(s-4)(s-5)x^{s-6} \frac{d^2Z}{dx^2} + \text{etc.} \dots\dots\dots(vi). \end{aligned}$$

Now all higher differentials of Z than the fifth vanish, and therefore we may cancel the first two terms of $\frac{d^7 Z'}{dx^7}$ and the first four of $\frac{d^9 Z'}{dx^9}$. The value of $\frac{d^{11} Z'}{dx^{11}}$ starts with the term $462s(s-1)(s-2)(s-3)(s-4)(s-5)x^{s-6}\frac{d^5 Z}{dx^5}$ and accordingly this and all terms beyond vanish for $s=0, 1, \dots, 5$, or this 11th differential of Z' is zero for our purposes.

We have now to give s in succession the values 0 to 5, and subtract the result for the first from those for the second terminal:

$$\begin{aligned}
 s=0: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-\frac{b_1}{h_p} - \frac{a_1}{h_0} \right) N, \quad \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(-\frac{b_3}{h_p^3} - \frac{a_3}{h_0^3} \right) N, \\
 & \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(-\frac{b_5}{h_p^5} - \frac{a_5}{h_0^5} \right) N, \quad \left[\frac{d^7 Z'}{dx^7} \right]_{x_0}^{x_p} = \left[\frac{d^9 Z'}{dx^9} \right]_{x_0}^{x_p} = 0; \\
 s=1: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-b_1 \frac{x_p}{h_p} - a_1 \frac{x_0}{h_0} - 1 \right) N, \\
 & \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(-\left(\frac{b_3}{h_p^2} \frac{x_p}{h_p} + \frac{a_3}{h_0^2} \frac{x_0}{h_0} \right) + 3 \left(\frac{b_2}{h_p^2} - \frac{a_2}{h_0^2} \right) \right) N, \\
 & \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(-\left(\frac{b_5}{h_p^4} \frac{x_p}{h_p} + \frac{a_5}{h_0^4} \frac{x_0}{h_0} \right) + 5 \left(\frac{b_4}{h_p^4} - \frac{a_4}{h_0^4} \right) \right) N, \quad \left[\frac{d^{2q+1} Z'}{dx^{2q+1}} \right]_{x_0}^{x_p} = 0 \text{ if } q > 2; \\
 s=2: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-\left(b_1 \frac{x_p^2}{h_p} + a_1 \frac{x_0^2}{h_0} \right) - 2x_0 \right) N, \\
 & \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(-\left(\frac{b_3}{h_p} \frac{x_p^2}{h_p^2} + \frac{a_3}{h_0} \frac{x_0^2}{h_0^2} \right) + 6 \left(\frac{b_2}{h_p} \frac{x_p}{h_p} - \frac{a_2}{h_0} \frac{x_0}{h_0} \right) - 6 \left(\frac{b_1}{h_p} + \frac{a_1}{h_0} \right) \right) N, \\
 & \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(-\left(\frac{b_5}{h_p^3} \frac{x_p^2}{h_p^2} + \frac{a_5}{h_0^3} \frac{x_0^2}{h_0^2} \right) + 10 \left(\frac{b_4}{h_p^3} \frac{x_p}{h_p} - \frac{a_4}{h_0^3} \frac{x_0}{h_0} \right) - 20 \left(\frac{b_3}{h_p^3} + \frac{a_3}{h_0^3} \right) \right) N, \\
 & \left[\frac{d^7 Z'}{dx^7} \right]_{x_0}^{x_p} = 42 \left(-\frac{b_5}{h_p^5} - \frac{a_5}{h_0^5} \right) N, \quad \left[\frac{d^{2q+1} Z'}{dx^{2q+1}} \right]_{x_0}^{x_p} = 0, \quad q > 3; \\
 s=3: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-b_1 \frac{x_p^3}{h_p} - a_1 \frac{x_0^3}{h_0} - 3x_0^2 \right) N, \\
 & \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(-\left(b_3 \frac{x_p^3}{h_p^3} + a_3 \frac{x_0^3}{h_0^3} \right) + 9 \left(b_2 \frac{x_p^2}{h_p^2} - a_2 \frac{x_0^2}{h_0^2} \right) - 18 \left(b_1 \frac{x_p}{h_p} + a_1 \frac{x_0}{h_0} \right) - 6 \right) N, \\
 & \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(-\left(b_5 \frac{x_p^3}{h_p^5} + a_5 \frac{x_0^3}{h_0^5} \right) + 15 \left(b_4 \frac{x_p^2}{h_p^4} - a_4 \frac{x_0^2}{h_0^4} \right) \right. \\
 & \quad \left. - 60 \left(b_3 \frac{x_p}{h_p^3} + a_3 \frac{x_0}{h_0^3} \right) + 60 \left(\frac{b_2}{h_p^2} - \frac{a_2}{h_0^2} \right) \right) N, \\
 & \left[\frac{d^7 Z'}{dx^7} \right]_{x_0}^{x_p} = \left(-126 \left(\frac{b_5}{h_p^4} \frac{x_p}{h_p} + \frac{a_5}{h_0^4} \frac{x_0}{h_0} \right) + 210 \left(\frac{b_4}{h_p^4} - \frac{a_4}{h_0^4} \right) \right) N, \\
 & \quad \left[\frac{d^{2q+1} Z'}{dx^{2q+1}} \right]_{x_0}^{x_p} = 0, \quad q > 3;
 \end{aligned}$$

$$\begin{aligned}
s = 4: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-b_1 \frac{x_p^4}{h_p} - a_1 \frac{x_0^4}{h_0} - 4x_0^3 \right) N, \\
& \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(- \left(b_3 \frac{x_p^4}{h_p^3} + a_3 \frac{x_0^4}{h_0^3} \right) + 12 \left(b_2 \frac{x_p^3}{h_p^2} - a_2 \frac{x_0^3}{h_0^2} \right) \right. \\
& \quad \left. - 36 \left(b_1 \frac{x_p^2}{h_p} + a_1 \frac{x_0^2}{h_0} \right) - 24x_0 \right) N, \\
& \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(- \left(b_5 \frac{x_p^4}{h_p^5} + a_5 \frac{x_0^4}{h_0^5} \right) + 20 \left(b_4 \frac{x_p^3}{h_p^4} - a_4 \frac{x_0^3}{h_0^4} \right) - 120 \left(b_3 \frac{x_p^2}{h_p^3} + a_3 \frac{x_0^2}{h_0^3} \right) \right. \\
& \quad \left. + 240 \left(b_2 \frac{x_p}{h_p^2} - a_2 \frac{x_0}{h_0^2} \right) - 120 \left(\frac{b_1}{h_p} + \frac{a_1}{h_0} \right) \right) N, \\
& \left[\frac{d^7 Z'}{dx^7} \right]_{x_0}^{x_p} = \left(-252 \left(b_5 \frac{x_p^2}{h_p^5} + a_5 \frac{x_0^2}{h_0^5} \right) + 840 \left(b_4 \frac{x_p}{h_p^4} - a_4 \frac{x_0}{h_0^4} \right) - 840 \left(\frac{b_3}{h_p^3} + \frac{a_3}{h_0^3} \right) \right) N, \\
& \left[\frac{d^9 Z'}{dx^9} \right]_{x_0}^{x_p} = 3024 \left(- \left(\frac{b_5}{h_p^5} + \frac{a_5}{h_0^5} \right) \right) N, \quad \left[\frac{d^{2q+1} Z'}{dx^{2q+1}} \right]_{x_0}^{x_p} = 0, \quad q > 4; \\
s = 5: \quad & \left[\frac{dZ'}{dx} \right]_{x_0}^{x_p} = \left(-b_1 \frac{x_p^5}{h_p} - a_1 \frac{x_0^5}{h_0} - 5x_0^4 \right) N, \\
& \left[\frac{d^3 Z'}{dx^3} \right]_{x_0}^{x_p} = \left(- \left(b_3 \frac{x_p^5}{h_p^3} + a_3 \frac{x_0^5}{h_0^3} \right) + 15 \left(b_2 \frac{x_p^4}{h_p^2} - a_2 \frac{x_0^4}{h_0^2} \right) \right. \\
& \quad \left. - 60 \left(b_1 \frac{x_p^3}{h_p} + a_1 \frac{x_0^3}{h_0} \right) - 60x_0^2 \right) N, \\
& \left[\frac{d^5 Z'}{dx^5} \right]_{x_0}^{x_p} = \left(- \left(b_5 \frac{x_p^5}{h_p^5} + a_5 \frac{x_0^5}{h_0^5} \right) + 25 \left(b_4 \frac{x_p^4}{h_p^4} - a_4 \frac{x_0^4}{h_0^4} \right) - 200 \left(b_3 \frac{x_p^3}{h_p^3} + a_3 \frac{x_0^3}{h_0^3} \right) \right. \\
& \quad \left. + 600 \left(b_2 \frac{x_p^2}{h_p^2} - a_2 \frac{x_0^2}{h_0^2} \right) - 600 \left(b_1 \frac{x_p}{h_p} + a_1 \frac{x_0}{h_0} \right) - 120 \right) N, \\
& \left[\frac{d^7 Z'}{dx^7} \right]_{x_0}^{x_p} = \left(-420 \left(b_5 \frac{x_p^3}{h_p^5} + a_5 \frac{x_0^3}{h_0^5} \right) + 2100 \left(b_4 \frac{x_p^2}{h_p^4} - a_4 \frac{x_0^2}{h_0^4} \right) \right. \\
& \quad \left. - 4200 \left(b_3 \frac{x_p}{h_p^3} + a_3 \frac{x_0}{h_0^3} \right) + 2520 \left(\frac{b_2}{h_p^2} - \frac{a_2}{h_0^2} \right) \right) N, \\
& \left[\frac{d^9 Z'}{dx^9} \right]_{x_0}^{x_p} = \left(-15120 \left(b_5 \frac{x_p}{h_p^5} + a_5 \frac{x_0}{h_0^5} \right) + 15120 \left(\frac{b_4}{h_p^4} - \frac{a_4}{h_0^4} \right) \right) N, \\
& \quad \left[\frac{d^{2q+1} Z'}{dx^{2q+1}} \right]_{x_0}^{x_p} = 0, \quad q > 4 \dots (\text{vii}).
\end{aligned}$$

(4) We have now to see the relation of the present integral $\int_{x_0}^{x_p} Zx^s dx$ to the moment-coefficients. Integrating by parts we have

$$\int_{x_0}^{x_p} Zx^s dx = \left[\frac{x^{s+1}}{s+1} Z \right]_{x_0}^{x_p} + \int_{x_0}^{x_p} \frac{y x^{s+1}}{s+1} dx = \frac{-Nx_0^{s+1}}{s+1} + \frac{N}{s+1} \mu'_{s+1},$$

where μ'_{s+1} is the $(s+1)$ th moment-coefficient about the arbitrary origin. Accordingly we have, changing s to $s-1$,

$$\mu'_s = x_0^s + \frac{s}{N} \int_{x_0}^{x_p} Zx^{s-1} dx \dots \dots \dots (\text{viii}).$$

Thus we can write

$$\mu_s' = x_0^s + \frac{s}{N} C_{s-1} - \frac{s}{N} L_{s-1},$$

where C_{s-1} is the "chordal area" term and L_{s-1} is the limit term of the Euler-Maclaurin series, or if $x_u = x_0 + uh$,

$$\begin{aligned} C_{s-1} &= h \left\{ \frac{1}{2} Z_0 x_0^{s-1} + Z_1 x_1^{s-1} + Z_2 x_2^{s-1} + \dots + \frac{1}{2} Z_p x_p^{s-1} \right\}, \\ L_{s-1} &= h \left[\frac{1}{12} h \frac{d(Zx^{s-1})}{dx} - \frac{1}{720} h^3 \frac{d^3(Zx^{s-1})}{dx^3} + \frac{h^5}{30240} \frac{d^5(Zx^{s-1})}{dx^5} \right. \\ &\quad \left. - \frac{h^7}{1209600} \frac{d^7(Zx^{s-1})}{dx^7} + \frac{h^9}{47900160} \frac{d^9(Zx^{s-1})}{dx^9} \right]_{x_0}^{x_p} \dots\dots(ix). \end{aligned}$$

We now turn to the evaluation of the chordal areas. We can obtain these by remembering that if n_q be the frequency on the q th subrange h ,

$$Z_q = \int_{x_q}^{x_p} y dx = n_{q+1} + n_{q+2} + \dots + n_p.$$

Thus it follows that

$$C_{s-1} = h \sum_{u=1}^p \left\{ \frac{1}{2} x_0^{s-1} + (x_0 + h)^{s-1} + (x_0 + 2h)^{s-1} + \dots + (x_0 + (u-1)h)^{s-1} \right\} n_u,$$

if we note that $Z_p = 0$.

But the series coefficient of n_u can itself be summed by the Euler-Maclaurin Theorem, i.e.

$$\begin{aligned} &h \left\{ \frac{1}{2} x_0^{s-1} + (x_0 + h)^{s-1} + \dots + (x_0 + (u-1)h)^{s-1} \right\} \\ &= \int_{x_0}^{x_0+uh} t^{s-1} dt - \frac{1}{2} h (x_0 + uh)^{s-1} + \left[\frac{1}{12} h^2 \frac{d(x^{s-1})}{dx} - \frac{1}{720} h^4 \frac{d^3(x^{s-1})}{dx^3} \right. \\ &\quad \left. + \frac{1}{30240} h^6 \frac{d^5(x^{s-1})}{dx^5} \right]_{x_0}^{x_0+uh} \\ &= -\frac{1}{2} h (x_0 + uh)^{s-1} + \frac{1}{s} (x_0 + uh)^s + (s-1) \frac{h^2}{12} (x_0 + uh)^{s-2} \\ &\quad - (s-1)(s-2)(s-3) \frac{1}{720} h^4 (x_0 + uh)^{s-4} \\ &\quad + (s-1)(s-2)(s-3)(s-4)(s-5) \frac{1}{30240} h^6 (x_0 + uh)^{s-6} - \dots \\ &\quad - \frac{1}{s} x_0^s - (s-1) \frac{h^2}{12} x_0^{s-2} + (s-1)(s-2)(s-3) \frac{1}{720} h^4 x_0^{s-4} \\ &\quad - (s-1)(s-2)(s-3)(s-4)(s-5) \frac{1}{30240} h^6 x_0^{s-6} + \dots \end{aligned}$$

We are now in the position to find the value of sC_{s-1} for the successive values of s . We have

$$\begin{aligned} \frac{1}{N} (sC_{s-1})_{s=1} &= \frac{1}{N} \sum_{u=1}^p \left\{ -\frac{1}{2} h + x_0 + uh - x_0 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ x_0 + (u - \frac{1}{2})h \right\} n_u - \frac{x_0}{N} \sum_{u=1}^p (n_u) = \nu_1' - x_0 \dots\dots\dots(x), \end{aligned}$$

where ν'_1 = moment about the origin of the subrange frequencies concentrated at their mid subrange points. Similarly

$$\begin{aligned}\frac{1}{N} (sC_{s-1})_{s=2} &= \frac{1}{N} \sum_{u=1}^p \left\{ -h(x_0 + uh) + (x_0 + uh)^2 + \frac{1}{6}h^2 - x_0^2 - \frac{1}{6}h^2 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ (x_0 + (u - \frac{1}{2})h)^2 - \frac{1}{4}h^2 - x_0^2 \right\} n_u \\ &= \nu'_2 - \frac{1}{4}h^2 - x_0^2 \dots\dots\dots(\text{xi});\end{aligned}$$

$$\begin{aligned}\frac{1}{N} (sC_{s-1})_{s=3} &= \frac{1}{N} \sum_{u=1}^p \left\{ -\frac{3h}{2}(x_0 + uh)^2 + (x_0 + uh)^3 + \frac{h^2}{2}(x_0 + uh) - x_0^3 - \frac{h^2}{2}x_0 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ (x_0 + (u - \frac{1}{2})h)^3 - \frac{1}{4}h^2(x_0 + (u - \frac{1}{2})h) - x_0^3 - \frac{1}{2}h^2x_0 \right\} n_u \\ &= \nu'_3 - \frac{1}{4}h^2\nu'_1 - \frac{1}{2}h^2x_0 - x_0^3 \dots\dots\dots(\text{xii});\end{aligned}$$

$$\begin{aligned}\frac{1}{N} (sC_{s-1})_{s=4} &= \frac{1}{N} \sum_{u=1}^p \left\{ -2h(x_0 + uh)^3 + (x_0 + uh)^4 + h^2(x_0 + uh)^2 \right. \\ &\quad \left. - \frac{1}{6}h^4 - x_0^4 - h^2x_0^2 + \frac{1}{6}h^4 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ (x_0 + (u - \frac{1}{2})h)^4 - \frac{1}{2}h^2(x_0 + (u - \frac{1}{2})h)^2 + \frac{1}{6}h^4 - x_0^4 - h^2x_0^2 \right\} n_u \\ &= \nu'_4 - \frac{1}{2}h^2\nu'_2 + \frac{1}{6}h^4 - x_0^4 - h^2x_0^2 \dots\dots\dots(\text{xiii});\end{aligned}$$

$$\begin{aligned}\frac{1}{N} (sC_{s-1})_{s=5} &= \frac{1}{N} \sum_{u=1}^p \left\{ -\frac{5}{2}h(x_0 + uh)^4 + (x_0 + uh)^5 + \frac{5}{3}h^2(x_0 + uh)^3 \right. \\ &\quad \left. - \frac{1}{6}h^4(x_0 + uh) - x_0^5 - \frac{5}{3}h^2x_0^3 + \frac{1}{6}h^4x_0 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ (x_0 + (u - \frac{1}{2})h)^5 - \frac{5}{6}h^2(x_0 + (u - \frac{1}{2})h)^3 \right. \\ &\quad \left. + \frac{7}{48}h^4(x_0 + (u - \frac{1}{2})h) - x_0^5 - \frac{5}{3}h^2x_0^3 + \frac{1}{6}h^4x_0 \right\} n_u \\ &= \nu'_5 - \frac{5}{6}h^2\nu'_3 + \frac{7}{48}h^4\nu'_1 - x_0^5 - \frac{5}{3}h^2x_0^3 + \frac{1}{6}h^4x_0 \dots\dots\dots(\text{xiv});\end{aligned}$$

$$\begin{aligned}\frac{1}{N} (sC_{s-1})_{s=6} &= \frac{1}{N} \sum_{u=1}^p \left\{ -3h(x_0 + uh)^5 + (x_0 + uh)^6 + \frac{5}{2}h^2(x_0 + uh)^4 \right. \\ &\quad \left. - \frac{1}{2}h^4(x_0 + uh)^2 + \frac{1}{24}h^6 - x_0^6 - \frac{5}{2}h^2x_0^4 + \frac{1}{2}h^4x_0^2 - \frac{1}{24}h^6 \right\} n_u \\ &= \frac{1}{N} \sum_{u=1}^p \left\{ (x_0 + (u - \frac{1}{2})h)^6 - \frac{5}{4}h^2(x_0 + (u - \frac{1}{2})h)^4 \right. \\ &\quad \left. + \frac{7}{16}h^4(x_0 + (u - \frac{1}{2})h)^2 - \frac{3}{64}h^6 - x_0^6 - \frac{5}{2}h^2x_0^4 + \frac{1}{2}h^4x_0^2 \right\} n_u \\ &= \nu'_6 - \frac{5}{4}h^2\nu'_4 + \frac{7}{16}h^4\nu'_2 - \frac{3}{64}h^6 - x_0^6 - \frac{5}{2}h^2x_0^4 + \frac{1}{2}h^4x_0^2 \dots\dots\dots(\text{xv}).\end{aligned}$$

(5) We can now put together the complete formulae for the corrected moment-coefficients about any origin from the values we have obtained for the component parts of (ix), but we may first simplify our notation slightly so as to abbreviate somewhat the lengthy resulting expressions. We write $\rho_p = h/h_p$, $\rho_0 = h/h_0$; these will very frequently be unity. Next we put $b'_s = b_s \rho_p^s$, $a'_s = a_s \rho_0^s$, and $x_p/h = x'_p$, $x_0/h = x'_0$; thus for terminal units the same as for the bulk of the frequency we

should have $b_s' = b_s$, $a_s' = a_s$, and for working units where h generally = 1, $x_p' = x_p$, $x_0' = x_0$. We find

$$\mu_1' = \nu_1' + h \left\{ \frac{1}{12} (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') + \frac{1}{12} (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xvi}).$$

$$\begin{aligned} \mu_2' = \nu_2' - \frac{1}{12} h^2 + h^2 \left\{ -\frac{1}{120} (a_2' - \frac{5}{126} a_4') + \frac{1}{120} (b_2' - \frac{5}{126} b_4') \right. \\ \left. + \frac{1}{6} x_0' (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') + \frac{1}{6} x_p' (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xvii}). \end{aligned}$$

$$\begin{aligned} \mu_3' = \nu_3' - \frac{1}{4} h^2 \nu_1' + h^3 \left\{ -\frac{1}{40} (a_1' - \frac{5}{63} a_3' + \frac{1}{240} a_5') - \frac{1}{40} (b_1' - \frac{5}{63} b_3' + \frac{1}{240} b_5') \right. \\ \left. - \frac{1}{40} x_0' (a_2' - \frac{5}{126} a_4') + \frac{1}{40} x_p' (b_2' - \frac{5}{126} b_4') + \frac{1}{4} x_0'^2 (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') \right. \\ \left. + \frac{1}{4} x_p'^2 (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xviii}). \end{aligned}$$

$$\begin{aligned} \mu_4' = \nu_4' - \frac{1}{2} h^2 \nu_2' + \frac{7}{240} h^4 + h^4 \left\{ \frac{1}{126} (a_2' - \frac{7}{80} a_4') - \frac{1}{126} (b_2' - \frac{7}{80} b_4') \right. \\ \left. - \frac{1}{16} x_0' (a_1' - \frac{5}{63} a_3' + \frac{1}{240} a_5') - \frac{1}{16} x_p' (b_1' - \frac{5}{63} b_3' + \frac{1}{240} b_5') \right. \\ \left. - \frac{1}{20} x_0'^2 (a_2' - \frac{5}{126} a_4') + \frac{1}{20} x_p'^2 (b_2' - \frac{5}{126} b_4') + \frac{1}{3} x_0'^3 (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') \right. \\ \left. + \frac{1}{3} x_p'^3 (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xix}). \end{aligned}$$

$$\begin{aligned} \mu_5' = \nu_5' - \frac{5}{6} h^2 \nu_3' + \frac{7}{48} h^4 \nu_1' + h^5 \left\{ \frac{5}{252} (a_1' - \frac{7}{40} a_3' + \frac{7}{440} a_5') + \frac{5}{252} (b_1' - \frac{7}{40} b_3' + \frac{7}{440} b_5') \right. \\ \left. + \frac{5}{126} x_0' (a_2' - \frac{7}{80} a_4') - \frac{5}{126} x_p' (b_2' - \frac{7}{80} b_4') - \frac{1}{4} x_0'^2 (a_1' - \frac{5}{63} a_3' + \frac{1}{240} a_5') \right. \\ \left. - \frac{1}{4} x_p'^2 (b_1' - \frac{5}{63} b_3' + \frac{1}{240} b_5') - \frac{1}{12} x_0'^3 (a_2' - \frac{5}{126} a_4') + \frac{1}{12} x_p'^3 (b_2' - \frac{5}{126} b_4') \right. \\ \left. + \frac{5}{12} x_0'^4 (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') + \frac{5}{12} x_p'^4 (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xx}). \end{aligned}$$

$$\begin{aligned} \mu_6' = \nu_6' - \frac{5}{4} h^2 \nu_4' + \frac{7}{16} h^4 \nu_2' - \frac{31}{1344} h^6 + h^6 \left\{ -\frac{1}{80} (a_2' - \frac{5}{33} a_4') + \frac{1}{80} (b_2' - \frac{5}{33} b_4') \right. \\ \left. + \frac{5}{42} x_0' (a_1' - \frac{7}{40} a_3' + \frac{7}{440} a_5') + \frac{5}{42} x_p' (b_1' - \frac{7}{40} b_3' + \frac{7}{440} b_5') + \frac{5}{42} x_0'^2 (a_2' - \frac{7}{80} a_4') \right. \\ \left. - \frac{5}{42} x_p'^2 (b_2' - \frac{7}{80} b_4') - \frac{1}{2} x_0'^3 (a_1' - \frac{5}{63} a_3' + \frac{1}{240} a_5') - \frac{1}{2} x_p'^3 (b_1' - \frac{5}{63} b_3' + \frac{1}{240} b_5') \right. \\ \left. - \frac{1}{8} x_0'^4 (a_2' - \frac{5}{126} a_4') + \frac{1}{8} x_p'^4 (b_2' - \frac{5}{126} b_4') + \frac{1}{2} x_0'^5 (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') \right. \\ \left. + \frac{1}{2} x_p'^5 (b_1' - \frac{1}{60} b_3' + \frac{1}{2520} b_5') \right\} \dots\dots\dots(\text{xxi}). \end{aligned}$$

The first series of terms outside the curled brackets are precisely the Sheppard's corrections for the moments which accordingly still remain essential portions of the corrective terms even when there are final terminal ordinates and any degree of abruptness in the slopes at the end of the range. We may speak of the a 's and b 's as the "abruptness coefficients." They are determined by equations (iii) and (iv).

It will be clear that the terms in the curled brackets repeat themselves, so that in working as we usually do to the fourth moment we have to deal only with eight functions of the abruptness coefficients.

The next stage is to consider how equations (xvi)—(xxi) may be most advantageously arranged for practical statistical work. In all such work the subrange h is taken as unity. Hence we may always write it 1. Further, the origin is at our choice, and it might seem desirable to take it at the mean. But there are two means, namely the true mean μ_1' of the data and the mean ν_1' of the concentrated groups; with abruptness these are no longer identical. If we take moments about the true mean, ν_1' is not zero and our calculations are not simplified by its vanishing. On the other hand if we take moments about the mean of the concentrated groups

we shall then have to transfer the μ 's to their true mean. Nothing is therefore gained practically by the process. Besides this, neither mean is a good working origin, and if we take such to calculate the moments in the first place we may have *two* transfers to the means, one for the ν 's and one for the μ 's. Further x_p and x_0 will both have to be calculated and all the terms used. We can, however, get rid of slightly less than half the corrective terms, if we take moments about one end of the range* and then transfer the μ 's to their mean. This appears to us in practice to be the best policy, for, although it involves taking the differences of large numbers, it is quite easy with modern mechanical calculators to retain the requisite number of figures for accurate results. Accordingly we will rewrite our formulae with these changes, remembering that x_p is now the range $l = ph = p$.

$$\mu'_1 = \nu'_1 + \left\{ \frac{1}{12} (a'_1 - \frac{1}{60} a'_3 + \frac{1}{2520} a'_5) + \frac{1}{12} (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxii).$$

$$\begin{aligned} \mu'_2 = \nu'_2 - \frac{1}{12} + \left\{ -\frac{1}{120} (a'_2 - \frac{5}{126} a'_4) + \frac{1}{120} (b'_2 - \frac{5}{126} b'_4) \right. \\ \left. + \frac{1}{6} p (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxiii). \end{aligned}$$

$$\begin{aligned} \mu'_3 = \nu'_3 - \frac{1}{4} \nu'_1 + \left\{ -\frac{1}{40} (a'_1 - \frac{5}{63} a'_3 + \frac{1}{240} a'_5) - \frac{1}{40} (b'_1 - \frac{5}{63} b'_3 + \frac{1}{240} b'_5) \right. \\ \left. + \frac{1}{40} p (b'_2 - \frac{5}{126} b'_4) + \frac{1}{4} p^2 (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxiv). \end{aligned}$$

$$\begin{aligned} \mu'_4 = \nu'_4 - \frac{1}{2} \nu'_2 + \frac{7}{240} + \left\{ \frac{1}{126} (a'_2 - \frac{7}{80} a'_4) - \frac{1}{126} (b'_2 - \frac{7}{80} b'_4) - \frac{1}{10} p (b'_1 - \frac{5}{63} b'_3 + \frac{1}{240} b'_5) \right. \\ \left. + \frac{1}{20} p^2 (b'_2 - \frac{5}{126} b'_4) + \frac{1}{3} p^3 (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxv). \end{aligned}$$

$$\begin{aligned} \mu'_5 = \nu'_5 - \frac{5}{6} \nu'_3 + \frac{7}{48} \nu'_1 + \left\{ \frac{5}{252} (a'_1 - \frac{7}{40} a'_3 + \frac{7}{440} a'_5) + \frac{5}{252} (b'_1 - \frac{7}{40} b'_3 + \frac{7}{440} b'_5) \right. \\ \left. - \frac{5}{126} p (b'_2 - \frac{7}{80} b'_4) - \frac{1}{4} p^2 (b'_1 - \frac{5}{63} b'_3 + \frac{1}{240} b'_5) + \frac{1}{12} p^3 (b'_2 - \frac{5}{126} b'_4) \right. \\ \left. + \frac{5}{12} p^4 (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxvi). \end{aligned}$$

$$\begin{aligned} \mu'_6 = \nu'_6 - \frac{5}{4} \nu'_4 + \frac{7}{16} \nu'_2 - \frac{3}{1344} + \left\{ -\frac{1}{80} (a'_2 - \frac{5}{33} a'_4) + \frac{1}{80} (b'_2 - \frac{5}{33} b'_4) \right. \\ \left. + \frac{5}{42} p (b'_1 - \frac{7}{40} b'_3 + \frac{7}{440} b'_5) - \frac{5}{42} p^2 (b'_2 - \frac{7}{80} b'_4) - \frac{1}{2} p^3 (b'_1 - \frac{5}{63} b'_3 + \frac{1}{240} b'_5) \right. \\ \left. + \frac{1}{3} p^4 (b'_2 - \frac{5}{126} b'_4) + \frac{1}{2} p^5 (b'_1 - \frac{1}{60} b'_3 + \frac{1}{2520} b'_5) \right\} \dots\dots\dots (xxvii). \end{aligned}$$

(6) We now propose to illustrate the degree of exactness with which it is possible to obtain the moment-coefficients of curves with marked degrees of abruptness, and further to investigate in practice the extent to which small terminal range elements may be of advantage. We will commence with some mathematical frequency distributions for which it is possible to calculate the exact values of the moment-coefficients.

Illustration I. Moment-coefficients of the common parabola $y = \sqrt{x} \times 100,000$ from $x = 0$ to 10. This is a good case for a test, for the curve rises vertically at $x_0 = 0$, and therefore, theoretically, our equations fail. At $x_0 = 10$, we have a finite ordinate and finite abruptness coefficients. We are hardly likely to get a case wherein the abruptness causes greater changes in the grouped frequency moments, or to which it is less possible *a priori* to apply merely Sheppard's corrections.

* The distances of the successive concentrated groups are $\frac{1}{2}h, \frac{3}{2}h, \frac{5}{2}h, \dots$. In taking moments it is convenient to use 1, 3, 5, etc. and then before substitution multiply ν'_s by $(\cdot 5)^s$.

We divide the space from $x = 0$ to $x = 10$ into ten subranges giving the following system of "frequencies":

	Absolute frequencies	Proportional frequencies
n_1	66,667	$n_1' = \cdot 031,623$
n_2	121,895	$n_2' = \cdot 057,820$
n_3	157,848	$n_3' = \cdot 074,874$
n_4	186,923	$n_4' = \cdot 088,665$
n_5	212,023	$n_5' = \cdot 100,571$
n_{p-4}	234,440	$n_{p-4}' = \cdot 111,205$
n_{p-3}	254,888	$n_{p-3}' = \cdot 120,904$
n_{p-2}	273,811	$n_{p-2}' = \cdot 129,880$
n_{p-1}	291,505	$n_{p-1}' = \cdot 138,273$
n_p	308,185	$n_p' = \cdot 146,185$
Total frequency	<u>$= 2,108,185$</u>	<u>$1,000,000$</u>

These lead to

	Abruptness Differentials and Ordinate	
	<i>Calculated</i>	<i>Actual</i>
$a_1 = -\cdot 0131,0643$	$y_0 = 27630\cdot 78$	0
$a_2 = -\cdot 0444,8167$	$y_0' = +93775\cdot 59$	∞
$a_3 = \cdot 0258,4150$	$y_0'' = -54478\cdot 66$	∞
$a_4 = -\cdot 0148,8400$	$y_0''' = +31378\cdot 23$	∞
$a_5 = \cdot 0045,0200$	$y_0^{iv} = -9491\cdot 05$	∞

Now it is clear that the ordinate of our auxiliary curve is not zero, but it looks larger than it really is relative to the ordinate at the other terminal which is 316227·77 so that the ratio is only ·087, or if the curve be actually drawn to any reasonable scale, the ordinate of the auxiliary curve at the vertex which is less than one-tenth of that at the other terminal, looks relatively small. We may also compare it with the ordinate of the actual curve at $x = 1$ which is 100,000, or between three and four times as great. Similarly the abruptness differentials are not infinite, but their values in the actual curve are very considerable at $x = 1$ and are then: $y_1' = 50,000$, $y_1'' = -25,000$, $y_1''' = 37,500$ and $y_1^{iv} = -93,750$. Thus the first two for the auxiliary curve are about double, the third of the same order, but the fourth is much less. Clearly all this is a result of the fact that we cannot expand $\sqrt{x}$ in a series of integer powers of x , and this is one of the reasons why we selected it. We want to determine whether the formulae give a very bad result for the moments even in the case of extreme abruptness. Accordingly our real test lies in the values of the deduced moment-coefficients and not in those of the abruptness differentials.

We pass now to the b 's and find:

	<i>Calculated</i>	<i>Actual</i>
$b_1 = \cdot 1499,9857$	$y_p = 316,224\cdot 74$	316,227·77
$b_2 = -\cdot 0074,9283$	$y_p' = -15,796\cdot 27$	- 15,811·39
$b_3 = -\cdot 0003,9450$	$y_p'' = -831\cdot 68$	- 790·57
$b_4 = -\cdot 0000,2600$	$y_p''' = -54\cdot 81$	- 118·59
$b_5 = -\cdot 0000,3800$	$y_p^{iv} = -80\cdot 11$	- 29·65

Here again the values of the terminal ordinate and the abruptness coefficients, although good in the former case, are only approximations and the real test must

depend on the moment-coefficients. If we go as far as μ_4' we have to calculate the following eight expressions:

$$\begin{aligned} a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5' &= -\cdot0131,0643 - \cdot0004,3069 + \cdot0000,0179 = -\cdot0135,3533, \\ a_2' - \frac{5}{126}a_4' &= -\cdot0444,8167 + \cdot0005,9063 = -\cdot0438,9104, \\ a_1' - \frac{5}{63}a_3' + \frac{1}{240}a_5' &= -\cdot0131,0643 - \cdot0020,5092 + \cdot0000,1876 = -\cdot0151,3859, \\ a_2' - \frac{7}{80}a_4' &= -\cdot0444,8167 + \cdot0013,0235 = -\cdot0431,7932. \end{aligned}$$

The quantities we require are

$$\begin{aligned} \frac{1}{12}(a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5') &= -\cdot0011,2794, & \frac{1}{120}(a_2' - \frac{5}{126}a_4') &= -\cdot0003,6576, \\ \frac{1}{40}(a_1' - \frac{5}{63}a_3' + \frac{1}{240}a_5') &= -\cdot0003,7846, & \frac{1}{126}(a_2' - \frac{7}{80}a_4') &= -\cdot0003,4269. \end{aligned}$$

It will be seen from these results that a_4' does not contribute very much and a_5' still less to the final corrections. We now take the b 's and find

$$\begin{aligned} (b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= \cdot1500,0513, & (b_2' - \frac{5}{126}b_4') &= -\cdot0074,9273, \\ (b_1' - \frac{5}{63}b_3' + \frac{1}{240}b_5') &= \cdot1500,2972, & (b_2' - \frac{7}{80}b_4') &= -\cdot0074,9055. \end{aligned}$$

Whence we deduce for the abruptness functions, since $p = 10$:

$$\begin{aligned} \frac{1}{12}(b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= \cdot0125,0043, & \frac{1}{6}p(b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= \cdot2500,0860, \\ \frac{1}{4}p^2(b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= 3\cdot7501,2825, & \frac{1}{3}p^3(b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= 50\cdot0017,1000, \\ \frac{1}{120}(b_2' - \frac{5}{126}b_4') &= -\cdot0000,6244, & \frac{1}{40}p(b_2' - \frac{5}{126}b_4') &= -\cdot0001,8731, \\ \frac{1}{20}p^2(b_2' - \frac{5}{126}b_4') &= -\cdot0374,6365, & \frac{1}{40}(b_1' - \frac{5}{63}b_3' + \frac{1}{240}b_5') &= \cdot0037,5074, \\ \frac{1}{10}p(b_1' - \frac{5}{63}b_3' + \frac{1}{240}b_5') &= \cdot1500,2972, & \frac{1}{126}(b_2' - \frac{7}{80}b_4') &= -\cdot0000,5945. \end{aligned}$$

We now give the values of the grouped moment-coefficients about the origin. Alongside them we place their values as corrected by Sheppard's terms. We then give the values as found by full correction formulae and lastly the actual values as deduced by integrating the parabola.

Raw moments		Values with Sheppard's corrections	Values with full corrections	Actual values		
ν_1'	5.9880	ν_1'	5.9880	μ_1'	5.9994	6.0000
ν_2'	42.6900	$\nu_2' - \frac{1}{12}$	42.6067	μ_2'	42.8570	42.8571
ν_3'	331.0854	$\nu_3' - \frac{1}{4}\nu_1'$	329.5884	μ_3'	333.3349	333.3333
ν_4'	2698.7735	$\nu_4' - \frac{1}{2}\nu_2 + \frac{7}{240}$	2677.4576	μ_4'	2727.2757	2727.2729

It will be seen that the fully corrected results are in most excellent accord with the actual values. Sheppard's corrections, although component parts of the general corrections, move if taken alone in the wrong direction, i.e. they lower moments, all of which need to be raised. Thus while Sheppard's correction lowers the fourth moment by about 21, our new corrections raise it by about 50, the result being the requisite raising by 29.

It seems to us unlikely that a more unfavourable case for our abruptness coefficients could be found. It certainly emphasises the point that to obtain very

* The a 's and the b 's will in this case be equal to the a 's and b 's.

good results it is quite unnecessary for the terminal ordinates and differential coefficients of the actual curve of frequency and the auxiliary terminal curve to be closely identical.

(7) *Illustration II.* Now let us take a normal curve containing 1,000,000 individuals with a standard deviation of unity, and let us suppose the frequency grouped on $0.5 \times$ standard deviation subranges, the first such subrange being central. Then, adjusting to units, we have the following system:

— .25 — + .25	197,414
+ .25 — + .75	174,666
+ .75 — + 1.25	120,977
+ 1.25 — + 1.75	65,591
+ 1.75 — + 2.25	27,834
+ 2.25 — + 2.75	9,245
+ 2.75 — + 3.25	2,402
+ 3.25 — + 3.75	489
+ 3.75 — + 4.25	78
+ 4.25 — + 4.75	10
+ 4.75 — + 5.25	1

To test the error introduced by our adjustments, take second moments for the complete curve about the centre of the group from $-.25$ to $+.25$. We have

$$\nu_1' = 0, \quad \nu_2' = 4.083,394.$$

Using Sheppard's correction as abruptness coefficients are zero, we have

$$\begin{aligned} \mu_2 &= 4.000,061 \quad \text{in working units,} \\ &= 1.000,0152 \quad \text{in actual units.} \end{aligned}$$

Accordingly $\sigma = 1.000,008$, which is a quite good approximation to unity. The error introduced by our adjustments for omitted decimals is therefore not great.

(a) We will start first with the singly truncated normal curve given below and $h_0 = h$, i.e. the area from $x = 1.25$ onwards,

Frequencies		Moments about stump
65,591	Total 105,650,	$\nu_1' = 1.029,513$
27,834		$\nu_2' = 1.693,994$
9,245		$\nu_3' = 3.883,416$
2,402		$\nu_4' = 10.974,937$
489		(in working units)
78		
10		
1		

and determine its moment-coefficients, as a frequency curve having high contact at one end and marked abruptness at the other. In this case all the b 's are zero and we only need to find the a 's. Clearly $a' = a$.

Using (iii) we determine from

$$\begin{array}{ll} n_1' = \cdot 6208,3294 & \text{the values: } a_1' = -\cdot 878,708, \\ n_2' = \cdot 2634,5480 & a_2' = \cdot 607,984, \\ n_3' = \cdot 0875,0592 & a_3' = -\cdot 296,843, \\ n_4' = \cdot 0227,3545 & a_4' = \cdot 081,723, \\ n_5' = \cdot 0046,2849 & a_5' = \cdot 005,736. \end{array}$$

Whence

$$\begin{array}{ll} a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5' = -\cdot 873,758, & a_2' - \frac{5}{126}a_4' = +\cdot 604,741, \\ a_1' - \frac{5}{63}a_3' + \frac{1}{240}a_5' = -\cdot 855,125, & a_2' - \frac{7}{80}a_4' = +\cdot 600,833. \end{array}$$

The corrections due to these abruptness coefficients by (xxii)—(xxv) are for ν_1' , ν_2' , ν_3' and ν_4' respectively

$$-\cdot 072,813, \quad -\cdot 005,040, \quad +\cdot 021,378 \quad \text{and} \quad +\cdot 004,769,$$

while the corresponding Sheppard's corrections are

$$0, \quad -\cdot 083,333, \quad -\cdot 257,378 \quad \text{and} \quad -\cdot 817,830.$$

Thus we deduce for moment-coefficients about stump:

	Raw moment	With Sheppard's corrections	Full corrections
ν_1'	1·029,513	1·029,513	·956,700
ν_2'	1·693,994	1·610,661	1·605,621
ν_3'	3·883,416	3·626,038	3·647,416
ν_4'	10·974,937	10·157,107	10·161,876

These values are in working units = 5 actual units. Hence in actual units we have

	Raw moments		With Sheppard's corrections		With full corrections	True values
ν_1'	·514,756	ν_1'	·514,756	μ_1'	·478,350	·478,8131
ν_2'	·423,498	$\nu_2' - \frac{1}{12}$	·402,665	μ_2'	·401,405	·401,4837
ν_3'	·485,427	$\nu_3' - \frac{1}{4}\nu_1'$	·453,255	μ_3'	·455,927	·455,7714
ν_4'	·685,934	$\nu_4' - \frac{1}{2}\nu_2' + \frac{7}{240}$	·634,819	μ_4'	·635,117	·634,7360

It will be seen from these results that our full correction values for the moments about the stump are in every case accurate to 1 in the 1000, while, if Sheppard's corrections only are made, we may be out nearly 1 in the 100. The change in the mean, second and third moment-coefficients is very noteworthy. In the case of the fourth moment we are out ·0004 in ·6347, while the Sheppard's correction alone is out only ·0001 in ·6347. The cause of this irregularity we have not been able to detect, although we have examined carefully the whole of our arithmetic. It seemed accordingly worth while inquiring what differences would occur when the moment-coefficients were taken about the mean and not about the stump.

	Raw moments about mean		With Sheppard's corrections		With full corrections	True values
ν_2	·158,524	$\nu_2 - \frac{1}{12}(\cdot 5)^2$	·137,691	μ_2	·172,587	·172,222
ν_3	·104,226	ν_3	·104,226	μ_3	·098,801	·098,612
ν_4	·149,090	$\nu_4 - \frac{1}{2}\nu_2(\cdot 5)^2 + \frac{7}{240}(\cdot 5)^4$	·131,097	μ_4	·156,767	·156,405 "

It will be seen that now Sheppard's corrections are wholly inadequate and our corrections are essential, even in the case of the fourth moment-coefficient. This confirms the view of Sheppard himself, who insisted on the importance of high contact at the terminals, if they are to be used *alone*. It is a convincing illustration of the fallacy of those "proofs" of Sheppard's corrections which do not appeal to the principle of high terminal contact.

We now propose to illustrate the degree of improvement in the exactness obtained, if we calculate the abruptness coefficients on smaller subranges. Accordingly we break up the terminal group 65591 on $\cdot 5\sigma$ base into five groups each on $\cdot 1\sigma$ base. These are

17142	leading to	$n_1' = \cdot 1622,5272$	and	$a_1 = -\cdot 1728,9281,$
14979		$n_2' = \cdot 1417,7946$		$a_2 = -\cdot 0216,4780,$
12959		$n_3' = \cdot 1226,5973$		$a_3 = -\cdot 0010,4603,$
11099		$n_4' = \cdot 1050,5442$		$a_4 = -\cdot 0002,3651,$
9412		$n_5' = \cdot 0890,8661$		$a_5 = \cdot 0000,3781,$

whence, remembering $a_s' = \left(\frac{h}{h_0}\right)^s a_s = 5^s a_s$, we find

$$\begin{aligned} a_1' &= -\cdot 864,464, \text{ and } \frac{1}{12} (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') = -\cdot 071,853, \\ a_2' &= \cdot 541,195, \quad -\frac{1}{120} (a_2' - \frac{5}{126} a_4') = -\cdot 004,559, \\ a_3' &= -\cdot 130,754, \quad -\frac{1}{40} (a_1' - \frac{5}{63} a_3' + \frac{1}{240} a_5') = \cdot 021,351, \\ a_4' &= -\cdot 147,818, \quad \frac{1}{124} (a_2' - \frac{7}{80} a_4') = \cdot 004,398, \\ a_5' &= \cdot 118,156. \end{aligned}$$

Thus the moment-coefficients become

$$\begin{aligned} \mu_1' &= \cdot 957,660 \text{ or in actual units } \cdot 478,830, \\ \mu_2' &= 1\cdot 606,102 \quad \cdot 401,526, \\ \mu_3' &= 3\cdot 647,389 \quad \cdot 455,924, \\ \mu_4' &= 10\cdot 161,505 \quad \cdot 635,094. \end{aligned}$$

Transferring to the mean we have

	On $\cdot 5$ subranges	On $\cdot 1$ subranges	Actual values
μ_1'	$\cdot 478,350$	$\cdot 478,830$	$\cdot 478,813$
μ_2'	$\cdot 172,587$	$\cdot 172,248$	$\cdot 172,222$
μ_3'	$\cdot 098,801$	$\cdot 098,705$	$\cdot 098,612$
μ_4'	$\cdot 156,767$	$\cdot 156,516$	$\cdot 156,405$

While the first column of values would be amply adequate for most statistical purposes, the second makes a still closer approximation to the actual values, the differences being only

$$-\cdot 000,017, \quad +\cdot 000,026, \quad +\cdot 000,093, \quad +\cdot 000,111$$

$$\text{as against} \quad -\cdot 000,463, \quad +\cdot 000,365, \quad +\cdot 000,189, \quad +\cdot 000,362$$

respectively. The greatest improvements are in the mean and standard deviation. Accordingly it is well worth using smaller terminal subranges, if they are available as in the cases of cricket scores, wages, house values, infant mortality and other frequency material.

(8) *Illustration II.* (b) We now propose to consider the moment-coefficients of a doubly truncated normal curve. We will take the portion of the above 1,000,000 distribution with unit standard deviation from variate value 1.25 to variate value 3.75 and divide it into five groups, i.e.

Absolute frequencies	Relative frequencies
65,591	·6213,5637
27,834	·2636,7693
9,245	·0875,7969
2,402	·0227,5462
489	·0046,3239
Total 105,561	Total 1.0000,0000

Using (iii) and (iv) which now involve all five groups we find

$$\begin{aligned}
 a_1 &= -\cdot8794,4917, & b_1 &= -\cdot0038,5527, \\
 a_2 &= \cdot6084,9651, & b_2 &= \cdot0258,2393, \\
 a_3 &= -\cdot2970,9362, & b_3 &= -\cdot0401,0477, \\
 a_4 &= \cdot0817,9157, & b_4 &= \cdot0530,8779, \\
 a_5 &= -\cdot0057,4076, & b_5 &= \cdot0057,4076.
 \end{aligned}$$

From these results, since a 's = a 's and b 's = b 's we have for the abruptness functions :

$$\begin{aligned}
 a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5' &= -\cdot8744,9989, & b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5' &= -\cdot0031,8458, \\
 a_2' - \frac{5}{126}a_4' &= \cdot6052,5081, & b_2' - \frac{5}{126}b_4' &= \cdot0237,1727, \\
 a_1' - \frac{5}{63}a_3' + \frac{1}{240}a_5' &= -\cdot8558,9423, & b_1' - \frac{5}{63}b_3' + \frac{1}{240}b_5' &= -\cdot0006,4843, \\
 a_2' - \frac{7}{80}a_4' &= \cdot6013,3975, & b_2' - \frac{7}{80}b_4' &= \cdot0211,7875
 \end{aligned}$$

About the first terminal we have for the raw moment-coefficients

$$v_1' = 1.025,630, \quad v_2' = 1.668,535, \quad v_3' = 3.733,743, \quad v_4' = 10.108,966,$$

and by (xxii)—(xxv) the corresponding corrective terms are

$$-\cdot0731,4037, \quad -\cdot0074,9994, \quad +\cdot0044,7459, \quad -\cdot0981,1557,$$

leading to the Sheppard's correction moment-coefficients *in actual units* :

$$\mu_1' = \cdot512,815, \quad \mu_2' = \cdot396,300, \quad \mu_3' = \cdot434,667, \quad \mu_4' = \cdot581,492,$$

and the full correction moment-coefficients :

$$\mu_1' = \cdot476,245, \quad \mu_2' = \cdot394,425, \quad \mu_3' = \cdot435,226, \quad \mu_4' = \cdot575,360.$$

We now transfer to the mean of the block and find

$$\mu_1' = \cdot476,245, \quad \mu_2' = \cdot167,616, \quad \mu_3' = \cdot087,7305, \quad \mu_4' = \cdot128,691,$$

while the values for the Sheppard's corrections only would be

$$\mu_1' = \cdot512,815, \quad \mu_2' = \cdot133,321, \quad \mu_3' = \cdot104,701, \quad \mu_4' = \cdot107,714.$$

The theoretical values for the normal curve block are

$$\mu_1' = \cdot476,930, \quad \mu_2' = \cdot168,025, \quad \mu_3' = \cdot089,730, \quad \mu_4' = \cdot133,743.$$

It will be seen that the Sheppard's corrections *alone* give very unsatisfactory results, and that while the full corrections for the first three moments are statistically satisfactory, the approximation of the fourth moment-coefficient would not for certain investigations be adequate. We are in fact using only five groups and trusting to these for the accuracy of our abruptness coefficients. We will accordingly now test what improvement arises when we divide our terminal groups into five subgroups and calculate the abruptness coefficients on these smaller subranges. Thus we have $h = \cdot 5$, $h_0 = h_p = \cdot 1$, and therefore $\rho_0 = \rho_p = 5$. Our subgroups are:

$n_1 = 17142$	therefore $n'_1 = \cdot 1623,8952$,	$n_{p-4} = 173$	therefore $n'_{p-4} = \cdot 0016,3886$,
$n_2 = 14979$	$n'_2 = \cdot 1418,9900$,	$n_{p-3} = 124$	$n'_{p-3} = \cdot 0011,7468$,
$n_3 = 12959$	$n'_3 = \cdot 1227,6314$,	$n_{p-2} = 88$	$n'_{p-2} = \cdot 0008,3364$,
$n_4 = 11099$	$n'_4 = \cdot 1051,4300$,	$n_{p-1} = 61$	$n'_{p-1} = \cdot 0005,7786$,
$n_5 = \frac{9412}{65591}$	$n'_5 = \cdot 0891,6172$,	$n_p = \frac{43}{489}$	$n'_p = \cdot 0004,0735$.

Whence

$a_1 = -\cdot 1730,3848$	and $a'_1 = -\cdot 8651,9242$,	$b_1 = \cdot 0003,5810$	and $b'_1 = \cdot 0017,9049$,
$a_2 = +\cdot 0216,6599$	$a'_2 = +\cdot 5416,4969$,	$b_2 = \cdot 0000,5368$	$b'_2 = \cdot 0013,4204$,
$a_3 = -\cdot 0010,4679$	$a'_3 = -\cdot 1308,4851$,	$b_3 = \cdot 0001,5157$	$b'_3 = \cdot 0189,4639$,
$a_4 = -\cdot 0002,3683$	$a'_4 = -\cdot 1480,1868$,	$b_4 = -\cdot 0000,7579$	$b'_4 = -\cdot 0473,6598$,
$a_5 = +\cdot 0000,3789$	$a'_5 = +\cdot 1184,1494$,	$b_5 = \cdot 0000,3789$	$b'_5 = \cdot 1184,1494$.

Determining the abruptness functions from these values, we have

$$\begin{aligned} a'_1 - \frac{1}{60}a'_3 + \frac{1}{2520}a'_5 &= -\cdot 8629,6462, & b'_1 - \frac{1}{60}b'_3 + \frac{1}{2520}b'_5 &= +\cdot 0015,2171, \\ a'_2 - \frac{5}{126}a'_4 &= \cdot 5475,2345, & b'_2 - \frac{5}{126}b'_4 &= +\cdot 0032,2164, \\ a'_1 - \frac{5}{63}a'_3 + \frac{1}{240}a'_5 &= -\cdot 8543,1522, & b'_1 - \frac{5}{63}b'_3 + \frac{1}{240}b'_5 &= +\cdot 0007,8021, \\ a'_2 - \frac{7}{80}a'_4 &= \cdot 5604,7508, & b'_2 - \frac{7}{80}b'_4 &= +\cdot 0054,8656. \end{aligned}$$

Working out the corrective terms for abruptness we find them

$$-\cdot 071,787, \quad -\cdot 003,368, \quad +\cdot 031,252, \quad +\cdot 071,446$$

in working units, leading to

$$\mu'_1 = \cdot 476,922, \quad \mu'_2 = \cdot 395,458, \quad \mu'_3 = \cdot 438,5735, \quad \mu'_4 = \cdot 585,957,$$

or transferring to the mean we have

$$\mu'_1 = \cdot 476,922, \quad \mu'_2 = \cdot 168,003, \quad \mu'_3 = \cdot 089,722, \quad \mu'_4 = \cdot 133,783,$$

as against the actual values

$$\mu_1 = \cdot 476,930, \quad \mu_2 = \cdot 168,025, \quad \mu_3 = \cdot 089,730, \quad \mu_4 = \cdot 133,743,$$

an eminently satisfactory agreement. It is thus clear that when possible it is desirable to obtain the abruptness corrections by small subranges—in this case $\frac{1}{10}$ of the standard deviation. Hence any terminal small range groupings such as are frequently provided in statistical data are useful from this standpoint. In fact if

the abruptness coefficients are found from such small groupings the remaining sub-ranges can safely be made fairly coarse, as in the above examples, where five divisions of the total range are clearly adequate.

(9) *Illustration III. Mean Age and Variability of Infants at Death.* It is very important in practical statistics to obtain the mean and standard deviation of J-shaped curves. A good illustration of such curves may be found in infantile mortality statistics. These have the advantage that in the early part of the year of infancy the frequencies are in certain cases given by much smaller intervals. Thus in the Prussian official statistics they are given for the first fortnight by *days*. Professor Raymond Pearl in a paper of 1906 (*Biometrika*, Vol. IV, p. 510) has endeavoured to ascertain the mean age at death of infants in the first year of life from the Prussian data. It will be of interest to determine what changes are likely to be made in his results by the use of our present abruptness corrections. He writes (p. 512):

It is evident that the grouping here [i.e. in the Prussian data] is sufficiently fine to make possible a very accurate determination of the mean age of death.... A standard month of 30 days was assumed: then with a unit of 30 days the first and second moment-coefficients about an arbitrary axis were determined. From these the position of the mean and the value of the second moment about it were easily found. Only the "rough" second moment was calculated, as it was deemed sufficiently accurate for present purposes, and furthermore it was difficult to determine the proper corrective terms to apply in this case. In the calculations each frequency element was for practical convenience centred at the midpoint of its range. The error made by so doing is negligible.

With our present corrections we can test how far the errors made by concentration at the midpoints of the subranges are really negligible. It is certainly right to concentrate at those points *provided* we allow for terminal abruptness which is very marked in this case. If we make the proper terminal corrections theory shows that quite considerable subranges, say in this case one month, may be used to determine the raw moments. It will be sufficient to illustrate the method on the Prussian male infant deaths.

We have deaths per 1000 infants born: For the birth terminal we have*:

Months	Deaths	Days	Deaths
0—1	63.99	0—3	18.25
1—2	22.59	3—6	6.58
2—3	18.58	6—9	7.89
3—4	15.96	9—12	5.65
4—5	13.30	12—15	5.82
5—6	11.51		
6—7	10.61		
7—8	9.30		
8—9	8.74		
9—10	8.29		
10—11	7.51		
11—12	6.94		

Total 197.32

$$\left. \begin{aligned} \nu_1' &= 3.759,224 \\ \nu_2' &= 25.809,801 \end{aligned} \right\} \text{ in months.}$$

* Three day intervals taken with a view to smoothing anomalous values.

The month subranges will be quite adequate at the childhood terminal.

As the results are based on 1877—1881 averages, we shall suppose the month to be 30·4375 days. Thus $h/h_0 = 10·145,833$, $h/h_p = 1$. We find

$$\begin{array}{lll} n'_1 = \cdot 0924,8936, & a_1 = - \cdot 1877,2637, & a'_1 = - \quad 1\cdot 904,640, \\ n'_2 = \cdot 0333,4685, & a_2 = \quad \cdot 2966,9657, & a'_2 = \quad 30\cdot 541,331, \\ n'_3 = \cdot 0399,8581, & a_3 = - \cdot 3909,0057, & a'_3 = - \quad 408\cdot 253,057, \\ n'_4 = \cdot 0286,3369, & a_4 = \quad \cdot 3117,2715, & a'_4 = \quad 3303\cdot 128,589, \\ n'_5 = \cdot 0294,9524, & a_5 = - \cdot 1139,7730, & a'_5 = - 12253\cdot 408,881; \\ \\ n'_{p-4} = \cdot 0471,3156, & b'_1 = b_1 = \quad \cdot 035,759, \\ n'_{p-3} = \cdot 0442,9353, & b'_2 = b_2 = - \cdot 004,823, \\ n'_{p-2} = \cdot 0420,1297, & b'_3 = b_3 = \quad \cdot 013,861, \\ n'_{p-1} = \cdot 0380,6000, & b'_4 = b_4 = - \cdot 012,670, \\ n'_p = \cdot 0351,7130, & b'_5 = b_5 = \quad \cdot 004,967. \end{array}$$

From these we deduce

$$\begin{array}{ll} \frac{1}{12}(a'_1 - \frac{1}{60}a'_3 + \frac{1}{2520}a'_5) = \cdot 003,093, & \frac{1}{12}(b'_1 - \frac{1}{60}b'_3 + \frac{1}{2520}b'_5) = \cdot 002,961, \\ \frac{1}{120}(a'_2 - \frac{5}{126}a'_4) = - \cdot 837,793, & \frac{1}{120}(b'_2 - \frac{5}{126}b'_4) = - \cdot 000,036, \end{array}$$

whence

$$\text{Total abruptness correction on } \nu'_1 = \cdot 006,054,$$

$$\text{,, ,, } \nu'_2 = \cdot 843,679.$$

$$\text{Thus } \mu'_1 = 3\cdot 765,278 \text{ months, } \mu'_2 = 26\cdot 570,147 \text{ (months)}^2,$$

using of course Sheppard's correction.

Finally we reach

$$\text{Mean} = 114\cdot 61 \text{ days as against }^* 113\cdot 07 \text{ days,}$$

$$\text{Standard Deviation} = 107\cdot 15 \quad \text{,,} \quad \text{,,} \quad 105\cdot 44 \quad \text{,,}$$

obtained from taking the raw moments of small elements of one day up to the end of the first fortnight. Thus, if we desire to get a mean within 1·5% of the correct value, it will be well to adopt abruptness corrections.

(10) *Illustration IV.* In view of the fact that in the previous illustration the infantile death-rate curve has probably an infinite initial ordinate it seems well to measure, in a case which can be tested, the degree with which our corrections give the actual values of the moment-coefficients in such a case.

$$\text{We choose the curve } y = \frac{1}{2}x^{-\frac{1}{2}},$$

and suppose ten subranges going up to the terminal $x = 10$, from $x = 0$.

* Pearl's results modified by taking the average month to be 30·4375, not 30 days.

We have for the "frequencies":

x	Frequency
0 to 1	1,000,0000
1 to 2	414,2136
2 to 3	317,8372
3 to 4	267,9492
4 to 5	236,0680
5 to 6	213,4217
6 to 7	196,2616
7 to 8	182,6758
8 to 9	171,5729
9 to 10	162,2777
Total	3,162,2777

Further, for small terminal subranges, we have:

x	Frequency
0 to .2	447,2136
.2 to .4	185,2419
.4 to .6	142,1412
.6 to .8	119,8305
.8 to 1.0	105,5728

It will be sufficient to take the subranges unity at the other terminal, or $h/h_0 = 5$ $h/h_p = 1$. Thus we have

$$\begin{aligned} \nu_1' &= 3,394,907, & \nu_2' &= 20,016,109, \\ n_1' &= 1414,2136, & a_1 &= -2332,9561, & a_1' &= -1166,4781, \\ n_2' &= 0585,7863, & a_2 &= +2583,1707, & a_2' &= +6457,9263, \\ n_3' &= 0449,4899, & a_3 &= -2657,4026, & a_3' &= -33,217,5325, \\ n_4' &= 0378,9373, & a_4 &= +1798,6052, & a_4' &= +112,412,8250, \\ n_5' &= 0333,8505, & a_5 &= -0586,1091, & a_5' &= -183,159,0938. \end{aligned}$$

		Actual values
$n'_{p-4} = 0674,8987,$	$b_1' = b_1 = +050,0105$	$+0500,0000,$
$n'_{p-3} = 0620,6337,$	$b_2' = b_2 = +002,4535$	$+0025,0000,$
$n'_{p-2} = 0577,6716,$	$b_3' = b_3 = +000,4817$	$+0003,7500,$
$n'_{p-1} = 0542,5612,$	$b_4' = b_4 = -000,0497$	$+0000,9375,$
$n'_p = 0513,1672,$	$b_5' = b_5 = +000,1316$	$+0000,3281.$

These values of a 's and b 's lead to the abruptness functions:

$$\begin{aligned} \frac{1}{12}(a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5') &= -057,1279, & -\frac{1}{120}(a_2' - \frac{5}{126}a_4') &= -016,6425, \\ \frac{1}{12}(b_1' - \frac{1}{60}b_3' + \frac{1}{2520}b_5') &= 004,1669, & \frac{1}{120}(b_2' - \frac{5}{126}b_4') &= 000,0205. \end{aligned}$$

Accordingly we find

$$\mu_1' = 3,394,9066 - 052,9616 = 3,341,9450$$

and $\mu_2' = 20,016,1090 - 083,3333 + 066,7155 = 19,999,4912,$

which gives

$$\mu_2 = 8,830,8953.$$

For comparison we have

	Raw moments	Using only Sheppard's corrections	Full corrections	True values
μ_1'	3,3949	3,3949	3,3419	3,3333
μ_2'	20,0161	19,9328	19,9995	20,0000
μ_2	8,4907	8,4074	8,8309	8,8889
σ	2,9139	2,8995	2,9718	2,9814

It will be seen that Sheppard's corrections *alone* are worse than the raw moment results. In other words they should certainly not be used alone for J-shaped curves; it would be better to take the raw moment results without any corrections. On the other hand the full corrections even in this extreme case—where (a) the Euler-

Maclaurin Theorem fails theoretically, (b) our auxiliary curve is unreasonable, for $x^{-\frac{1}{2}}$ cannot be expanded at the origin terminal in powers of x —are found to give results within $\frac{1}{3}\%$ of the true values for both mean and standard deviation.

The variety of illustrations we have taken seems to suggest that for most practical statistical problems—even with J- or U-shaped distributions—we shall obtain reasonable results from the system developed in the first part of this paper. At the same time the method adopted indicates that for the best possible results in asymptotic frequency curves it may be needful to use a more suitable auxiliary curve for the asymptotic terminal. This leads us directly to the second part of our paper.

Part II. Cases of Asymptotic Frequency.

(11) In selecting our auxiliary curve to give the first five frequencies we must remember that it has (i) to give an infinite ordinate but a finite frequency, (ii) it must be of such a character that its constants can be readily determined.

If we adopt

$$Z = N(1 + x^q(A + Bx + Cx^2 + Dx^3 + Ex^4)),$$

where q is chosen less than unity, we have the adequate number of constants and

$$y = -dZ/dx \text{ is infinite when } x = 0.$$

If we leave q undetermined, however, we should have six not five constants and might then omit E . But the process of determining A, B, C, D and q would be very laborious and involve a troublesome series of approximations. We are accordingly thrown back on the retention of E and an arbitrary choice of q . Clearly to give an infinite ordinate and finite area we may give q any value from slightly over zero to slightly under unity, and the size of q measures so to speak the intensity of the asymptoting. This is probably rather an important feature of the frequency curve, but as we see no way of determining it accurately without very great labour, we give q its mean value $\frac{1}{2}$. Accordingly our problem becomes that of determining A, B, C, D and E so as to give the first five frequencies or the values of $Nn_1', Nn_2', Nn_3', Nn_4', Nn_5'$ as before. After a good deal of work they are found to be

$$(xxviii) \left\{ \begin{array}{l} A = -1.64964,84755n_1' + 3.35035,15245n_2' - 3.72071,62874n_3' \\ \quad + 2.05278,64045n_4' - .44721,35955n_5', \\ B = + .91328,76419n_1' - 5.50337,90247n_2' + 7.10669,19065n_3' \\ \quad - 4.15163,83427n_4' + .93169,49906n_5', \\ C = - .31317,72759n_1' + 2.64515,60574n_2' - 4.30806,06243n_3' \\ \quad + 2.76448,01733n_4' - .65218,64934n_5', \\ D = + .05299,17797n_1' - .53034,15536n_2' + 1.00172,31390n_3' \\ \quad - .73032,76686n_4' + .18633,89981n_5', \\ E = - .00345,36703n_1' + .03821,29964n_2' - .07963,81338n_3' \\ \quad + .06469,94335n_4' - .01863,38998n_5'. \end{array} \right.$$

The large number of decimals is requisite owing to the high coefficients they have to be multiplied by in ascertaining the values of the abruptness coefficients.

Now our scheme of action is of the following kind: we shall obtain the abruptness coefficients at $x = 1$, or at the finite ordinate of the first trapezette, for here they will be finite. We shall then trust to our auxiliary curve to give the moments of this trapezette about $x = 1$, using the integral

$$n_1\mu_s'' = \int_0^1 (x-1)^s y dx = - \int_0^1 (x-1)^s \frac{dZ}{dx} dx.$$

And lastly we shall determine the moments and the corrections of the remainder of the curve by the process already discussed as if it had to be applied from $x = 1$ onwards*. The moments for the trapezette before $x = 1$, and for the remainder of the curve, must then be added together to get the total moments and so the moment-coefficients about $x = 1$. The transference to the centroid then proceeds in the usual manner.

Moments of first trapezette n_1 about non-infinite ordinate:

$$(xxix) \quad \begin{cases} n_1\mu_1'' = 2N(\frac{1}{3}A + \frac{1}{5}B + \frac{1}{7}C + \frac{1}{9}D + \frac{1}{11}E), \\ n_1\mu_2'' = -8N(\frac{1}{15}A + \frac{1}{35}B + \frac{1}{63}C + \frac{1}{99}D + \frac{1}{143}E), \\ n_1\mu_3'' = 16N(\frac{1}{35}A + \frac{1}{105}B + \frac{1}{231}C + \frac{1}{429}D + \frac{1}{715}E), \\ n_1\mu_4'' = -128N(\frac{1}{315}A + \frac{1}{1155}B + \frac{1}{3003}C + \frac{1}{6435}D + \frac{1}{12155}E). \end{cases}$$

Again remembering that

$$a_s = \frac{1}{N} \left(\frac{d^s Z}{dx^s} \right)_{x=1},$$

we find

$$(xxx) \quad \begin{cases} a_1 = \frac{1}{2}(A + 3B + 5C + 7D + 9E), \\ a_2 = \frac{1}{4}(-A + 3B + 15C + 35D + 63E), \\ a_3 = \frac{3}{8}(A - B + 5C + 35D + 105E), \\ a_4 = \frac{3}{16}(-5A + 3B - 5C + 35D + 315E), \\ a_5 = \frac{3}{32}(35A - 15B + 15C - 35D + 315E). \end{cases}$$

If we now substitute (xxviii) in (xxix) and (xxx) we shall obtain the moments of the first trapezette and the abruptness coefficients at $x = 1$ in terms of the first five sub-frequencies. We have

$$(xxxi) \quad \begin{cases} n_1\mu_1'' = -812,7818n_1 + 677,0691n_2 - 660,5497n_3 + 347,1889n_4 - 073,7827n_5, \\ n_1\mu_2'' = 706,7407n_1 - 824,1137n_2 + 830,5586n_3 - 441,5218n_4 + 094,3572n_5, \\ n_1\mu_3'' = -634,7502n_1 + 857,2689n_2 - 880,7900n_3 + 471,4747n_4 - 101,1083n_5, \\ n_1\mu_4'' = 581,4517n_1 - 854,1149n_2 + 888,8688n_3 - 478,0407n_4 + 102,7607n_5, \end{cases}$$

* The abruptness coefficients in the previous case were determined from the five frequencies following the initial ordinate; here they are found from the four frequencies following and the one preceding it.

and again

$$(xxxii) \left\{ \begin{array}{l} a_1' = - \cdot 067,9063n_1' - 1\cdot651,2396n_2' + 1\cdot177,1875n_3' \\ \quad - \cdot 554,8634n_4' + \cdot 111,8034n_5', \\ a_2' = \cdot 332,2458n_1' + \cdot 915,5792n_2' - 2\cdot384,2525n_3' \\ \quad + 1\cdot368,5243n_4' - \cdot 298,1424n_5', \\ a_3' = - \cdot 988,7796n_1' + 2\cdot823,7204n_2' - 2\cdot126,0271n_3' \\ \quad + \cdot 472,0491n_4' - \cdot 027,9508n_5', \\ a_4' = 2\cdot497,6496n_1' - 9\cdot939,8504n_2' + 13\cdot394,6734n_3' \\ \quad - 7\cdot822,9490n_4' + 1\cdot677,0510n_5', \\ a_5' = - 7\cdot413,4958n_1' + 25\cdot320,8792n_2' - 33\cdot899,3138n_3' \\ \quad + 20\cdot768,5399n_4' - 4\cdot856,4601n_5'. \end{array} \right.$$

Here as before $n_s' = n_s/N$.

We have accordingly to add the values given by (xxxi) to the expressions for the moments for the remainder of the frequency corrected for the abruptness by means of the series (xxxii). We propose to illustrate our results on one or two numerical examples.

(12) *Illustration V.* The following data provide the years of survival for 10,000 persons, male and female, born in England and Wales with congenital malformations*.

Age at death	Male	Female
Years 0—1	8762	8753
1—2	393	339
2—3	140	150
3—4	95	80
4—5	86	69
5—10	185	184
10—15	90	132
15—20	86	86
20—25	63	52
25—30	45	40
30—35	9	40
35—40	18	17
40—45	9	6
45—50	9	23
50—55	5	11
55—60	—	6
60—65	5	—
65—70	—	6
70—75	—	6
Totals	10,000	10,000

Now consider how we should endeavour to find the mean and standard deviation of such series under the old method. We clearly cannot use Sheppard's corrections. If we concentrate the deaths in the first year of life at 0·5, we shall certainly get too high a mean. Now Pearl has shown by taking Prussian statistics (*Biometrika*,

* Registrar-General's *Annual Report*, p. 207, 1913.

Vol. IV, p. 515) that as deduced from data registered at short intervals of days, the mean of the *total population* of infants dying in the first year of life should be concentrated at 0.3 instead of 0.5 year of life. But our infants with congenital malformations undoubtedly die earlier than the great bulk of normal infants. We might therefore hazard a concentration at 0.2; but this would be mere guesswork*, and what is more would not provide the proper corrections for concentrating in the case of other years of life. We obtain, however, by this process the following results:

First Year concentrated at :	Male		Female	
	0.2	0.3	0.2	0.3
Mean	1.2436	1.3313	1.5077	1.5952
Standard Deviation	4.5932	4.5734	5.7750	5.7532

The differences between the 0.2 and the 0.3 results are considerable and it will be found from the sequel that the 0.2 results are closest to the corrected results for both mean and standard deviation in the case of the male and the female. Indeed a quite reasonable result might have been reached by centring the deaths in the first year of life at 0.2. But such *a priori* guesses must be at best risky. When we proceed to apply our method by cutting off the first year of life, we note at once that in this case, as in many other of a like J-distribution character, a grave difficulty arises, namely we have starting from the group 1—2 not got the groupings in year or five year ranges, for we have cut off the first of our five year groups. We cannot therefore straight away apply our formulae based on the Euler-Maclaurin theory for equal subranges. The suggestion that at once occurs is to take year groupings for our material. This of course would make no change in the first raw moment ν'_1 , which would be the same whether we grouped into year or five year subranges on the supposition that we simply split up our frequencies into five *equal* groups for the five year periods. But there will be a change for the second and higher moments. For the second moment the total frequency of the five year group (nx) centred at x has to be multiplied by $x^2 + 2h^2$, where $h = \frac{1}{5}$ of the subrange = one year, and similar corrections can be easily obtained for the higher moments. Of course this distribution of each five year frequency into five *equal* one year frequency groups is not satisfactory, but with the irregular data as given it is, perhaps, as good a result as we can hope to get, until official statisticians recognise the difficulty and table their statistics in a manner to meet it, i.e. in this case, it would mean either proceeding by four year groups after the 4—5, or giving the 5—6 frequency and then proceeding by five year groups 6—11, 11—16, etc.

* Actually our auxiliary curve gives 0.210 for males and 0.205 for females for means of deaths in the first year of life.

Assuming the legitimacy for the present purposes of this redistribution in year groups we find for moments round the end of the first year of life :

Males	Females
$1238\nu_1''' = 9446$	$1247\nu_1''' = 12079\cdot5$
$1238\nu_2''' = 207,007\cdot25$	$1247\nu_2''' = 331545\cdot75$

Again we have for males :

$n_1 = 8762$, and by (xxxii) $a_1' \times 1238 = -$	$1122\cdot222,8440$,
$n_2 = 393$,	$a_2' \times 1238 = \quad 3041\cdot534,5373$,
$n_3 = 140$,	$a_3' \times 1238 = - \quad 9518\cdot206,0741$,
$n_4 = 95$,	$a_4' \times 1238 = \quad 19254\cdot345,0950$,
$n_5 = 86$,	$a_5' \times 1238 = - 58196\cdot492,8841$.

Our abruptness functions are thus found to be

$$1238 \frac{1}{12} (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') = - 83\cdot056,6602,$$

$$1238 \frac{1}{120} (a_2' - \frac{5}{126} a_4') = \quad 18\cdot978,9435.$$

These provide for the moments about 1 :

$$1238\mu_1''' = 9446 - 83\cdot056,6602 = 9362\cdot943,3398,$$

$$1238\mu_2''' = 207,007\cdot25 - 103\cdot166,6667 - 18\cdot978,9435 = 206885\cdot104,3898.$$

We now find from (xxxi) the values of

$$8762\mu_1'' = - 6921\cdot345,3000,$$

and

$$8762\mu_2'' = \quad 5951\cdot033,6815.$$

Thus : $10,000\mu_1' = 8762\mu_1'' + 1238\mu_1''' = \quad 2441\cdot598,0398,$

$$10,000\mu_2' = 8762\mu_2'' + 1238\mu_2''' = 212836\cdot138,0713,$$

or

$$\mu_1' = 244,160, \quad \mu_2' = 21\cdot283,614.$$

Thus finally the Mean = 1·2442 years and the Standard Deviation = 4·6069 years.

We now turn to the female deaths and find with the same notation :

$$1247\nu_1''' = 12079\cdot5, \quad 1247\nu_2''' = 331,545\cdot75.$$

Here $n_1 = 8753$, leading by (v) to $1247a_1' = - \quad 1014\cdot250,5807$,

$$n_2 = 339, \quad 1247a_2' = \quad 2949\cdot801,0796,$$

$$n_3 = 150, \quad 1247a_3' = - \quad 7980\cdot615,3654,$$

$$n_4 = 80, \quad 1247a_4' = \quad 19991\cdot399,2722,$$

$$n_5 = 69, \quad 1247a_5' = - 60065\cdot060,3135.$$

Hence we deduce :

$$1247 \times \frac{1}{12} (a_1' - \frac{1}{60} a_3' + \frac{1}{2520} a_5') = -75.422,972,$$

$$1247 \times \frac{1}{120} (a_2' - \frac{5}{126} a_4') = 17.970,765,$$

and

$$1247 \mu_1''' = 12004.077,028,$$

$$1247 \mu_2''' = 331545.75 - 103.916,667 - 17.970,765, \\ = 331423.862,568.$$

Again by (xxxi):

$$8753 \mu_1'' = -6961.151,020, \quad 8753 \mu_2'' = 6002.499,428.$$

Thus :

$$\mu_1' = (8753 \mu_1'' + 1247 \mu_1''')/10,000 = .504,293,$$

$$\mu_2' = (8753 \mu_2'' + 1247 \mu_2''')/10,000 = 33.742,636.$$

Accordingly we have for females :

$$\text{Mean} = 1.5043 \text{ years,}$$

$$\text{Standard Deviation} = 5.7869 \text{ years.}$$

These are both in fairly good accord with the result that would have been obtained by the *a priori* guess of 0.2 for centring the first sub-frequency.

(13) *Illustration VI.* It is not without interest to inquire if this centring of 0.2 maintains itself when we turn to other material for congenital malformations. We can use the material provided in the United States Census for 1899—1900, Vol. IV, p. 670. From the data there given we deduce that for 10,000 congenitally malformed individuals of either sex born :

Died in year of life	Males	Females
0—1	9626	9543
1—2	129	204
2—3	61	57
3—4	27	49
4—5	14	25
5—10	54	41
10—15	34	41
15—20	20	8
20—25	14	8
25—30	—	8
30—35	7	—
35—40	—	—
40—45	—	—
45—50	—	—
50—55	—	8
55—60	14	—
60—65	—	—
65—70	—	8
Totals	10,000	10,000

We have as before : ♂'s, $374 \nu_1''' = 2657$, ♀'s, $457 \nu_1''' = 2595.5$,
 $374 \nu_2''' = 71127.5$, $457 \nu_2''' = 76280.25$.

We now turn to the abruptness coefficients at the end of the first year of life and find:

Males

$$n_1=9626, \text{ and by (xxxii)} \quad 374a_1' = -808\cdot283,5789$$

$$n_2=129 \quad 374a_2' = 3203\cdot644,5476$$

$$n_3=61 \quad 374a_3' = -9271\cdot066,1366$$

$$n_4=27 \quad 374a_4' = 23389\cdot468,5164$$

$$n_5=14 \quad 374a_5' = -69671\cdot015,1599$$

$$374 \times \frac{1}{12} (a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5') = -56\cdot784,418,$$

$$374 \times \frac{1}{120} (a_2' - \frac{5}{126}a_4') = 18\cdot962,592.$$

Thus: $374\mu_1''' = 2600\cdot215,572,$
 $374\mu_2''' = 71077\cdot370,741.$

From (xxxi) we have:
 $9626\mu_1'' = -7768\cdot448,082,$
 or, $1 - \mu_1'' = \cdot1930,$
 $9626\mu_2'' = 6736\cdot839,298.$

Thus: $10,000\mu_1' = -5168\cdot232,510,$
 and $\mu_1' = -\cdot516,823;$
 $10,000\mu_2' = 77814\cdot210,039,$
 $\mu_2' = 7\cdot781,421;$

or, finally
 Mean = $\cdot4832$ year,
 Standard Deviation = $2\cdot7412$ years.

Females

$$n_1=9543, \text{ and by (xxxii)} \quad 457a_1' = -942\cdot176,2334$$

$$n_2=204 \quad 457a_2' = 3281\cdot101,5644$$

$$n_3=57 \quad 457a_3' = -8958\cdot636,6700$$

$$n_4=49 \quad 457a_4' = 22229\cdot438,8090$$

$$n_5=25 \quad 457a_5' = -66617\cdot544,9966$$

$$457 \times \frac{1}{12} (a_1' - \frac{1}{60}a_3' + \frac{1}{2520}a_5') = -68\cdot275,096,$$

$$457 \times \frac{1}{120} (a_2' - \frac{5}{126}a_4') = 19\cdot991,508.$$

Thus: $457\mu_1''' = 2527\cdot224,904,$
 $457\mu_2''' = 76222\cdot175,159.$

From (xxxi) we have:
 $9543\mu_1'' = -7640\cdot738,2653,$
 or, $1 - \mu_1'' = \cdot1993,$
 $9543\mu_2'' = 6604\cdot373,5073.$

Thus: $10,000\mu_1' = -5113\cdot513,361,$
 and $\mu_1' = -\cdot511,3513;$
 $10,000\mu_2' = 82826\cdot548,666,$
 $\mu_2' = 8\cdot282,655;$

or, finally
 Mean = $\cdot4886$ year,
 Standard Deviation = $2\cdot8322$ years.

It is clear that in both cases the centring of those who die in the first year of life is a little under 0·2, instead of slightly over 0·2 as in the English data. It is worth while inquiring what the effect of concentrating the deaths in the first year of life at 0·2 and then simply determining the crude moments will be. We find:

	Concentration at 0·2		Concentration at actual centres*		Complete corrections	
	Male	Female	Male	Female	Male	Female
Mean	·496	·496	·489	·495	·483	·489
Standard Deviation	2·729	2·821	2·730	2·822	2·741	2·832

This process then gives quite a reasonable value for the mean and standard deviation. Thus all we have to do for a rough practical value is to use the μ_1'' of the first equation of (xxxi) to obtain the centring of the first group and then find the *raw* moments only. For a very high first group this is considerably better than applying our first non-asymptotic method and of course better than mere raw moments. The following are values found *from year groups*:

* That is at ·1930 and ·1993.

	1st Method of this paper		Raw moments	
	Male	Female	Male	Female
Mean	·609	·610	·592	·592
Standard Deviation	2·73	2·82	2·72	2·81

The means are inadequate, but it is remarkable how close the standard deviations are to the corrected values.

(14) The reader may occasionally be puzzled to settle whether a frequency distribution has really a finite or infinite initial ordinate and therefore be in doubt as to whether he should apply the first or second method of this paper. Our Illustration III may be taken as a possible example of this, although the exaggeration of the first frequency is nothing like so marked as in the case of congenital malformations.

If we apply the first equation of (xxx) to the first three days' period we find :

$$\begin{array}{l}
 0-3 \text{ days } n_1 = 18\cdot25 \\
 3-6 \text{ " } n_2 = 6\cdot58 \\
 6-9 \text{ " } n_3 = 7\cdot89 \\
 9-12 \text{ " } n_4 = 5\cdot65 \\
 12-15 \text{ " } n_5 = 5\cdot82
 \end{array}
 \left\{ \begin{array}{l}
 \text{whence } 18\cdot25\mu_1''' = -14\cdot057,688 \\
 \text{or } \mu_1'' = -\cdot77, \\
 \text{or remembering our three days' unit,} \\
 \text{Mean} = \cdot69 \text{ day.}
 \end{array} \right.$$

Our table now becomes :

0-3 days	18·25	centred at	·69 days
3-6 "	6·58		4·5 "
6-9 "	7·89		7·5 "
9-12 "	5·65		10·5 "
12-15 "	5·82		13·5 "
15-1 months	19·80		·75 months
1-2 "	22·59		1·5 "
2-3 "	18·58		2·5 "
3-4 "	15·96		3·5 "
4-5 "	13·30		4·5 "
5-6 "	11·51		5·5 "
6-7 "	10·61		6·5 "
7-8 "	9·30		7·5 "
8-9 "	8·74		8·5 "
9-10 "	8·29		9·5 "
10-11 "	7·51		10·5 "
11-12 "	6·94		11·5 "
Total		197·32	

Hence by raw moments we find :

Mean = 112·98 days as against 114·61 days,

Standard Deviation = 105·53 " " 107·15 "

found by the first method of this paper.

Here we have not used our full second method but the results are in fairly close accord, especially in view of the fact that we have not corrected for the curtailment abruptly at the end of the 12 months. Accordingly the suggestion made is that in *doubtful* cases both methods will give fairly closely the same values, and therefore we need not worry over which is the more correct one to apply.

PECCAVIMUS !

This paper is devoted to a number of slips recently made by the Biometric School and which it is desirable to correct at once, before the formulae which need correction pass into general use. Some of these slips are due to war haste, others to neglect of terms which ought to have been included in our approximations and some to printers' errors. We have to thank Professor Tchouproff of Petrograd for indicating the existence of several of these mistakes.

(I) *Biometrika*, Vol. XI, p. 215. *On the Probable Error of a Coefficient of Contingency without Approximation.* By Andrew W. Young and Karl Pearson.

Down to p. 222, equation (xii), this has been again checked without discovery of any error. But on that page the authors "take M to be very large compared with N and make $\chi_1 = \chi_2 = \chi_3 = \chi_4 = 1$ " by an oversight. The values of the χ 's are given on p. 217, equation (vi), and clearly when M is very large compared with N , $\chi_1 = \chi_2 = \chi_4 = 1$ and $\chi_3 = 1 - 2/N$. Accordingly equations (xiii) and (xiv) of p. 222 for samples from "an infinite population" require modification and should be* :

$$\begin{aligned}\sigma_{\phi^2}^2 = & \frac{4}{N} \left[S \left(\frac{\bar{n}_s^3}{N\lambda_s^2} \right) - \left\{ S \left(\frac{\bar{n}_s^2}{N\lambda_s} \right) \right\}^2 \right] \\ & + \frac{1}{N^2} \left[6S \left(\frac{\bar{n}_s^2}{\lambda_s^2} \right) - 4S \left(\frac{\bar{n}_s^2}{N\lambda_s} \right) S \left(\frac{\bar{n}_s}{\lambda_s} \right) + 10S \left(\frac{\bar{n}_s^2}{N\lambda_s} \right) - 12S \left(\frac{\bar{n}_s^3}{N\lambda_s^2} \right) \right] \\ & + \frac{1}{N^3} \left[S \left(\frac{N\bar{n}_s}{\lambda_s^2} \right) - \left\{ S \left(\frac{\bar{n}_s}{\lambda_s} \right) \right\}^2 - 6S \left(\frac{\bar{n}_s^2}{\lambda_s^2} \right) + 4S \left(\frac{\bar{n}_s^2}{N\lambda_s} \right) S \left(\frac{\bar{n}_s}{\lambda_s} \right) \right. \\ & \quad \left. + 8S \left(\frac{\bar{n}_s^3}{N\lambda_s^2} \right) - 6 \left\{ S \left(\frac{\bar{n}_s^2}{N\lambda_s} \right) \right\}^2 \right] \dots\dots (xiii),\end{aligned}$$

$$\begin{aligned}\sigma_{\phi^2}^2 = & \frac{4}{N} \left[S \left(\frac{\Psi_s^3}{N\lambda_s^2} \right) + \phi^2 - \phi^4 \right] \\ & + \frac{1}{N^2} \left[6S \left(\frac{\Psi_s^2}{\lambda_s^2} \right) + (8 - 4\phi^2) S \left(\frac{\Psi_s}{\lambda_s} \right) - 12S \left(\frac{\Psi_s^3}{N\lambda_s^2} \right) \right. \\ & \quad \left. - (2 - 4\phi^2)c - 16\phi^2 + 10\phi^4 - 2 \right] \\ & + \frac{1}{N^3} \left[S \left(\frac{N\Psi_s}{\lambda_s^2} \right) + S \left(\frac{N}{\lambda_s} \right) - \left\{ S \left(\frac{\Psi_s}{\lambda_s} \right) \right\}^2 - 6S \left(\frac{\Psi_s^2}{\lambda_s^2} \right) + 8S \left(\frac{\Psi_s^3}{N\lambda_s^2} \right) \right. \\ & \quad \left. - S \left(\frac{\Psi_s}{\lambda_s} \right) (2c - 4\phi^2 + 8) - 6\phi^4 + 12\phi^2 + 4c\phi^2 - c^2 - 2c + 2 \right] \\ & \dots\dots (xiv).\end{aligned}$$

* The changes due to χ_3 affect the term in $1/N^3$, but the original (xiii) has a wrong sign to the third term in $1/N^2$.

We may now turn to the numerical illustrations. It will be sufficient to show the correct values of σ_{ϕ^2} in the table on p. 224.

	First and second terms of (xiv)	All terms of (xiv)
Old Values	·02709	·02729
Corrected Values	·02725	·02744

For practical purposes, these would all be taken as ·027, and accordingly the errors, although sufficiently distressing, do not modify the conclusions, that for a sample over 1000 the first and second terms of (xiv) are adequate. In the second example, p. 227, more serious changes are made, chiefly owing to the error in the sign of the second-order term $(2 - 4\phi^2)c$, which becomes of greater importance now that N is reduced from 1801 in the first illustration to the 218 of the second illustration. We have for σ_{ϕ^2} :

	First and second terms of (xiv)	All terms of (xiv)
Old Values	·0798	·0823
Corrected Values	·0693	·0719

Thus for practical purposes the ·069 of the first and second-order terms is only raised to ·072, if we include the third-order term. We may therefore conclude that 250 cases marks something like the limit at which we need to consider the third-order term as well as the first- and the second-order terms.

We now turn to the test for zero-contingency. Equation (xvii) of the original paper is correct, but the wrong value of χ_3 was inserted to obtain (xviii); it should of course be $1 - 2/N$. This leads to

$$\sigma_{\phi^2}^2 = \frac{1}{N^2} \left\{ \frac{c}{H} + \frac{c(c-2) - 2(c^2-1)}{N} + 2(c-1) \right\} \dots\dots\dots(\text{xviii}),$$

or perhaps as it is better expressed:

$$\sigma_{\phi^2}^2 = \frac{1}{N^2} \left\{ S \left(\frac{1}{n_s} \right) + 2 \left(1 - \frac{1}{N} \right) (c-1) - \frac{c^2}{N} \right\} \dots\dots\dots(\text{xviii}) \text{ bis.}$$

The formulae summarised on p. 229 must be altered to accord with the results given above. (C) must be (xiv) of the present paper. (D) must have $-2c$ and not $+2c$ for its last term. (C') must be (xviii) above.

(II) The object of our next note is to make some additions and corrections provided by Dr Isserlis himself to his paper: "On the Conditions under which the 'Probable Errors' of Frequency Distributions have a real significance" (*R. S. Proc.* Vol. 92, A,

pp. 23—41, 1915). In that paper he gave the values of the frequency constants B'_1 and B'_2 (formulae (19) and (23), pp. 30 and 31) of the distribution of the moment-coefficient of any order u about a fixed origin for a sample of size n drawn from a population of size N . These formulae are exact and no alterations are proposed here in them nor in any conclusions drawn from them. In the latter part of the paper Dr Isserlis deals with the value of the β -constants for moment-coefficients referred to the mean of the sample. These latter values were approximate and intended to be correct to terms in $\frac{1}{n}$. We are indebted to Professor Tchouproff for pointing out that there is an error in the approximation, for one of the neglected terms rises. When the correction is made, however, the statement (p. 24) remains true that "for coefficients of high order the sample has to be an inconveniently large fraction of the population itself if β_1 and β_2 are to approach even approximately their Gaussian values" (i.e. 0 and 3). The results in the paper cited are exact and correct* until section 5 (p. 35) is reached. In that section, formulae (38), (39) and (41) are approximations and for the purposes of the paper should be given correct to terms in $\frac{1}{n}$ for (38) and to terms in $\frac{1}{n^2}$ for (41). The use of the incomplete value

$$d\mu_u = \frac{1}{n} S (dn_s X_s^u) - u \mu_{u-1} d\bar{x}$$

in equation (37) has introduced an error in the value of M_3 given by equation (39). We proceed to amend this error.

$$\begin{aligned} \text{We have } \mu_u &= \frac{1}{n} S \{n_s (x_s - \bar{x})^u\} = \frac{1}{n} S (n_s X_s^u), \quad dX_s = -d\bar{x}; \\ \therefore d\mu_u &= \frac{1}{n} S \{dn_s X_s^u - un_s X_s^{u-1} d\bar{x}\} \\ &\quad + \frac{1}{n} S \left\{ -u dn_s d\bar{x} X_s^{u-1} + \frac{u(u-1)}{2} n_s X_s^{u-2} d\bar{x}^2 \right\} + \dots \\ &= A + B + \text{terms of third and higher orders in } dn_s, d\bar{x}. \end{aligned}$$

Now it is well known that the mean value of fifth and higher powers of dn_s , $d\bar{x}$, ... contains no terms of lower degree than the third in $1/n$.

In the formulae (38), (39) and (41) the values of M_2 , M_3 , M_4 were obtained as the mean values of A^2 , A^3 and A^4 respectively. The inclusion of the neglected terms does not affect M_2 which is given correct to $\frac{1}{n}$ nor M_4 , for the only term of the fourth order in dn_s , $d\bar{x}$ in $(A + B + \dots)^4$ is A^4 .

But $(d\mu_u)^3 = A^3 + 3A^2B + \text{fifth-order terms in } dn_s, d\bar{x}$.

* There are some obvious printers' errors overlooked in proof, of which the omission of the factor $(\mu'_u)^4 - 2\mu'_{2u}(\mu'_u)^2 + (\mu'_{2u})^2$ in the first line of equation (21) is most likely to mislead. It may also be noted that the factor μ_2^2 is missing in the first term of (26) and the factor 3 in the first term of (41).

Hence the correction to be applied to the value of M_3 in formula (39) is the mean value of $3A^2B$.

Now
$$A = \frac{1}{n} S (dn_s X_s^u) - u \mu_{u-1} d\bar{x}$$

and
$$B = \frac{1}{n} S (-u X_s^{u-1} dn_s d\bar{x}) + \frac{u(u-1)}{2} \mu_{u-2} d\bar{x}^2,$$

so that

$$A^2 = \frac{1}{n^2} S (dn_s^2 X_s^{2u} + 2dn_s dn_t X_s^u X_t^u) - \frac{2u\mu_{u-1}}{n} S (dn_s d\bar{x} X_s^u) + u^2 \mu_{u-1}^2 d\bar{x}^2.$$

Let us write $A^2 = L + M + N$ and $B = H + K$.

$$HL = -\frac{u}{n^3} \left[S (X_s^{3u-1} dn_s^3 d\bar{x}) + S (X_s^{2u} X_t^{u-1} dn_s^2 dn_t d\bar{x}) \right. \\ \left. + 2S (X_s^{2u-1} X_t^u dn_s^2 dn_t d\bar{x}) + 2S (dn_s dn_t dn_p d\bar{x} X_s^{u-1} X_t^u X_p^u) \right].$$

Denoting the mean value of HL by $\overline{HL}$ we have (see (31), (32), (33) of paper cited)

$$\begin{aligned} \overline{HL} &= -\frac{u}{n^3} \left[S \left\{ X_s^{3u-1} \frac{X}{n} \left[3\phi n_s \left(1 - \frac{n_s}{n} \right) + \phi' \right] n_s X_s \right\} \right. \\ &\quad \left. + S \left\{ (X_t^{u-1} X_s^{2u} + 2X_t^u X_s^{2u-1}) \chi \frac{n_s n_t}{n} \left[\left(\phi \left(1 - \frac{n_s}{n} \right) + \frac{\phi'}{n} \right) X_t - \left(2 \frac{n_s}{n} \phi + \frac{\phi'}{n} \right) X_s \right] \right\} \right. \\ &\quad \left. - 2S \left\{ \chi \phi \frac{n_s n_t n_p}{n^2} (X_s + X_t + X_p) (X_s^{u-1} X_t^u X_p^u) \right\} \right] \\ &= -\frac{u\chi\phi}{n^3} \left[3S \left\{ \left(\frac{n_s^2}{n} - \frac{n_s^3}{n^2} \right) X_s^{3u} \right\} + S \left\{ \left(\frac{n_s n_t}{n} - \frac{n_s^2 n_t}{n^2} \right) (X_s^{2u} X_t^u + 2X_s^{2u-1} X_t^{u+1}) \right\} \right. \\ &\quad \left. - 2S \left\{ \left(\frac{n_s^2 n_t}{n^2} \right) (X_s^{2u+1} X_t^{u-1} + 2X_s^{2u} X_t^u) \right\} \right. \\ &\quad \left. - 2S \left\{ \frac{n_s n_t n_p}{n^2} (X_s + X_t + X_p) X_s^{u-1} X_t^u X_p^u \right\} \right] \\ &\quad - \frac{u\chi\phi'}{n^3} \left[S \left\{ \frac{n_s}{n} X_s^{3u} \right\} \right. \\ &\quad \left. + S \left\{ \frac{n_s n_t}{n^2} (X_s^{2u} X_t^u + 2X_s^{2u-1} X_t^{u+1} - X_s^{2u+1} X_t^{u-1} - 2X_s^{2u} X_t^u) \right\} \right] \\ &= -\frac{u\chi\phi}{n^3} \left[-3S \left\{ \frac{n_s^3 X_s^{3u}}{n^2} \right\} - S \left\{ \frac{n_s^2 n_t}{n^2} (X_s^{2u} X_t^u + 2X_s^{2u-1} X_t^{u+1}) \right\} \right. \\ &\quad \left. - S \left\{ \frac{n_s^2 n_t}{n^2} (2X_s^{2u+1} X_t^{u-1} + 4X_s^{2u} X_t^u) \right\} \right. \\ &\quad \left. - 2S \left\{ \frac{n_s n_t n_p}{n^2} (3X_s^u X_t^u X_p^u + X_s^{u+1} X_t^{u-1} X_p^u) \right\} \right. \\ &\quad \left. + 3S \left\{ \frac{n_s^2}{n} X_s^{3u} \right\} + S \left\{ \frac{n_s n_t}{n} (X_s^{2u} X_t^u + 2X_s^{2u-1} X_t^{u+1}) \right\} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{u\chi\phi'}{n^3}\left[\mu_{3u}+S\left\{\frac{n_s n_t}{n^2}(X_s^{2u}X_t^u+2X_s^{2u-1}X_t^{u+1}-2X_s^{2u}X_t^u-X_s^{2u+1}X_t^{u-1})\right\}\right] \\
& =-\frac{u\chi\phi}{n^3}\left[-n\mu_u^3-2n\mu_{u-1}\mu_u\mu_{u+1}\right] \\
& \quad -\frac{u\chi\phi'}{n^3}[\mu_{3u}+2\mu_{2u-1}\mu_{u+1}-\mu_{2u}\mu_u-\mu_{2u+1}\mu_{u-1}];
\end{aligned}$$

$$\begin{aligned}
\text{or mean value of } HL^* &= \frac{u\chi}{n^2}\left\{\phi\left(\frac{\mu_u^3+2\mu_{u-1}\mu_u\mu_{u+1}}{-\mu_{2u}\mu_u-2\mu_{2u-1}\mu_{u+1}}\right)\right. \\
& \quad \left.-\frac{\phi'}{n}(\mu_{3u}+2\mu_{2u-1}\mu_{u+1}-\mu_{2u}\mu_u-\mu_{2u+1}\mu_{u-1})\right\},
\end{aligned}$$

$$HM = \frac{2u^2\mu_{u-1}}{n^2}[S\{X_s^{2u-1}dn_s^2d\bar{x}^2\}+S\{X_s^uX_t^{u-1}dn_sdn_t d\bar{x}^2\}],$$

so that, using (34), (35) of the original paper

$$\begin{aligned}
\overline{HM} &= \frac{2u^2\mu_{u-1}}{n^2}\left[S\left\{X_s^{2u-1}\frac{n_s}{n}\chi\left[\mu_2\left(\phi\left(1-\frac{n_s}{n}\right)+\frac{\phi'}{n}\right)+X_s^2\left(2\phi\frac{n_s}{n}+\frac{\phi'}{n}\right)\right]\right\}\right. \\
& \quad \left.+S\left\{X_s^uX_t^{u-1}\chi\frac{n_s n_t}{n^2}\left[\phi(2X_sX_t-\mu_2)-(X_s-X_t)^2\frac{\phi'}{n}\right]\right\}\right] \\
&= \frac{2u^2\mu_{u-1}}{n^2}\chi\phi\left[\mu_2S\left\{\left(\frac{n_s}{n}-\frac{n_s^2}{n^2}\right)X_s^{2u-1}\right\}+S\left\{2\frac{n_s^2}{n^2}X_s^{2u+1}\right\}\right. \\
& \quad \left.+2S\left\{\frac{n_s n_t}{n^2}X_s^{u+1}X_t^u\right\}-\mu_2S\left\{\frac{n_s n_t}{n^2}X_s^uX_t^{u-1}\right\}\right] \\
& \quad +\frac{2u^2\mu_{u-1}}{n^2}\chi\frac{\phi'}{n}\left[\mu_2S\left\{\frac{n_s}{n}X_s^{2u-1}\right\}+S\left\{\frac{n_s}{n}X_s^{2u+1}\right\}\right. \\
& \quad \left.-S\left\{\frac{n_s n_t}{n^2}(X_s^{u+2}X_t^{u-1}-X_s^{u+1}X_t^u)\right\}\right] \\
&= \frac{2u^2\mu_{u-1}}{n^2}\chi\left[\phi(\mu_2\mu_{2u-1}-\mu_{2u}\mu_u\mu_{u-1}+2\mu_u\mu_{u+1})\right. \\
& \quad \left.+\frac{\phi'}{n}(\mu_2\mu_{2u-1}+\mu_{2u+1}-\mu_{u-1}\mu_{u+2}+\mu_u\mu_{u+1})\right].
\end{aligned}$$

$$\text{Again} \quad HN = -\frac{u^3\mu_{u-1}^2}{n}S(X_s^{u-1}dn_s d\bar{x}^3),$$

so that using (36)

$$\begin{aligned}
\overline{HN} &= -\frac{u^3\mu_{u-1}^2}{n}\left[S\left\{X_s^{u-1}\left(3\chi\phi\frac{n_s}{n^2}\mu_2X_s+\chi\phi'\frac{n_s}{n^3}(X_s^3+3\mu_2X_s-\mu_3)\right)\right\}\right] \\
&= -\frac{u^3\mu_{u-1}^2}{n^2}\chi\left[3\phi\mu_2\mu_u+\frac{\phi'}{n}(\mu_{u+2}+3\mu_2\mu_u-\mu_3\mu_{u-1})\right],
\end{aligned}$$

$$KL = \frac{u(u-1)\mu_{u-2}}{2n^2}[S\{X_s^{2u}dn_s^2d\bar{x}^2\}+2S\{X_s^uX_t^udn_sdn_t d\bar{x}^2\}].$$

* It must be remembered that ϕ and $\frac{\phi'}{n}$ are of same order in $\frac{1}{n}$. Cf. equation (5), p. 26, *l.c.*

Therefore, using (34) and (35),

$$\begin{aligned}
 \overline{KL} &= \frac{u(u-1)\mu_{u-2}}{2n^2} \left[S \left\{ X_s^{2u} \frac{n_s}{n} \chi \left[\mu_2 \left(\phi \left(1 - \frac{n_s}{n} + \frac{\phi'}{n} \right) + X_s^2 \left(2\phi \frac{n_s}{n} + \frac{\phi'}{n} \right) \right] \right\} \right. \\
 &\quad \left. + 2S \left\{ X_s^u X_t^u \chi \frac{n_s n_t}{n^2} \left[\phi (2X_s X_t - \mu_2) - (X_s - X_t)^2 \frac{\phi'}{n} \right] \right\} \right. \\
 &= \frac{u(u-1)\mu_{u-2}}{2n^2} \left[\chi \phi \mu_2 S \left\{ \left(\frac{n_s}{n} - \frac{n_s^2}{n^2} \right) X_s^{2u} - 2 \frac{n_s n_t}{n^2} X_s^u X_t^u \right\} \right. \\
 &\quad \left. + \chi \phi S \left\{ 2 \frac{n_s^2}{n^2} X_s^{2u+2} + 4 \frac{n_s n_t}{n^2} X_s^{u+1} X_t^{u+1} \right\} \right. \\
 &\quad \left. + \chi \frac{\phi'}{n} \left[\mu_2 S \left\{ \frac{n_s}{n} X_s^{2u} \right\} + S \left\{ \frac{n_s}{n} X_s^{2u+2} \right\} \right. \right. \\
 &\quad \left. \left. - 2S \left\{ \frac{n_s n_t}{n^2} (X_s^{u+2} X_t^u - 2X_s^{u+1} X_t^{u+1} + X_s^u X_t^{u+2}) \right\} \right] \right] \\
 &= \frac{u(u-1)\mu_{u-2}}{2n^2} \chi \left[\phi \{ \mu_2 (\mu_{2u} - \mu_u^2) + 2\mu_{u+1}^2 \} \right. \\
 &\quad \left. + \frac{\phi'}{n} \{ \mu_2 \mu_{2u} + \mu_{2u+2} + 2\mu_{u+1}^2 - 2\mu_u \mu_{u+2} \} \right],
 \end{aligned}$$

$$KM = \frac{u(u-1)\mu_{u-2}}{2} \left(-\frac{2u\mu_{u-1}}{n} \right) S \{ dn_s d\bar{x}^3 X_s^u \}.$$

Therefore, by (36),

$$\overline{KM} = -\frac{u^2(u-1)\mu_{u-1}\mu_{u-2}}{n} S \left\{ X_s^u \left(3\chi \phi \frac{n_s}{n^2} \mu_2 X_s + \chi \phi' \frac{n_s}{n^3} (X_s^3 + 3\mu_2 X_s - \mu_3) \right) \right\}$$

$$\text{or } \overline{KM} = -\frac{u^2(u-1)\mu_{u-1}\mu_{u-2}}{n^2} \chi \left\{ 3\phi \mu_2 \mu_{u+1} + \frac{\phi'}{n} (\mu_{u+3} + 3\mu_2 \mu_{u+1} - \mu_3 \mu_u) \right\},$$

$$KN = \frac{u^3(u-1)\mu_{u-1}^2\mu_{u-2}}{2} d\bar{x}^4;$$

and using (26) of the original paper therefore

$$\text{Mean } KN = \frac{u^3(u-1)\mu_{u-1}^2\mu_{u-2}}{2n^2} \left[3\chi \phi \mu_2^2 + \chi \frac{\phi'}{n} (\mu_4 + 3\mu_2^2) \right].$$

Adding these various terms to the mean value of M_3 as given by (39) of paper cited, we find for the corrected value:

$$\begin{aligned}
 M_3 = & \frac{\chi \chi'}{n^2} [2\mu_u^3 - 3\mu_u \mu_{2u} + \mu_{3u} - 3u \mu_{u-1} (\mu_{2u+1} - 2\mu_{u+1} \mu_u) \\
 & + 3u^2 \mu_{u-1}^2 (\mu_{u+2} - \mu_2 \mu_u) - u^3 (\mu_{u-1}^3 \mu_3)]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{3\chi}{n^2} \left\{ \phi \left[(\mu_u^3 + 2\mu_{u-1}\mu_u\mu_{u+1} - \mu_{2u}\mu_u - 2\mu_{u+1}\mu_{2u-1})u \right. \right. \\
& \quad + 2u^2\mu_{u-1}(\mu_2\mu_{2u-1} - \mu_2\mu_{u-1}\mu_u + 2\mu_u\mu_{u+1}) \\
& \quad - u^3\mu_{u-1}^2(3\mu_2\mu_u) + \frac{u(u-1)}{2}\mu_{u-2}(\mu_2\mu_{2u} - \mu_2\mu_u^2 + 2\mu_{u+1}^2) \\
& \quad \left. - u^2(u-1)\mu_{u-2}\mu_{u-1}(3\mu_2\mu_{u+1}) + \frac{u^3(u-1)}{2}\mu_{u-1}^2\mu_{u-2}(3\mu_2^2) \right] \\
& \quad + \frac{\phi'}{n} \left[-u(\mu_{3u} + 2\mu_{u+1}\mu_{2u-1} - \mu_u\mu_{2u} - \mu_{u-1}\mu_{2u+1}) \right. \\
& \quad + 2u^2\mu_{u-1}(\mu_2\mu_{2u-1} + \mu_{2u+1} - \mu_{u-1}\mu_{u+2} + \mu_u\mu_{u+1}) \\
& \quad - u^3\mu_{u-1}^2(\mu_{u+2} + 3\mu_2\mu_u - \mu_3\mu_{u-1}) \\
& \quad + \frac{u(u-1)}{2}\mu_{u-2}(\mu_2\mu_{2u} + \mu_{2u+2} + 2\mu_{u+1}^2 - 2\mu_u\mu_{u+2}) \\
& \quad - u^2(u-1)\mu_{u-1}\mu_{u-2}(\mu_{u+3} + 3\mu_2\mu_{u+1} - \mu_3\mu_u) \\
& \quad \left. \left. + \frac{u^3(u-1)}{2}\mu_{u-1}^2\mu_{u-2}(\mu_4 + 3\mu_2^2) \right] \right\},
\end{aligned}$$

and (40), (46) and (47) must be modified accordingly. It remains true that for a *normal* population M_3 vanishes when u is odd, and that in all cases $B_1 \propto \frac{1}{n}$.

If we write this value of M_3 in the form

$$M_3 = \frac{\chi\chi'R}{n^2} + \frac{3\chi\phi K}{n^2} + \frac{3\chi\phi'T}{n^3},$$

then in order to obtain B_1 correct to $\frac{1}{n}$, the third term may be omitted, R has the same value as in Equation (47) of the paper cited and K is zero for normal distributions when u is odd. For even values of u in normal distributions the value of K is

$$u(\mu_u^3 - \mu_{2u}\mu_u) + \frac{u(u-1)}{2}\mu_{u-2}(\mu_2\mu_{2u} - \mu_2\mu_u^2),$$

which easily reduces to $-\frac{1}{2}u \times P \times \mu_u$, where $P = \mu_{2u} - \mu_u^2$ as in Equation (52).

We may therefore add to Table I on p. 39 of the original paper the following column:

u	K
2	$-2\mu_2^3$
3	0
4	$-576\mu_2^6$
5	0
6	$-457,650\mu_2^{12}$

In Table II on p. 39, the third column is unaltered, the second column becomes (the coefficients of ϕ being approximated)

u	B_1
2	$\frac{8 (\chi' - 0.75\phi)^2}{n \chi}$
3	0
4	$\frac{102 (\chi' - 0.108\phi)^2}{n \chi}$
5	0
6	$\frac{1099 (\chi' - 0.040\phi)^2}{n \chi}$

The corrected form of Table III (p. 40 of paper cited) is now as follows:

Table III. *Approximate values of β_1, β_2 for samples of 1000 out of a population of 1,000,000.*

u	β_1	β_2
2	0.001	3.012
3	0.000	3.090
4	0.081	3.204

Thus the effect of the correction is to change the values of β_1 for $u=2$ and $u=4$ from the values 0.008 and 0.102 to 0.001 and 0.081 respectively, but it remains true that the frequency of the fourth moment-coefficient differs appreciably from the normal distribution.

(III) Dr Isserlis also wishes to make the following emendations in his paper in the last number of *Biometrika*, Vol. XII, p. 134. On p. 138 near the foot + ABC has been dropped from the bracket ($8FGH + 2AF^2 + 2BG^2 + 2CH^2$). Also in l. 6 of the same page for "on Q " read "and Q ."

(IV) The point indicated by Professor Tchouproff, namely: that fourth-order mean products are of the same order finally in $\frac{1}{N}$ as third-order mean products and cannot be neglected therefore in comparison with third-order mean products, is of great importance in investigations into the probable errors of frequency constants in the case of small samples. In expanding functions of the deviations from mean values of subfrequencies such as δn_p we cannot neglect products of the fourth order in the δn_p 's compared with products of the third order. In obtaining results true to products of an *odd* order in the "statistical differentials" we must proceed to products of the next highest even order to reach correctness.

This principle, which is almost self-obvious, was, however, overlooked by Pearson in his paper "On the Application of 'Goodness of Fit' Tables to test Regression Curves and Theoretical Curves used to describe observational or experimental Data," in *Biometrika*, Vol. XI, pp. 239—261.

One of the objects of that paper was to investigate the probable errors and frequency distributions of errors in the mean and standard-deviation of an array. If we have an array of a first variate corresponding to a small subrange of a second variate in a sample of N , the law of distribution of the means and standard-deviations of such arrays when many samples of N are taken had not been investigated at the time Pearson wrote. If there be n_p individuals in such a sample, then the problem differs from the ordinary problem of the distribution of means and standard-deviations in a sample of size n_p , in the fact that n_p in the case of the array varies from sample to sample. Hence we cannot straight away assume that if $\bar{n}_p$ be the mean number in the array then $\bar{\sigma}_{\bar{n}_p}/\sqrt{\bar{n}_p}$ and $\bar{\sigma}_{\bar{n}_p}/\sqrt{2\bar{n}_p}$ will be the standard-deviations of the distributions of means and of standard-deviations of the arrays; still less do we know how far it is legitimate to suppose these distributions approximate to the Gaussian or normal type. As the problem is an exceedingly important one the writer asked Miss Eleanor Pairman to revise his work of 1916 by introducing where needful the fourth-order products. This she has done with certain additions and expansions.

(a) From the equation on p. 239 we have:

$$\text{mean } (\delta m_p) = \text{mean } \sum (-)^{\alpha} S \left\{ \frac{\bar{n}_{qp} x_q}{\bar{n}_p} \left(\frac{\delta n_p}{\bar{n}_p} \right)^{\alpha} - \frac{\delta n_{qp} x_q}{\bar{n}_p} \left(\frac{\delta n_p}{\bar{n}_p} \right)^{\alpha-1} \right\},$$

where Σ is a summation for every value of α from 1 to ∞ .

But $\text{mean } \delta n_{qp} \delta n_p = \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N} \right)$

and the regression relation is accordingly

$$\delta n_{qp} = \frac{\bar{n}_{qp}}{\bar{n}_p} \delta n_p.$$

Substituting this we see that every term vanishes and accordingly $\delta m_p = 0$, not merely to a high order of approximation, but absolutely. In other words the mean of the means of any array—notwithstanding that the number* in that array will vary—is equal to the mean of that array in the sampled population.

(b) We have for the p th array:

$$\begin{aligned} \bar{m}_p + \delta m_p &= \frac{S(\bar{n}_{qp} + \delta n_{qp}) x_q}{\bar{n}_p + \delta n_p}, \\ \delta m_p &= \frac{S(\delta n_{qp} \bar{n}_p x_q) - S(\bar{n}_{qp} x_q) \delta n_p}{\bar{n}_p (\bar{n}_p + \delta n_p)}, \end{aligned}$$

but

$$\delta n_p = S(\delta n_{qp}) \text{ and } S(\bar{n}_{qp} x_q) = \bar{n}_p \bar{m}_p,$$

and accordingly

$$\delta m_p = \frac{S\{\delta n_{qp}(x_q - \bar{m}_p)\}}{\bar{n}_p + \delta n_p} = \frac{S\{\delta n_{qp}\tilde{x}_q\}}{\bar{n}_p + \delta n_p},$$

where $\tilde{x}_q$ is measured from the sampled population array mean.

Now we desire to obtain the various moment-coefficients of δm_p , or mean $(\delta m_p)^s$, which for convenience may be written $\{\delta m_p^s\}$.

There are two ways at first sight of doing this:

(i) We may expand $(\bar{n}_p + \delta n_p)^s$ in terms of $\delta n_p/\bar{n}_p$ and then take the mean values of products such as:

$$(\delta n_p)^a (\delta n_{qp})^b (\delta n_{q'p})^{b'} (\delta n_{q''p})^{b''} \dots\dots\dots$$

This was the process adopted in the original memoir. It is very laborious and the algebra so lengthy as to lead easily to slips. Still on the present occasion we went to terms of a high order (O_s) and some of the results obtained will be so useful in other investigations on probable errors of frequency constants that it seems worth while placing them on record here. The fourth-order mean products in δn_p and δn_{qp} may be added to those given on p. 245 of the original memoir.

They are:

$$\begin{aligned} \frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^3) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N}\right) \left\{1 + 3 \left(1 - \frac{2}{N}\right) \bar{n}_{qp} \left(1 - \frac{\bar{n}_{qp}}{N}\right)\right\}, \\ \frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^2 \delta n_{q'p}) &= \left(1 - \frac{2}{N}\right) \bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N}\right) \left(1 - \frac{3\bar{n}_{qp}}{N}\right), \\ \frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp} \delta n_{q'p} \delta n_{q''p}) &= -3 \left(1 - \frac{2}{N}\right) \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{N} \left(1 - \frac{\bar{n}_p}{N}\right), \\ \frac{1}{\lambda} \Sigma (\delta n_p^2 \delta n_{qp}^2) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N}\right) \left\{1 + \left(1 - \frac{2}{N}\right) \left[2\bar{n}_{qp} + \bar{n}_p \left(1 - \frac{3\bar{n}_{qp}}{N}\right)\right]\right\}, \\ \frac{1}{\lambda} \Sigma (\delta n_p^2 \delta n_{qp} \delta n_{q'p}) &= \left(1 - \frac{2}{N}\right) \bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N}\right) \left(2 - \frac{3\bar{n}_p}{N}\right), \\ \frac{1}{\lambda} \Sigma (\delta n_p^3 \delta n_{qp}) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N}\right) \left\{1 + 3 \left(1 - \frac{2}{N}\right) \bar{n}_p \left(1 - \frac{\bar{n}_p}{N}\right)\right\} \\ &\dots\dots\dots(A). \end{aligned}$$

For the *fifth*-order mean products, Miss Pairman also provided the following values*:

$$\begin{aligned} \frac{1}{\lambda} \Sigma (\delta n_{qp}^5) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_{qp}}{N}\right) \left(1 - \frac{2\bar{n}_{qp}}{N}\right) \left\{1 + 2 \left(5 - \frac{6}{N}\right) \bar{n}_{qp} \left(1 - \frac{\bar{n}_{qp}}{N}\right)\right\}, \\ \frac{1}{\lambda} \Sigma (\delta n_{qp}^4 \delta n_{q'p}) &= \frac{\bar{n}_{qp} \bar{n}_{q'p}}{N} \left(1 - \frac{2\bar{n}_{qp}}{N}\right) \left\{1 + 2 \left(5 - \frac{6}{N}\right) \bar{n}_{qp} \left(1 - \frac{\bar{n}_{qp}}{N}\right)\right\}, \\ \frac{1}{\lambda} \Sigma (\delta n_{qp}^3 \delta n_{q'p}^2) &= \bar{n}_{pq} \bar{n}_{pq'} \left\{1 - \frac{1}{N} - \frac{\bar{n}_{pq}}{N} - \left(1 - \frac{2}{N}\right) \frac{\bar{n}_{pq'}}{N} \right. \\ &\quad \left. - \left(5 - \frac{6}{N}\right) \frac{\bar{n}_{pq}}{N} \left[1 - \frac{\bar{n}_{pq}}{N} - \frac{\bar{n}_{pq'}}{N} \left(3 - \frac{4\bar{n}_{pq}}{N}\right)\right]\right\}, \end{aligned}$$

* These results are of course perfectly general, that is to say we can suppress p and suppose them the mean variation values of elements n_q , $n_{q'}$, $n_{q''}$, $n_{q'''}$ and n_{qiv} of any frequency distribution.

$$\begin{aligned}
\frac{1}{\lambda} \Sigma (\delta n_{qp}^3 \delta n_{q'p} \delta n_{q''p}) &= -\frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{N} \left\{ 1 - \frac{2}{N} - \left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp}}{N} \left(3 - \frac{4\bar{n}_{qp}}{N} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_{qp}^2 \delta n_{q'p}^2 \delta n_{q''p}) &= -\frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{N} \left\{ 2 \left(1 - \frac{1}{N} \right) - \left(5 - \frac{6}{N} \right) \left(\frac{\bar{n}_{qp} + \bar{n}_{q'p}}{N} - \frac{4\bar{n}_{qp} \bar{n}_{q'p}}{N^2} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_{qp}^2 \delta n_{q'p} \delta n_{q''p} \delta n_{q'''p}) &= \left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p} \bar{n}_{q'''p}}{N^2} \left(1 - \frac{4\bar{n}_{qp}}{N} \right), \\
\frac{1}{\lambda} \Sigma (\delta n_{qp} \delta n_{q'p} \delta n_{q''p} \delta n_{q'''p} \delta n_{q^{ivp}}) &= -4 \left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p} \bar{n}_{q'''p} \bar{n}_{q^{ivp}}}{N^3} \\
&\dots\dots\dots(B).
\end{aligned}$$

Alongside these we give the fifth order combinations of δn_p and δn_{qp} , which are deduced from these and were required for our purposes:

$$\begin{aligned}
\frac{1}{\lambda} \Sigma (\delta n_p^4 \delta n_{qp}) &= N \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N} \right) \left(1 - \frac{2\bar{n}_p}{N} \right) \left\{ 1 + 2 \left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp}}{N} \left(1 - \frac{\bar{n}_{qp}}{N} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p^3 \delta n_{qp}^2) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 1 - \bar{n}_p + \left(1 - \frac{2}{N} \right) \bar{n}_{qp} \right. \\
&\quad \left. + \left(5 - \frac{6}{N} \right) \left(1 - \frac{\bar{n}_p}{N} \right) \left[\bar{n}_p + \bar{n}_{qp} \left(1 - \frac{4\bar{n}_p}{N} \right) \right] \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p^3 \delta n_{qp} \delta n_{q'p}) &= \bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 1 - \frac{2}{N} + \left(5 - \frac{6}{N} \right) \left(1 - \frac{\bar{n}_p}{N} \right) \left(1 - \frac{4\bar{n}_p}{N} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p^2 \delta n_{qp}^3) &= \bar{n}_{qp} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 1 + \left(1 - \frac{2}{N} \right) \bar{n}_p - 6 \left(1 - \frac{1}{N} \right) \bar{n}_{pq} \right. \\
&\quad \left. + \left(5 - \frac{6}{N} \right) \bar{n}_{pq} \left[3 \left(1 - \frac{\bar{n}_p}{N} \right) - \left(3 - \frac{4\bar{n}_p}{N} \right) \frac{\bar{n}_{qp}}{N} \right] \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p^2 \delta n_{qp}^2 \delta n_{q'p}) &= -\bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 2 \left(1 - \frac{1}{N} \right) \right. \\
&\quad \left. - \left(5 - \frac{6}{N} \right) \left[1 - \frac{\bar{n}_p}{N} - \frac{\bar{n}_{qp}}{N} \left(3 - \frac{4\bar{n}_p}{N} \right) \right] \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p^2 \delta n_{qp} \delta n_{q'p} \delta n_{q''p}) &= -\left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{N} \left(1 - \frac{\bar{n}_p}{N} \right) \left(3 - \frac{4\bar{n}_p}{N} \right), \\
\frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^4) &= \bar{n}_{qp} \left(1 - \frac{2\bar{n}_{qp}}{N} \right) \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 1 + 2 \left(5 - \frac{6}{N} \right) \bar{n}_{qp} \left(1 - \frac{\bar{n}_{qp}}{N} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^3 \delta n_{q'p}) &= \bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ \left(1 - \frac{2}{N} \right) - \left(5 - \frac{6}{N} \right) \frac{\bar{n}_{qp}}{N} \left(3 - \frac{4\bar{n}_{qp}}{N} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^2 \delta n_{q'p}^2) &= \bar{n}_{qp} \bar{n}_{q'p} \left(1 - \frac{\bar{n}_p}{N} \right) \left\{ 2 \left(1 - \frac{1}{N} \right) \right. \\
&\quad \left. - \left(5 - \frac{6}{N} \right) \left(\frac{\bar{n}_{qp} + \bar{n}_{q'p}}{N} - \frac{4\bar{n}_{qp} \bar{n}_{q'p}}{N^2} \right) \right\}, \\
\frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp}^2 \delta n_{q'p} \delta n_{q''p}) &= -\frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{N} \left(1 - \frac{\bar{n}_p}{N} \right) \left(5 - \frac{6}{N} \right) \left(1 - \frac{4\bar{n}_{qp}}{N} \right), \\
\frac{1}{\lambda} \Sigma (\delta n_p \delta n_{qp} \delta n_{q'p} \delta n_{q''p} \delta n_{q'''p}) &= \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p} \bar{n}_{q'''p}}{N^2} \left(1 - \frac{\bar{n}_p}{N} \right) 4 \left(5 - \frac{6}{N} \right) \\
&\dots\dots\dots(C).
\end{aligned}$$

Also we give here additional moments* of the binomial $(p+q)^n$ about its mean:

$$\begin{aligned}\mu_1 &= 0, & \mu_2 &= npq, & \mu_3 &= npq(p-q), \\ \mu_4 &= npq\{1+3(n-2)pq\}, & \mu_5 &= npq(p-q)\{1+2(5n-6)pq\}, \\ \mu_6 &= npq\{1+5(5n-6)pq(1-4pq)+15n(n-2)p^2q^2\}, \\ \mu_7 &= npq(p-q)\{1+4pq(14n-15)+p^2q^2(105n^2-462n+360)\}, \\ \mu_8 &= npq\{1+7pq(17n-18)+14p^2q^2(35n^2-154n+120) \\ &\quad +7p^3q^3(15n^3-340n^2+1044n-720)\} \\ &\quad \dots\dots\dots(D).\end{aligned}$$

The values obtained in the above laborious manner for $\{\delta m_p^s\}$ agreed as far as we proceeded with those obtained by the following or second method.

(ii) This second method consisted in first summing for δn_{qp} on the assumption that δn_p was constant, and then summing for δn_p . This involved some new results which will be useful in other problems and are recorded here.

For constant $\bar{n}_p + \delta n_p$:

$$\begin{aligned}\text{Mean } (\delta n_{qp})^2 &= \left(1 + \frac{\delta n_p}{\bar{n}_p}\right) \frac{\bar{n}_{qp}}{\bar{n}_p} (\bar{n}_p - \bar{n}_{qp}) + \frac{\bar{n}_{qp}^2}{\bar{n}_p^2} \delta n_p^2, \\ \text{Mean } (\delta n_{qp} \delta n_{q'p}) &= -\left(1 + \frac{\delta n_p}{\bar{n}_p}\right) \frac{\bar{n}_{qp} \bar{n}_{q'p}}{\bar{n}_p} + \frac{\bar{n}_{qp} \bar{n}_{q'p}}{\bar{n}_p^2} \delta n_p^2, \\ \text{Mean } (\delta n_{qp})^3 &= \left(1 + \frac{\delta n_p}{\bar{n}_p}\right) \frac{\bar{n}_{qp}}{\bar{n}_p} (\bar{n}_p - \bar{n}_{qp}) \left\{1 - \frac{2\bar{n}_{qp}}{\bar{n}_p} + 3\delta n_p \frac{\bar{n}_{qp}}{\bar{n}_p}\right\} + \frac{\bar{n}_{qp}^3}{\bar{n}_p^3} \delta n_p^3, \\ \text{Mean } (\delta n_{qp}^2 \delta n_{q'p}) &= -\left(1 + \frac{\delta n_p}{\bar{n}_p}\right) \frac{\bar{n}_{qp} \bar{n}_{q'p}}{\bar{n}_p} \left(1 - \delta n_p - 2 \frac{\bar{n}_{qp}}{\bar{n}_p} + 3\delta n_p \frac{\bar{n}_{qp}}{\bar{n}_p}\right) + \frac{\bar{n}_{qp}^2 \bar{n}_{q'p}}{\bar{n}_p^3} \delta n_p^3, \\ \text{Mean } (\delta n_{qp} \delta n_{q'p} \delta n_{q''p}) &= \frac{\bar{n}_{qp} \bar{n}_{q'p} \bar{n}_{q''p}}{\bar{n}_p^3} \{(\bar{n}_p + \delta n_p)(2 - 3\delta n_p) + \delta n_p^3\} \\ &\quad \dots\dots\dots(E).\end{aligned}$$

Now the value of this method was at once obvious, for proceeding to the summations in the moment-coefficients of δm_p , for constant $\bar{n}_p + \delta n_p$ we found that they corresponded with values to be found for the distribution in a sample n_p of constant size. In other words we reached a conclusion, which should have been obvious at first sight, namely that to find the value of

$$\text{Mean } (\delta m_p)^s$$

all that we have to do is to write down the known value $n_p = \bar{n}_p + \delta n_p$ constant and then sum for δn_p . We might have pulled down the scaffolding in this correctional paper and simply started from this result, but as several of the means reached in processes (i) and (ii) seemed likely to be of value, we have preferred to indicate the steps which led us to the final method.

* A simple reduction formula for the moments of a binomial about its *mean* was sought in vain. After a good deal of energy had been spent on the problem, we believe that μ_s being the *s*th moment about the mean

$$\mu_s = \left[\frac{d^s}{dx^s} (qe^{qx} + pe^{-qx})^n \right]_{x=0}$$

is, perhaps, the easiest expression for reaching these moment-coefficients by successive differentiation.

(iii) For a sample of constant size n_p the following are the moment-coefficients of the variation in the mean* :

$$\begin{aligned}(\delta\mu_1')^2 &= \frac{p\mu_2}{n_p} = \frac{\sigma^2 \bar{n}_p}{n_p}, \\(\delta\mu_1')^3 &= \frac{p\mu_3}{n_p^2}, \\(\delta\mu_1')^4 &= \frac{p\mu_4 - 3p\mu_2^2}{n_p^3} + \frac{3p\mu_2^2}{n_p^2}.\end{aligned}$$

Now let $S\left(\frac{1}{n_p^s}\right)$ equal the sum of $\frac{1}{n_p^s}$ for all values of p which may occur in the samples of N . If therefore f_{n_p} is the frequency with which n_p occurs, the whole problem reduces to finding the values of

$$S(f_{n_p}/n_p^s),$$

for various values of s .

Now the frequencies of the n_p are simply the terms of the binomial. The term in which $n_p=0$ must not we think be taken into consideration, for in this case there is no variability in μ_1' as there is no frequency in the array, i.e. $p\mu_2$ must be put zero. Thus in the notation of a binomial $(p+q)^n$ we require to find :

$$\frac{np^{n-1}q}{1^s} + \frac{n(n-1)p^{n-2}q^2}{1.2} \frac{1}{2^s} + \frac{n(n-1)(n-2)p^{n-3}q^3}{1.2.3} \frac{1}{3^s} + \dots, \dots\dots(F)$$

and to divide the result by $(p+q)^n - p^n = 1 - p^n$. This finite series we have not succeeded in summing. Before indicating how we may approximate to it by the mean-powers of δn_p , we can look at the problem from two other standpoints.

(i) If $\bar{n}_p/N=q$ be not small the binomial approximates to a Gaussian of standard-deviation squared $\sigma^2=npq=\bar{n}_p(1-\bar{n}_p/N)$. Hence

$$\begin{aligned}S\left(\frac{1}{n_p^s}\right) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \frac{1}{(\bar{n}_p+x)^s} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx \\&= \frac{1}{\sqrt{2\pi}\bar{n}_p^s\sigma} \int_{-\infty}^{+\infty} \left(1 - \frac{sx}{\bar{n}_p} + \frac{s(s+1)}{1.2} \frac{x^2}{\bar{n}_p^2} - \dots\right) e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx \\&= \frac{1}{\bar{n}_p^s} \left(1 + \frac{s(s+1)}{1.2} \frac{\sigma^2}{\bar{n}_p^2} + \frac{s(s+1)(s+2)(s+3)}{1.2.3.4} \frac{3\sigma^4}{\bar{n}_p^4} + \dots\right).\end{aligned}$$

Thus

$$\left. \begin{aligned}S\left(\frac{1}{n_p}\right) &= \frac{1}{\bar{n}_p} \left\{1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + 3\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}, \\S\left(\frac{1}{n_p^2}\right) &= \frac{1}{\bar{n}_p^2} \left\{1 + 3\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + 15\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}, \\S\left(\frac{1}{n_p^3}\right) &= \frac{1}{\bar{n}_p^3} \left\{1 + 6\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + 45\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}.\end{aligned} \right\} \dots\dots\dots(G)$$

* Here $p\mu_s$ is the s th moment-coefficient of the $\bar{n}_p$ array about its mean in the sampled population.

(ii) Another method is to assume a Pearson Type III curve: $y = y_0 x^\alpha e^{-\gamma x}$, which is known to give a better approximation than the Gaussian to the binomial. We assume it to start at the beginning of the first subrange of the binomial and to have the same mean and standard-deviation. These conditions involve

$$\frac{\alpha + 1}{\gamma} = \bar{n}_p, \quad \sigma^2 = \bar{n}_p \left(1 - \frac{\bar{n}_p}{N}\right) = \frac{\alpha + 1}{\gamma^2}.$$

Accordingly
$$S\left(\frac{1}{\bar{n}_p^s}\right) = \frac{1}{\lambda} \int_0^\infty y_0 x^{\alpha-s} e^{-\gamma x} dx = \frac{y_0}{\lambda} \frac{1}{\gamma^{\alpha-s+1}} \Gamma(\alpha - s + 1),$$

where $\lambda = \text{total frequency} = \frac{y_0}{\gamma^{\alpha+1}} \Gamma(\alpha + 1).$

Hence

$$S\left(\frac{1}{\bar{n}_p}\right) = \frac{\gamma}{\alpha} = \frac{1}{\bar{n}_p} \frac{1}{1 - \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)},$$

$$S\left(\frac{1}{\bar{n}_p^2}\right) = \frac{\gamma^2}{\alpha(\alpha-1)} = \frac{1}{\bar{n}_p^2} \frac{1}{\left\{1 - \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)\right\} \left\{1 - 2\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)\right\}},$$

$$S\left(\frac{1}{\bar{n}_p^3}\right) = \frac{\gamma^3}{\alpha(\alpha-1)(\alpha-2)} = \frac{1}{\bar{n}_p^3} \frac{1}{\left\{1 - \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)\right\} \left\{1 - 2\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)\right\} \left\{1 - 3\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)\right\}},$$

which are exact. Or, approximating

$$\left. \begin{aligned} S\left(\frac{1}{\bar{n}_p}\right) &= \frac{1}{\bar{n}_p} \left\{1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + \left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}, \\ S\left(\frac{1}{\bar{n}_p^2}\right) &= \frac{1}{\bar{n}_p^2} \left\{1 + 3\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + 7\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}, \\ S\left(\frac{1}{\bar{n}_p^3}\right) &= \frac{1}{\bar{n}_p^3} \left\{1 + 6\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right) + 25\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2 + \dots\right\}. \end{aligned} \right\} \dots\dots\dots(\text{H})$$

Both methods agree to the terms in $\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)$, but the Gaussian appears to exaggerate in the terms in $\left(\frac{1}{\bar{n}_p} - \frac{1}{N}\right)^2$.

(iii) We will now proceed to approximate on the basis of the moment-coefficients of the binomial. We have

$$S\frac{1}{(\bar{n}_p + \delta n_p)^s} = \frac{1}{\bar{n}_p^s} \left(1 - s \frac{S(\delta n_p)}{\bar{n}_p} + \frac{s(s+1)}{1 \cdot 2} \frac{S(\delta n_p^2)}{\bar{n}_p^2} - \frac{s(s+1)(s+2)}{1 \cdot 2 \cdot 3} \frac{S(\delta n_p^3)}{\bar{n}_p^3} + \dots\right).$$

Here $S(\delta n_p) = 0$, and we will keep terms up to the order $\frac{1}{\bar{n}_p^{3+s}}$ which involves proceeding to the fourth moment-coefficient. We find

$$S \frac{1}{(\bar{n}_p + \delta n_p)^s} = \frac{1}{\bar{n}_p^s} \left\{ 1 + \frac{s(s+1)}{1 \cdot 2} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) - \frac{s(s+1)(s+2)}{1 \cdot 2 \cdot 3} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{2}{N} \right) \right. \\ + \frac{s(s+1)(s+2)(s+3)}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left[3 \left(1 - \frac{2}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) + \frac{1}{\bar{n}_p^2} \right] \\ - \frac{s(s+1)(s+2)(s+3)(s+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} 10 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(\frac{1}{\bar{n}_p} - \frac{2}{N} \right) \\ \left. + \frac{s(s+1)(s+2)(s+3)(s+4)(s+5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} 15 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 - \text{etc.} \right\}$$

as far as terms of cubic order in the curled brackets.

Hence we find

$$\left. \begin{aligned} S\left(\frac{1}{\bar{n}_p}\right) &= \frac{1}{\bar{n}_p} \left\{ 1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(1 + \frac{1}{N} + \frac{1}{N^2} \right) \right. \\ &\quad \left. + 2 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(1 + \frac{3}{N} \right) + 6 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 + \dots \right\}, \\ S\left(\frac{1}{\bar{n}_p^2}\right) &= \frac{1}{\bar{n}_p^2} \left\{ 1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(3 + \frac{4}{N} + \frac{5}{N^2} \right) \right. \\ &\quad \left. + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(11 + \frac{40}{N} \right) + 50 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 + \dots \right\}, \\ S\left(\frac{1}{\bar{n}_p^3}\right) &= \frac{1}{\bar{n}_p^3} \left\{ 1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(6 + \frac{10}{N} + \frac{15}{N^2} \right) \right. \\ &\quad \left. + 5 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(7 + \frac{30}{N} \right) + 225 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 + \dots \right\}. \end{aligned} \right\} \dots\dots(I)$$

It will be seen that these values agree to the first term in $\left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)$ with those given by either hypothesis (i) or (ii). For the terms in $\left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2$ they appear to be intermediate between (i) and (ii). The additional terms which do not occur in either (i) or (ii) are those in powers of $\frac{1}{N}$ in the second- and third-order terms.

Using these results we deduce :

$${}_pM_2 = \text{mean } (\delta\mu_1')^2 \\ = \frac{n\mu_2}{\bar{n}_p} \left\{ 1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(1 + \frac{1}{N} + \frac{1}{N^2} \right) + 2 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(1 + \frac{3}{N} \right) + 6 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \\ \dots\dots\dots(J).$$

Thus the probable error of the mean of an array of mean size $\bar{n}_p$ in a sample of N is :

$$.67449 \frac{\sigma \bar{n}_p}{\sqrt{\bar{n}_p}} \left\{ 1 + \frac{1}{2} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(1 + \frac{1}{N} + \frac{1}{N^2} \right) + \frac{1}{8} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(7 + \frac{22}{N} \right) + \frac{41}{16} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \\ \dots\dots\dots(J^{bis}).$$

Again:

$${}_pM_3 = \frac{{}_p\mu_3}{\bar{n}_p^2} \left\{ 1 + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(3 + \frac{4}{N} + \frac{5}{N^2} \right) + \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \left(11 + \frac{40}{N} \right) + 50 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \dots\dots\dots(\text{K}),$$

and if
$$\zeta = \frac{1}{\bar{n}_p} - \frac{1}{N},$$

$${}_pB_1 = \frac{{}_pM_3}{{}_pM_2^3} = \frac{{}_p\beta_1}{\bar{n}_p} \frac{\left\{ 1 + \zeta \left(3 + \frac{4}{N} + \frac{5}{N^2} \right) + \zeta^2 \left(11 + \frac{40}{N} \right) + 50\zeta^3 \right\}^2}{\left\{ 1 + \zeta \left(1 + \frac{1}{N} + \frac{1}{N^2} \right) + \zeta^2 \left(2 + \frac{6}{N} \right) + 6\zeta^3 \right\}^3},$$

which after reduction gives

$${}_pB_1 = \frac{{}_p\beta_1}{\bar{n}_p} \left\{ 1 + \left(3 + \frac{5}{N} + \frac{7}{N^2} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) + \left(13 + \frac{56}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 + 69 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \dots\dots\dots(\text{L}).$$

Clearly if ${}_p\beta_1$ were as large as 0.5, ${}_pB_1$ for an array of 30 would be of the order .02 and thus the array means have an approximately symmetrical distribution.

We now turn to the value of ${}_pB_2$, and find with the same value of ζ as before

$${}_pM_4 - 3{}_pM_2^2 = \frac{{}_p\mu_4 - 3{}_p\mu_2^2}{\bar{n}_p^3} \left\{ 1 + \left(6 + \frac{10}{N} + \frac{15}{N^2} \right) \zeta + \left(35 + \frac{150}{N} \right) \zeta^2 + 225\zeta^3 \right\} \\ + \frac{3{}_p\mu_2^2}{\bar{n}_p^2} \left\{ \left(1 + \frac{2}{N} + \frac{3}{N^2} \right) \zeta + \left(6 + \frac{26}{N} \right) \zeta^2 + 34\zeta^3 \right\}.$$

$$\text{Further, } ({}_pM_2)^{-2} = \frac{\bar{n}_p^2}{{}_p\mu_2^2} \left\{ 1 - \left(2 + \frac{2}{N} + \frac{2}{N^2} \right) \zeta - \left(1 + \frac{6}{N} \right) \zeta^2 - 4\zeta^3 \right\}.$$

Multiplying the previous result by this we have

$${}_pB_2 - 3 = \frac{{}_p\beta_2 - 3}{\bar{n}_p} \left\{ 1 + \left(4 + \frac{8}{N} + \frac{13}{N^2} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) + \left(22 + \frac{112}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \right. \\ \left. + 145 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \\ + 3 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left\{ 1 + \frac{2}{N} + \frac{3}{N^2} + 4 \left(1 + \frac{5}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) + 21 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \right\} \dots\dots\dots(\text{M}).$$

For example, in an array of 25 in a sample of 1000, if ${}_p\beta_2$ were as high as 3.8, we should have ${}_pB_2$ slightly less than 3.2. Accordingly the constant ${}_pB_2$ of an array is not as approximately normal as ${}_pB_1$, or, we have the material thrown out further towards the tails than in the normal distribution.

It is probably, however, adequate to speak of the means of an array of variable size as roughly following a Gaussian curve and give the usual meaning to the "probable error" of the mean of such an array. Its value however is more accurately

$$.67449 \frac{\sigma_{\bar{n}_p}}{\sqrt{\bar{n}_p - 1 + \frac{\bar{n}_p}{N}}}, \text{ than } .67449 \sigma_{\bar{n}_p} / \sqrt{\bar{n}_p - 1}.$$

We may adopt a similar process to find the standard-deviation of the second moment of an array in a sample. For an array of constant size n_p we have

$$\text{Mean } (\delta_p \mu_2)^2 = \frac{1}{n_p} \left(1 - \frac{1}{n_p}\right) \left\{ \left(1 - \frac{1}{n_p}\right) {}_p\mu_4 - \left(1 - \frac{3}{n_p}\right) {}_p\mu_2^2 \right\},$$

where ${}_p\mu_4$ and ${}_p\mu_2$ refer to the values in the sampled population. Thus

$$\text{Mean } (\delta_p \mu_2)^2 = \frac{{}_p\mu_2^2}{n_p} \left(1 - \frac{1}{n_p}\right) \left\{ \left(1 - \frac{1}{n_p}\right) ({}_p\beta_2 - 3) + 2 \right\},$$

and, summing for all values of p ,

$$\sigma_{{}_p\mu_2}^2 = {}_p\mu_2^2 \left[2 \left\{ S\left(\frac{1}{n_p}\right) - S\left(\frac{1}{n_p^2}\right) \right\} + ({}_p\beta_2 - 3) \left\{ S\left(\frac{1}{n_p}\right) - 2S\left(\frac{1}{n_p^2}\right) + S\left(\frac{1}{n_p^3}\right) \right\} \right].$$

Accordingly we require to find $S\left(\frac{1}{n_p}\right) - S\left(\frac{1}{n_p^2}\right)$ and $S\left(\frac{1}{n_p^2}\right) - S\left(\frac{1}{n_p^3}\right)$. Writing ζ as before $= \frac{1}{\bar{n}_p} - \frac{1}{N}$, we have $\frac{1}{\bar{n}_p} = \zeta + \frac{1}{N}$ and after some reductions

$$\begin{aligned} S\left(\frac{1}{n_p}\right) - S\left(\frac{1}{n_p^2}\right) &= \frac{1}{\bar{n}_p} \left\{ 1 - \frac{1}{N} - \zeta \left(\frac{2}{N} + \frac{3}{N^2} \right) - \zeta^2 \left(1 + \frac{9}{N} \right) - 5\zeta^3 \right\}, \\ S\left(\frac{1}{n_p^2}\right) - S\left(\frac{1}{n_p^3}\right) &= \frac{1}{\bar{n}_p^2} \left\{ 1 - \frac{1}{N} + \zeta \left(2 - \frac{2}{N} - \frac{5}{N^2} \right) + \zeta^2 \left(5 - \frac{5}{N} \right) + 15\zeta^3 \right\}. \end{aligned}$$

Accordingly if ${}_p\mu_2$ be the second moment of the n_p array in a sample of N ,

$$\begin{aligned} \sigma_{{}_p\mu_2}^2 &= \frac{2{}_p\mu_2^2}{\bar{n}_p} \left[\left\{ 1 - \frac{1}{N} - \left(\frac{2}{N} + \frac{3}{N^2} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) - \left(1 + \frac{9}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 - 5 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \right. \\ &\quad \left. + \frac{1}{2} ({}_p\beta_2 - 3) \left\{ \left(1 - \frac{1}{N} \right)^2 - \left(1 + \frac{3}{N} + \frac{1}{N^2} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) - \left(3 + \frac{12}{N} \right) \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 \right. \right. \\ &\quad \left. \left. - 10 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^3 \right\} \right] \dots\dots(N). \end{aligned}$$

It is usually given the value $\frac{2{}_p\mu_2^2}{\bar{n}_p}$, and further the assumption is made that $(\delta\sigma_{n_p})^2$ may be neglected in $2\sigma_{n_p}\delta\sigma_{n_p} + (\delta\sigma_{n_p})^2 = \delta{}_p\mu_2$, so that we obtain the value

$$\sigma_{\sigma_{n_p}} = \frac{\sigma_{\bar{n}_p}}{\sqrt{2\bar{n}_p}} \dots\dots(O).$$

Now whatever may be said for this result the method by which it is reached is distinctly defective and this not merely because it assumes normality. We have in fact for any distribution of size M

$$\sigma = \sqrt{\mu_2}.$$

Now let us measure σ from the mean value $\bar{\sigma}$ of the sampled population and μ_2

from the mean value $\tilde{\mu}_2$ of μ_2 in the samples. Then $\tilde{\mu}_2$ will not be $\bar{\mu}_2$ but be equal to $\left(1 - \frac{1}{M}\right)\bar{\mu}_2$. Accordingly

$$\begin{aligned}\bar{\sigma} + \delta\sigma &= \sqrt{\left(1 - \frac{1}{M}\right)\bar{\mu}_2 + \delta\mu_2} \\ &= \bar{\sigma} \sqrt{1 - \left(\frac{1}{M} - \frac{\delta\mu_2}{\bar{\mu}_2}\right)},\end{aligned}$$

or expanding

$$\begin{aligned}\delta\sigma &= \bar{\sigma} \left\{ -\frac{1}{2M} \left(1 + \frac{1}{4M} + \frac{1}{8M^2} + \frac{5}{64M^3}\right) \right. \\ &\quad + \frac{1}{2} \frac{\delta\mu_2}{\bar{\mu}_2} \left(1 + \frac{1}{2M} + \frac{3}{8M^2} + \frac{5}{16M^3}\right) \\ &\quad - \frac{1}{8} \left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^2 \left(1 + \frac{3}{2M} + \frac{15}{8M^2}\right) \\ &\quad \left. + \frac{1}{16} \left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^3 \left(1 + \frac{5}{2M}\right) - \frac{5}{128} \left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^4 + \dots \right\} \dots\dots\dots(\text{P}).\end{aligned}$$

Now we need first the mean value of $\delta\sigma$ and for this purpose require the mean powers of $\frac{\delta\mu_2}{\bar{\mu}_2}$. These will be about the mean value of μ_2 in samples, i.e. $\left(1 - \frac{1}{M}\right)\bar{\mu}_2$ and are*, if we use curved brackets to represent means:

$$\begin{aligned}\left\{\left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)\right\} &= 0, \quad \left\{\left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^2\right\} = \left[\frac{1}{M} \left(1 - \frac{1}{M}\right)^2 \bar{\beta}_2 - \frac{1}{M} \left(1 - \frac{1}{M}\right) \left(1 - \frac{3}{M}\right)\right], \\ \left\{\left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^3\right\} &= \frac{1}{M^2} \left[(\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2) - \frac{1}{M} (3\bar{\beta}_4 - 21\bar{\beta}_2 - 18\bar{\beta}_1 + 26) \right. \\ &\quad \left. + \frac{1}{M^2} (3\bar{\beta}_4 - 33\bar{\beta}_2 - 22\bar{\beta}_1 + 54) + \dots \right], \\ \left\{\left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^4\right\} &= \frac{1}{M^2} \left[3(\bar{\beta}_2 - 1)^2 + \frac{1}{M} (\bar{\beta}_6 - 4\bar{\beta}_4 - 15\bar{\beta}_2^2 - 24\bar{\beta}_3 + 48\bar{\beta}_2 + 96\bar{\beta}_1 - 30) \right. \\ &\quad \left. - \frac{1}{M^2} (4\bar{\beta}_6 - 40\bar{\beta}_4 - 54\bar{\beta}_2^2 - 96\bar{\beta}_3 + 336\bar{\beta}_2 + 528\bar{\beta}_1 - 306) + \dots \right] \dots(\text{Q}),\end{aligned}$$

where as usual

$$\bar{\beta}_{2r} = \bar{\mu}_{2r+2}/\bar{\mu}_2^{r+1}, \text{ and } \bar{\beta}_{2r+1} = \bar{\mu}_{2r+3} \times \bar{\mu}_3/\bar{\mu}_2^{r+3},$$

and have reference to the sampled population.

Substituting in (P) we have

Mean $\sigma = \bar{\sigma} + \{\delta\sigma\} = \bar{\sigma} (1 - \lambda\sigma)$ say

$$= \bar{\sigma} \left[1 - \frac{1}{8M} (\bar{\beta}_2 + 3) + \frac{1}{128} \frac{1}{M^2} (8\bar{\beta}_4 - 15\bar{\beta}_2^2 + 14\bar{\beta}_2 - 48\bar{\beta}_1 - 55) \right] \dots(\text{R}).$$

* The value for $\left\{\left(\frac{\delta\mu_2}{\bar{\mu}_2}\right)^2\right\}$ is well-known, the two later values have been recently given by Professor Tchouproff, *Biometrika*, Vol. XII, p. 194.

In the special case of a normal distribution this reduces to

$$\text{Mean } \sigma = \bar{\sigma} \left[1 - \frac{3}{4M} - \frac{7}{32} \frac{1}{M^2} - \dots \right] \dots\dots\dots(\text{S}).$$

This agrees with the value given in *Biometrika*, Vol. x, p. 526, Equation (xv), which is now generalised in (R) for a sample from material following any frequency distribution given by $\bar{\beta}_4$, $\bar{\beta}_2$ and $\bar{\beta}_1$.

We must now adapt the result (R) for the array n_p of a sample of size N . We need only to replace $\frac{1}{M}$ and $\frac{1}{M^2}$ by $S\left(\frac{1}{n_p}\right)$ and $S\left(\frac{1}{n_p^2}\right)$ of our p. 273 and retain up to terms in $1/\bar{n}_p$. We find

$$\text{Mean } \sigma_{n_p} = \sigma_{\bar{n}_p} \left[1 - \frac{1}{8} \frac{\bar{\beta}_2 + 3}{\bar{n}_p} \left(1 - \frac{1}{N} \right) + \frac{1}{128} \frac{1}{\bar{n}_p^2} (8\bar{\beta}_4 - 15\bar{\beta}_2^2 - 2\bar{\beta}_2 - 48\bar{\beta}_1 - 103) \right] \dots\dots\dots(\text{T}).$$

This becomes in the case of normal frequency

$$\sigma_{n_p} = \sigma_{\bar{n}_p} \left[1 - \frac{3}{4\bar{n}_p} - \frac{31}{32\bar{n}_p^2} - \dots \right] \dots\dots\dots(\text{T}^{bis}).$$

We can now find σ_σ from (P). Subtracting the mean value $\{\delta\sigma\} = -\bar{\sigma}\lambda_\sigma$ from $\delta\sigma$ we have, if $\lambda_{\sigma'} = \lambda_\sigma - \frac{1}{2M} + \frac{1}{8M^2}$,

$$\begin{aligned} \delta\sigma - \{\delta\sigma\} = \bar{\sigma} \left\{ \lambda_{\sigma'} + \frac{1}{2} \frac{\delta\mu_2}{\bar{\mu}_2} \left(1 + \frac{1}{2M} + \frac{3}{8M^2} \right) \right. \\ \left. - \frac{1}{8} \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^2 \left(1 + \frac{3}{2M} \right) \right. \\ \left. + \frac{1}{16} \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^3 - \frac{5}{128} \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^4 + \dots \right\} \dots\dots\dots(\text{U}). \end{aligned}$$

Hence squaring and taking means we find

$$\begin{aligned} \sigma_\sigma^2 = \bar{\sigma}^2 \left[\lambda_{\sigma'}^2 - \frac{1}{4} \lambda_{\sigma'} \left\{ \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^2 \right\} + \frac{1}{4} \left\{ \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^2 \right\} \left(1 + \frac{1}{M} \right) \right. \\ \left. - \frac{1}{8} \left\{ \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^3 \right\} \left(1 + \frac{2}{M} \right) + \frac{5}{64} \left\{ \left(\frac{\delta\mu_2}{\bar{\mu}_2} \right)^4 \right\} + \dots \right] \\ = \bar{\sigma}^2 \left[\frac{\bar{\beta}_2 - 1}{4M} - \frac{1}{32M^2} (4\bar{\beta}_4 - 7\bar{\beta}_2^2 + 10\bar{\beta}_2 - 24\bar{\beta}_1 - 23) \right] \dots\dots\dots(\text{V}); \end{aligned}$$

$$\text{or} \quad \sigma_\sigma = \frac{\bar{\sigma} \sqrt{\bar{\beta}_2 - 1}}{2 \sqrt{M}} \left[1 - \frac{1}{16M} \frac{4\bar{\beta}_4 - 7\bar{\beta}_2^2 + 10\bar{\beta}_2 - 24\bar{\beta}_1 - 23}{\bar{\beta}_2 - 1} \right] \dots\dots\dots(\text{W}).$$

For a normal distribution this becomes*

$$\sigma_\sigma = \frac{\bar{\sigma}}{\sqrt{2M}} \left(1 - \frac{1}{8M} \right) = \frac{\bar{\sigma}}{\sqrt{2} \left(M + \frac{1}{4} \right)} \text{ nearly } \dots\dots\dots(\text{X}).$$

* The value is in agreement with that given in *Biometrika*, Vol. x, p. 526, Equation (xvi).

Formula (X) shows that even for small values of M we have a value only slightly less than the usual $\frac{\bar{\sigma}}{\sqrt{2M}}$.

Turning to the array n_p in a sample of N we require the mean values of

$$S\left(\frac{1}{n_p}\right)^{\frac{1}{2}} \text{ and } S\left(\frac{1}{n_p}\right)^{\frac{3}{2}},$$

$$\text{or } S\left(\frac{1}{n_p}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\bar{n}_p}} \left\{ 1 + \frac{3}{8} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(1 + \frac{5}{6N} \right) + \frac{65}{128} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 + \dots \right\},$$

$$S\left(\frac{1}{n_p}\right)^{\frac{3}{2}} = \frac{1}{\bar{n}_p \sqrt{\bar{n}_p}} \left\{ 1 + \frac{15}{8} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \left(1 + \frac{7}{6N} \right) + \frac{665}{128} \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right)^2 + \dots \right\} \dots (Y).$$

Substituting in (X) we find

$$\bar{\sigma}_{\sigma_{n_p}} = \frac{\sigma_{\bar{n}_p} \sqrt{\bar{\beta}_2 - 1}}{2 \sqrt{\bar{n}_p}} \left(1 - \frac{3}{8N} - \frac{1}{16\bar{n}_p} \frac{4\bar{\beta}_4 - 7\bar{\beta}_2^2 + 4\bar{\beta}_2 - 24\bar{\beta}_1 - 17}{\bar{\beta}_2 - 1} + \dots \right) \dots (Z).$$

For a normal distribution we have*

$$\bar{\sigma}_{\sigma_{n_p}} = \frac{\sigma_{\bar{n}_p}}{\sqrt{2\bar{n}_p}} \left(1 - \frac{3}{8N} + \frac{1}{2\bar{n}_p} \right) \dots \dots \dots (AA).$$

Thus the usual value $\sigma_{\bar{n}_p}/\sqrt{2\bar{n}_p}$ will be about 2% in defect in an array of 25 in a sample of 1000.

It will be realised therefore that if we do not take arrays of less than 25 in samples of 1000, the usual values of the mean standard-deviation of an array n_p and the standard-deviation of these standard-deviations will not lead us badly astray. We have finally to ask what degree of weight we can give to the "probable error" of this standard-deviation, i.e. to $67449\bar{\sigma}_{\sigma_{n_p}}$. This is only determining how far σ_{n_p} follows the normal law of distribution, that is to say, how nearly are ${}_pB_1'$ and ${}_pB_2' - 3$ zero for the distribution of σ_{n_p} , these representing the first two β -coefficients for this distribution. Before we do this, however, we may find from (Q), p. 276, the values of ${}_MB_1'$, ${}_MB_2'$ for the μ_2 of samples of constant size M to the second order of approximation. They are

$${}_MB_1' = \frac{1}{M} \frac{(\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2)^2}{(\bar{\beta}_2 - 1)^3} \left[1 + \frac{6}{M} \left(\frac{\bar{\beta}_2 - 2}{\bar{\beta}_2 - 1} - \frac{3\bar{\beta}_4 - 21\bar{\beta}_2 - 18\bar{\beta}_1 + 26}{3\bar{\beta}_4 - 9\bar{\beta}_2 - 18\bar{\beta}_1 + 6} \right) \right] \dots \dots (BB),$$

reducing with a Gaussian distribution to†

$${}_MB_1' = \frac{8}{M} \left(1 + \frac{1}{M} \right) \dots \dots \dots (CC),$$

* The reader should note that $\bar{\sigma}_{\sigma_{n_p}}$ is not the same thing as if we put

$$\bar{\sigma}_{\sigma_{n_p}} = \frac{1}{2\sigma_{\bar{n}_p}}, \quad \sigma_{\mu_2} = \frac{\sigma_{\bar{n}_p}}{\sqrt{2\bar{n}_p}} \left\{ 1 - \frac{1}{2N} + \frac{1}{N^2} - \frac{1}{\bar{n}_p N} - \frac{1}{\bar{n}_p^2} \right\}$$

from (N) on p. 275.

† These values are identical to the degree of approximation adopted with those given by "Student," in *Biometrika*, Vol. VI, p. 4. Professor Tchouproff (*Ibid.*, Vol. XII, p. 192) does scant justice to "Student." The only misprint we can see is that $1/n$ appears in the first term instead of $1/n^3$, the power having probably been 'drawn' in printing; it re-appears in the next equation. Further "Student" gives not only Tchouproff's (19) but his (22); this as "Student" himself indicates (p. 4) involved Tchouproff's lengthy equations (20) and (21), which he refrained owing to their length from publishing.

and
$${}_MB_2' - 3 = \frac{1}{M} \frac{\bar{\beta}_6 - 4\bar{\beta}_4 - 3\bar{\beta}_2^2 - 24\bar{\beta}_3 + 12\bar{\beta}_2 + 96\bar{\beta}_1 - 6}{(\bar{\beta}_2 - 1)^2} \dots\dots(\text{DD}),$$

reducing with a Gaussian distribution to*

$${}_MB_2' - 3 = \frac{12}{M}.$$

It will be observed that the approach to the normal curve is by no means close for fairly small samples. For example, if $M=24$, we might easily have ${}_MB_1' = \cdot 3$ and ${}_MB_2' = 3\cdot 5$. In other words the distribution of μ_2 in samples of size M is far from as close to a normal curve as the distribution of means μ_1' .

We should anticipate accordingly that the distribution of the second moment-coefficient in the n_p array of a sample N would be even further removed.

We find by replacing $1/M^s$ by $S\left(\frac{1}{n_p^s}\right)$ that

$$\begin{aligned} \left\{\left(\frac{d\mu_2}{\bar{\mu}_2}\right)^2\right\} &= \frac{\bar{\beta}_2 - 1}{\bar{n}_p} \left\{\left(1 - \frac{1}{N}\right) - \frac{1}{\bar{n}_p} \frac{\bar{\beta}_2 - 3}{\bar{\beta}_2 - 1}\right\}, \\ \left\{\left(\frac{d\mu_2}{\bar{\mu}_2}\right)^3\right\} &= \frac{\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2}{\bar{n}_p^2} \left(1 - \frac{3}{N} + \frac{4}{\bar{n}_p} \frac{3\bar{\beta}_2 - 5}{\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2}\right), \\ \left\{\left(\frac{d\mu_2}{\bar{\mu}_2}\right)^4\right\} &= \frac{(\bar{\beta}_2 - 1)^3}{\bar{n}_p^2} \left(3 - \frac{9}{N} + \frac{1}{\bar{n}_p} \frac{\bar{\beta}_6 - 4\bar{\beta}_4 - 6\bar{\beta}_2^2 - 24\bar{\beta}_3 + 30\bar{\beta}_2 + 96\bar{\beta}_1 - 21}{(\bar{\beta}_2 - 1)^2}\right) \end{aligned} \dots\dots\dots(\text{EE}).$$

Hence

$${}_{n_p}B_1' = \frac{1}{\bar{n}_p} \frac{\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2}{(\bar{\beta}_2 - 1)^3} \left\{1 - \frac{3}{N} + \frac{1}{\bar{n}_p} \left(\frac{3(\bar{\beta}_2 - 3)}{\bar{\beta}_2 - 1} + \frac{8(3\bar{\beta}_2 - 5)}{\bar{\beta}_4 - 3\bar{\beta}_2 - 6\bar{\beta}_1 + 2}\right)\right\} \dots\dots\dots(\text{FF}),$$

giving for a Gaussian distribution

$${}_{n_p}B_1' = \frac{8}{\bar{n}_p} \left(1 + \frac{4}{\bar{n}_p} - \frac{3}{N}\right);$$

and

$${}_{n_p}B_2' - 3 = \frac{1}{\bar{n}_p} \frac{\bar{\beta}_6 - 4\bar{\beta}_4 - 24\bar{\beta}_3 + 6\bar{\beta}_2^2 + 96\bar{\beta}_1 - 3}{(\bar{\beta}_2 - 1)^2} - \frac{3}{N} \dots\dots\dots(\text{GG}),$$

giving for a Gaussian distribution

$${}_{n_p}B_2' - 3 = \frac{15}{\bar{n}_p} - \frac{3}{N}.$$

Thus for a small array of, say, 25 we might easily have ${}_{n_p}B_1' = \cdot 37$ and ${}_{n_p}B_2' = 3\cdot 6$, values very remote from a Gaussian distribution.

It is clear therefore that the "probable error" of a second moment-coefficient has no very illuminating meaning in the case of the arrays of small or even moderate size in the case of a sample of size N . It may, however, be remarked that the distribution of μ_2 is one thing and that of σ_{n_p} which is what we usually require is another. In order to obtain this we must raise the expression in (U)

* See the second footnote on preceding page.

for $\delta\sigma - \{\delta\sigma\}$ to the third and fourth powers. But if we keep only terms of the two lowest orders in our results we require to ascertain the values of

$$\left\{\left(\frac{d\mu_2}{\bar{\mu}_2}\right)^5\right\} \text{ and } \left\{\left(\frac{d\mu_2}{\bar{\mu}_2}\right)^6\right\}.$$

The necessary term in the latter is $\frac{15}{\bar{n}_p^3}(\beta_2 - 1)^3$, but the finding of the former is far more troublesome and we have not so far succeeded in determining it. But it is possible to obtain some idea of the deviation of the σ_{n_p} curve from normality by considering what its β 's are in the case of the sample being made on a population following the normal law. We are able to do this, if we carry a stage further the work outlined in a paper in *Biometrika*, Vol. x, p. 526. We require the values of Σ , σ_{Σ^2} and μ_3 , μ_4 on p. 526 carried to a higher approximation by the introduction of the additional term in the Stirling's Theorem expression for the factorial. Miss Pairman has carried this out and finds

$$\bar{\Sigma}^2 = \sigma^2 \left(1 - \frac{3}{2n} + \frac{1}{8n^2} + \frac{3}{16n^3}\right) \dots\dots\dots(\text{HH})$$

which leads, having regard to (xi) on p. 525, to

$$\Sigma\mu_2 = \sigma_{\Sigma^2} = \mu_2' - \bar{\Sigma}^2 = \frac{\sigma^2}{2n} \left(1 - \frac{1}{4n} - \frac{3}{8n^2}\right) \dots\dots\dots(\text{II}).$$

We must now turn to the equations on pp. 527—8 to determine μ_3 and μ_4 . We find*:

$$\begin{aligned} \Sigma\mu_3 &= \frac{\sigma^2 \bar{\Sigma}}{4n^2} \left(1 + \frac{3}{2n}\right) = \frac{\sigma^3}{4n^2} \left(1 + \frac{3}{2n}\right) \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3}\right) \\ &= \frac{\sigma^3}{4n^2} \left(1 + \frac{3}{4n}\right) \text{ as far as our approximation is valid.} \dots\dots\dots(\text{KK}). \end{aligned}$$

$$\begin{aligned} \text{Again } \Sigma\mu_4 &= \sigma^4 \left\{ \frac{5}{4n^2} - \frac{1}{2n} \left(4 - \frac{3}{n}\right) \left(\frac{1}{4n} + \frac{3}{8n^2}\right) - \frac{3}{4n^2} \left(\frac{1}{4n} + \frac{3}{8n^2}\right)^2 \right\} \\ &= \frac{3\sigma^4}{4n^2} \left(1 - \frac{1}{2n}\right) \dots\dots\dots(\text{LL}). \end{aligned}$$

We must now replace n by $\bar{n}_p + \delta n_p$ and sum as we have frequently had occasion to do.

$$\begin{aligned} \text{We have: } \sigma_{n_p \mu_2} &= \frac{\sigma^2 \bar{n}_p}{2\bar{n}_p} \left(1 + \frac{3}{4\bar{n}_p} - \frac{1}{N}\right) \dots\dots\dots(\text{MM}), \\ \sigma_{n_p \mu_3} &= \frac{\sigma^3 \bar{n}_p}{4\bar{n}_p^2} \left(1 + \frac{15}{4\bar{n}_p} - \frac{3}{N}\right) \dots\dots\dots(\text{NN}), \\ \sigma_{n_p \mu_4} &= \frac{3\sigma^4 \bar{n}_p}{4\bar{n}_p^2} \left(1 + \frac{5}{2\bar{n}_p} - \frac{3}{N}\right) \dots\dots\dots(\text{OO}). \end{aligned}$$

* These values may be used to determine the nature of the distribution of Σ in samples of constant size n . We have:

$$\Sigma B_1 = \frac{1}{2n} \left(1 + \frac{9}{4n}\right), \quad \Sigma B_2 = 3 + \frac{0}{n}.$$

The term in $1/n^2$ in ΣB_2 could not be determined unless we went to a still higher order term in $\bar{\Sigma}$. Clearly for a sample of 25 ΣB_1 approaches close to the Gaussian and ΣB_2 still closer. The non-approximate values are .0219 and .0014 (*loc. cit.*, p. 529).

Hence

$$\sigma_{n_p} B_1 = \frac{1}{2\bar{n}_p} \left(1 + \frac{21}{4\bar{n}_p} - \frac{3}{N} \right) \dots\dots\dots (PP),$$

$$\sigma_{n_p} B_2 = 3 + 3 \left(\frac{1}{\bar{n}_p} - \frac{1}{N} \right) \dots\dots\dots (QQ).$$

Accordingly we see that when samples N are taken from normal material the array n_p of varying size in these samples will not differ very greatly from normality. For example if $\bar{n}_p = 25$ and the sample N be 1000, we shall have $\sigma_{n_p} B_1 = .024$ and $\sigma_{n_p} B_2 = 3.12$, showing no great deviation from normality, although more than in the case of a sample of constant size. It is probable that the deviation from normality will be somewhat greater when the sampled population itself is not normal. Still it is important to note that the distribution of σ_{n_p} for all cases is likely to be far closer to the normal, and the "probable error" of σ_{n_p} therefore more intelligible, than is the case with $n_p \mu_2$. In the same way it is extremely probable that the distributions of $(n_p \mu_3)^{\frac{1}{3}}$ and $(n_p \mu_4)^{\frac{1}{4}}$ are more nearly normal than those of $n_p \mu_3$ and $n_p \mu_4$.

(V) *Mathematical Contributions to the Theory of Evolution.* XIX. *Second Supplement to a Memoir on Skew Variation.* *Phil. Trans.* Series A—Vol. 216, pp. 429—457.

There are one or two corrections to be made in this paper by Pearson:

(a) p. 439, l. 18. The printer has drawn the solidus and the $3\beta_1 - 2\beta_2 + 6$, which followed it. Thus

$$4 - m = 2(\beta_2 + 3)$$

should be read as

$$4 - m = 2(\beta_2 + 3)/(3\beta_1 - 2\beta_2 + 6);$$

(b) p. 441, l. 18, about the middle of the page the equation

$$\gamma = 12(\sec \theta - \operatorname{cosec} \theta)$$

is given. It should be

$$\gamma = 3 \times 12(\sec \theta - \operatorname{cosec} \theta)^2 / \sec \theta,$$

but no use has been made of the equation in the paper.

QUADRATURE COEFFICIENTS.

In a large amount of recent quadrature work we have found that Sheppard's formula (c) given in *Biometrika*, 1, p. 276, gives very satisfactory results, and Mr P. F. Everitt has tabled the values of the three coefficients. His manuscript table has proved so useful that we reproduce it here, as others may also find it a help. It will eventually appear in the *Tables for Statisticians and Biometricians*. The formula supposes the quadrated area to be divided into p trapezettes on bases of equal size h . Then A_C the chordal area is given by

$$A_C = h \left(\frac{1}{2} z_0 + z_1 + z_2 + \dots + z_{p-1} + \frac{1}{2} z_p \right),$$

where $z_0, z_1, z_2 \dots z_{p-1}, z_p$ are the equally spaced ordinates.

The required area of the curve is then

$$\begin{aligned} \text{Area} = A_C &+ C_1 \{ (z_1 - z_0) - (z_p - z_{p-1}) \} h \\ &- C_2 \{ (z_2 - z_1) - (z_{p-1} - z_{p-2}) \} h \\ &+ C_3 \{ (z_3 - z_2) - (z_{p-2} - z_{p-3}) \} h. \end{aligned}$$

Here C_1, C_2, C_3 are certain functions of p and are provided for each value of p in the accompanying Table. They are selected to give the best result, provided we stop at third terminal differences.

Quadrature Coefficients for Sheppard's Formula (c). - Biom. Vol. I, p. 276.

<i>p</i>	C_1	C_2	C_3	<i>p</i>	C_1	C_2	C_3
6	+·2071429	+·3357143	+·4714286	54	+·1557006	+·1034971	+·0311473
7	·1957755	·2532407	·2108218	55	·1556195	·1033186	·0310489
8	·1883296	·2124868	·1369312	56	·1555414	·1031470	·0309545
9	·1830517	·1882653	·1037946	57	·1554662	·1029820	·0308639
10	·1791068	·1722411	·0854119	58	·1553936	·1028231	·0307767
11	·1760430	·1608670	·0738621	59	·1553235	·1026699	·0306929
12	·1735931	·1523810	·0659864	60	·1552559	·1025223	·0306122
13	·1715884	·1458097	·0602962	61	·1551906	·1023799	·0305345
14	·1699171	·1405724	·0560045	62	·1551274	·1022424	·0304596
15	·1685022	·1363012	·0526583	63	·1550663	·1021096	·0303874
16	·1672888	·1327519	·0499796	64	·1550071	·1019813	·0303177
17	·1662365	·1297561	·0477890	65	·1549498	·1018571	·0302503
18	·1653151	·1271939	·0459655	66	·1548944	·1017370	·0301852
19	·1645017	·1249776	·0444247	67	·1548406	·1016207	·0301223
20	·1637782	·1230418	·0431061	68	·1547884	·1015081	·0300614
21	·1631305	·1213364	·0419654	69	·1547378	·1013990	·0300025
22	·1625472	·1198227	·0409690	70	·1546887	·1012931	·0299445
23	·1620193	·1184701	·0400913	71	·1546410	·1011905	·0298902
24	·1615391	·1172542	·0393126	72	·1545946	·1010909	·0298366
25	·1611004	·1161553	·0386169	73	·1545496	·1009941	·0297846
26	·1606982	·1151573	·0379919	74	·1545058	·1009002	·0297341
27	·1603280	·1142469	·0374272	75	·1544632	·1008089	·0296852
28	·1599861	·1134131	·0369147	76	·1544218	·1007202	·0296376
29	·1596694	·1126466	·0364473	77	·1543814	·1006339	·0295914
30	·1593752	·1119396	·0360195	78	·1543422	·1005499	·0295465
31	·1591013	·1112855	·0356265	79	·1543039	·1004682	·0295029
32	·1588455	·1106784	·0352641	80	·1542666	·1003886	·0294604
33	·1586062	·1101136	·0349290	81	·1542303	·1003111	·0294190
34	·1583817	·1095867	·0346181	82	·1541948	·1002356	·0293788
35	·1581708	·1090941	·0343290	83	·1541603	·1001621	·0293396
36	·1579723	·1086325	·0340595	84	·1541265	·1000903	·0293015
37	·1577850	·1081991	·0338076	85	·1540936	·1000204	·0292643
38	·1576082	·1077914	·0335717	86	·1540615	·0999521	·0292280
39	·1574408	·1074072	·0333502	87	·1540301	·0998856	·0291926
40	·1572822	·1070444	·0331420	88	·1539995	·0998206	·0291582
41	·1571317	·1067014	·0329458	89	·1539695	·0997571	·0291245
42	·1569887	·1063765	·0327607	90	·1539403	·0996951	·0290917
43	·1568527	·1060684	·0325857	91	·1539116	·0996346	·0290596
44	·1567231	·1057759	·0324201	92	·1538837	·0995754	·0290283
45	·1565996	·1054976	·0322630	93	·1538563	·0995175	·0289977
46	·1564816	·1052328	·0321139	94	·1538295	·0994610	·0289678
47	·1563688	·1049803	·0319722	95	·1538034	·0994057	·0289386
48	·1562610	·1047393	·0318373	96	·1537777	·0993516	·0289101
49	·1561577	·1045091	·0317088	97	·1537526	·0992987	·0288821
50	·1560587	·1042890	·0315861	98	·1537280	·0992469	·0288548
51	·1559637	·1040783	·0314690	99	·1537040	·0991962	·0288281
52	·1558725	·1038765	·0313571	100	·1536804	·0991465	·0288020
53	·1557849	·1036829	·0312449				

ON GENERALISED TCHEBYCHEFF THEOREMS IN THE MATHEMATICAL THEORY OF STATISTICS.

BY KARL PEARSON, F.R.S.

(1) *Single Variate.*

Let $y = \phi(x)$ be any law of frequency and let the limits of the distribution be a and b , then if N be the total frequency,

$$N = \int_a^b \phi(x) dx,$$

and if $\bar{x}$ be the mean value of the variate,

$$N\bar{x} = \int_a^b x \phi(x) dx.$$

Generally, if μ_s be the s th moment-coefficient about the mean,

$$N\mu_s = \int_a^b (x - \bar{x})^s \phi(x) dx.$$

Now consider

$$\mu_{2s} = \frac{1}{N} \int_a^b (x - \bar{x})^{2s} \phi(x) dx,$$

and let ϵ be any value of $x - \bar{x}$, then

$$\mu_{2s}/\epsilon^{2s} = \frac{1}{N} \int_a^b \frac{(x - \bar{x})^{2s}}{\epsilon^{2s}} \phi(x) dx.$$

Now pick out all the values for which $x - \bar{x}$ is greater than ϵ , and let us suppose $b > a$; then

$$\mu_{2s}/\epsilon^{2s} > \frac{1}{N} \int_{\epsilon + \bar{x}}^b \frac{(x - \bar{x})^{2s}}{\epsilon^{2s}} \phi(x) dx,$$

and therefore

$$\mu_{2s}/\epsilon^{2s} > \frac{1}{N} \int_{\epsilon + \bar{x}}^b \phi(x) dx,$$

since $(x - \bar{x})/\epsilon$ is always greater than unity.

But $\frac{1}{N} \int_{\epsilon + \bar{x}}^b \phi(x) dx$ is the chance of an individual occurring with a deviation greater than ϵ from the mean $= 1 - P$ where P is the chance of an individual occurring with a deviation less than ϵ . Hence

$$P > 1 - \frac{\mu_{2s}}{\epsilon^{2s}}.$$

Now let $\epsilon = \lambda\sigma$, where $\sigma = \sqrt{\mu_2}$ is the standard deviation of the distribution.

Thus the chance of a deviation being of less magnitude than $\lambda\sigma$ is

$$P > 1 - \frac{1}{\lambda^{2s}} \cdot \frac{\mu_{2s}}{\mu_2^s} \dots\dots\dots (i).$$

If we put $s = 1$, the chance P of a deviation less than $\lambda\sigma$ is limited by

$$P > 1 - \frac{1}{\lambda^2} \dots\dots\dots (ii).$$

This special case is Tchebycheff's Theorem*.

Inequality (i) gives our first generalisation for a single variate of Tchebycheff's Theorem in (ii)†. We can now compare the accuracy of (i) and (ii) by supposing them applied to a normal distribution of frequency for the cases of deviations 1, 2, 3 and 4 times the standard deviation. In this case

$$\mu_{2s} = (2s-1)(2s-3) \dots 1 \mu_2^s.$$

TABLE I.

Values of Lower Limit for P given by $1 - \frac{(2s-1)(2s-3) \dots 1}{\lambda^{2s}}$.

s	$\lambda=1.5$	$\lambda=2$	$\lambda=3$	$\lambda=4$
1	·5556	·7500	·8889	·9375
2	·4074	·8125	·9630	·9883
3	— ·3169	·7656	·9794	·9963
4	—	—	·9840	·9984
5	—	—	·9840	·9991
6	—	—	·9804	·9994
7	—	—	—	·99950
8	—	—	—	·99953
9	—	—	—	·99950
10	—	—	—	·99940
Actual value of P	·8664	·9545	·9970	·99994

Clearly the maximum for any λ will be found by making $(2s-1)/\lambda^2$ equal to unity, or if λ^2 = an odd number, $s = \frac{1}{2}(\lambda^2 + 1)$ and $\frac{1}{2}(\lambda^2 + 1) - 1$ will give equal limits. If λ^2 be an even number then $s = \frac{1}{2}\lambda^2$ will give the highest limit.

* It was first proved in the *Recueil des sciences mathématiques*, T. II, according to Liouville, but I cannot trace this reference at all. It was translated from Russian into French in Liouville's *Journal de mathématiques*, Vol. XII, pp. 177—184, Paris, 1867. The proof there given is somewhat lengthy and at first sight the result might appear more general than (ii); but this is not so. Assume $x = u + v + w + \dots$ and suppose u, v, w uncorrelated, so that $\sigma_x^2 = \sigma_u^2 + \sigma_v^2 + \sigma_w^2 + \dots$ then we have with minor differences of notation and terminology (especially the use of the words "mathematical expectation" for our moments) Tchebycheff's own phrasing of his theorem. The remark of Dr Anderson (*Biometrika*, Vol. x, p. 269) with regard to the neglect of the theory of "mathematical expectation" by the English statistical school seems based on a misunderstanding of the moment method.

† This generalised form of Tchebycheff's Theorem was given by me in a paper for the Honours degree of the University of London in Statistics, October, 1915.

(2) *Two Variates; Limit to the Frequency within an Elliptic Area round the Mean as Centre.*

Let the law of frequency be $z = \phi(x, y)$ and let the standard deviations of x and y be σ_1, σ_2 , and r be the coefficient of correlation between x and y . Let us take as our ellipse,

$$\frac{1}{1-r^2} \left(\theta_{11} \frac{x^2}{\sigma_1^2} - 2\theta_{12} \frac{rxy}{\sigma_1\sigma_2} + \theta_{22} \frac{y^2}{\sigma_2^2} \right) = \chi^2;$$

x and y being measured as deviations from the mean.

Then by giving special values to $\theta_{11}, \theta_{12}, \theta_{22}$ and χ^2 we can get any ellipse we please. Further since the curve is to be an ellipse $r^2\theta_{12}^2 < \theta_{11}\theta_{22}$ * and we shall take θ_{11} and θ_{22} always positive. Thus χ^2 and all its powers will invariably be positive.

Now consider; if $N = \iint \phi(xy) dx dy$,

$$I_s = \frac{1}{N} \iint \phi(xy) \left\{ \frac{1}{1-r^2} \left(\theta_{11} \frac{x^2}{\sigma_1^2} - 2\theta_{12} \frac{rxy}{\sigma_1\sigma_2} + \theta_{22} \frac{y^2}{\sigma_2^2} \right) \right\}^s dx dy,$$

the integration extending all over space covered by the frequency surface. Divide both sides by χ_0^{2s} ,

$$\frac{I_s}{\chi_0^{2s}} = \frac{1}{N} \iint \phi(xy) \left(\frac{\chi}{\chi_0} \right)^{2s} dx dy.$$

Take out all the values for which χ is greater than χ_0 , then

$$\frac{I_s}{\chi_0^{2s}} > \frac{1}{N} \iint \phi(xy) \left(\frac{\chi}{\chi_0} \right)^{2s} dx dy,$$

when the integral extends over the area for which χ is $> \chi_0$. Hence

$$\begin{aligned} \frac{I_s}{\chi_0^{2s}} &> \frac{1}{N} \iint \phi(xy) dx dy \\ &> \text{chance of an observation falling outside the} \\ &\quad \text{ellipse } \chi_0. \end{aligned}$$

Let P be the chance of an observation falling inside this ellipse, then we have at once

$$P > 1 - \frac{I_s}{\chi_0^{2s}} \dots \dots \dots \text{(iii).}$$

Now we define

$$\begin{aligned} p_{ss'} &= \frac{1}{N} \iint \phi(xy) (x - \bar{x})^s (y - \bar{y})^{s'} dx dy \\ &= \frac{1}{N} \iint \phi(xy) x^s y^{s'} dx dy \end{aligned}$$

in our case, as the s, s' th product moment-coefficient about the mean. And it is very convenient to write

$$q_{ss'} = p_{ss'} / (\sigma_1^s \sigma_2^{s'}) \dots \dots \dots \text{(iv)}$$

and term $q_{ss'}$ a reduced product moment-coefficient.

* We shall generally wish to have symmetry of expression between x and y , and in this case we take $\theta_{22} = \theta_{11} = \theta$ say and write $\theta_{12}r/\theta = \rho$ and we shall have as necessary condition for the ellipse $\rho < 1$.

It is clear that by simple expansion of the trinomial expression, we can always find I_s in terms of $q_{ss'}$.

We have accordingly to study the expansion of

$$\frac{1}{(1-r^2)^s} (\theta_{11}x^2 - 2r\theta_{12}xy + \theta_{22}y^2)^s \\ = \frac{1}{(1-r^2)^s} \sum_{u=0}^{u=s} \sum_{m=0}^{m=s-u} \left\{ (-1)^u 2^u r^u \theta_{12}^u \theta_{11}^{s-m-u} \theta_{22}^m \frac{s!}{(s-u-m)! m! u!} x^{2s-u-2m} y^{2m+u} \right\},$$

and if this value be substituted in the integral expression for I_s we find

$$I_s = \frac{1}{(1-r^2)^s} \sum_{u=0}^{u=s} \sum_{m=0}^{m=s-u} \left\{ (-1)^u 2^u r^u \theta_{12}^u \theta_{11}^{s-m-u} \theta_{22}^m \frac{s!}{(s-u-m)! m! u!} q_{2s-u-2m, 2m+u} \right\} \\ \dots\dots(v).$$

The lower values can be equally readily found by the expansion of

$$\left(\theta_{11} \frac{x^2}{\sigma_1^2} - 2\theta_{12}r \frac{xy}{\sigma_1\sigma_2} + \theta_{22} \frac{y^2}{\sigma_2^2} \right)^s$$

in powers of r by aid of the binomial.

The first few cases are

$$I_1 = \frac{1}{(1-r^2)} \{ \theta_{11}q_{20} - 2\theta_{12}r q_{11} + \theta_{22}q_{02} \}, \\ I_2 = \frac{1}{(1-r^2)^2} \{ \theta_{11}^2 q_{40} + \theta_{22}^2 q_{04} + 2\theta_{11}\theta_{22}q_{22} - 4\theta_{12}r(\theta_{11}q_{31} + \theta_{22}q_{13}) + 4\theta_{12}^2 r^2 q_{22} \}, \\ I_3 = \frac{1}{(1-r^2)^3} \{ \theta_{11}^3 q_{60} + \theta_{22}^3 q_{06} + 3\theta_{11}\theta_{22}(\theta_{11}q_{42} + \theta_{22}q_{24}) \\ - 6\theta_{12}r(\theta_{11}^2 q_{51} + \theta_{22}^2 q_{15} + 2\theta_{11}\theta_{22}q_{33}) \\ + 12\theta_{12}^2 r^2(\theta_{11}q_{42} + \theta_{22}q_{24}) - 8\theta_{11}^3 r^3 q_{33} \}, \\ I_4 = \frac{1}{(1-r^2)^4} \{ \theta_{11}^4 q_{80} + \theta_{22}^4 q_{08} + 4\theta_{11}\theta_{22}(\theta_{11}^2 q_{62} + \theta_{22}^2 q_{26}) + 6\theta_{11}^2 \theta_{22}^2 q_{44} \\ - 8\theta_{12}r(\theta_{11}^3 q_{71} + 3\theta_{11}\theta_{22}(\theta_{11}q_{53} + \theta_{22}q_{35}) + \theta_{22}^3 q_{17}) \\ + 24\theta_{12}^2 r^2(\theta_{11}^2 q_{62} + \theta_{22}^2 q_{26} + 2\theta_{11}\theta_{22}q_{44}) \\ - 32\theta_{12}^3 r^3(\theta_{11}q_{53} + \theta_{22}q_{35}) + 16r^4 \theta_{12}^4 q_{44} \} \\ \dots\dots(vi).$$

These expressions simplify for various cases, but it is clear that for the general case of unknown type of distribution we shall have to find very high product moments from the observations in order to use our generalised Tchebycheff's Theorem. Otherwise we shall have to make assumptions as to the relations between high order and low order q 's.

Since generally $q_{20} = q_{02} = 1$ and $q_{11} = r$, we have

$$I_1 = \frac{1}{1-r^2} (\theta_{11} + \theta_{22} - 2\theta_{12}r^2).$$

This suggests that for all cases we are likely to get simplified results, if we take $\theta_{11} = \theta_{22} = \theta_{12} = 1$ when we find $I_1 = 2$. In other words, simplification arises if we make

our ellipse that of the normal contours, although of course for the general case this will not be a contour of equal probability, although it may roughly approximate to it.

Thus we find for this case,

$$\begin{aligned} I_1 &= 2, \\ I_2 &= \frac{1}{(1-r^2)^2} \{q_{40} + q_{04} + 2q_{22} - 4r(q_{31} + q_{13}) + 4r^2 q_{22}\}, \\ I_3 &= \frac{1}{(1-r^2)^3} \{q_{60} + q_{06} + 3(q_{42} + q_{24}) - 6r(q_{51} + q_{15} + 2q_{33}) + 12r^2(q_{42} + q_{24}) - 8r^3 q_{33}\}, \\ I_4 &= \frac{1}{(1-r^2)^4} \{q_{80} + q_{08} + 4(q_{62} + q_{26}) + 6q_{44} - 8r(q_{71} + q_{17} + 3q_{53} + 3q_{35}) \\ &\quad + 24r^2(q_{62} + q_{26} + 2q_{44}) - 32r^3(q_{53} + q_{35}) + 16r^4 q_{44}\} \dots\dots\dots(\text{vii}), \end{aligned}$$

and the general value of I_s will be

$$I_s = \frac{1}{(1-r^2)^s} \sum_{u=0}^{u=s} \sum_{m=0}^{m=s-u} S \left\{ (-1)^u 2^u r^u \frac{s!}{(s-u-m)! m! u!} q_{2s-u-2m, 2m+u} \right\}.$$

For the case of a normal distribution the q 's are all given in terms of r (*Biometrika*, Vol. XII, p. 87) and on substitution we find

$$I_1 = 2, \quad I_2 = 8, \quad I_3 = 48, \quad I_4 = 384;$$

generally $I_s = 2s(2s-2)(2s-4) \dots 2$, which can be shown directly, thus:

$$\begin{aligned} I_s &= \iint \phi(x, y) \frac{1}{1-r^2} \left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right)^s dx dy \\ &= \int_0^\infty e^{-\frac{1}{2}\chi^2} \chi^{2s} \chi d\chi = 2s I_{s-1}, \end{aligned}$$

if we integrate by parts,

$$\begin{aligned} &= 2s(2s-2)(2s-4) \dots 2 \times \int_0^\infty e^{-\frac{1}{2}\chi^2} \chi d\chi \\ &= 2s(2s-2)(2s-4) \dots 2. \end{aligned}$$

Accordingly our generalised Tchebycheff's limit becomes

$$P > 1 - \frac{2s(2s-2)(2s-4) \dots 2}{\chi_0^{2s}} \dots\dots\dots(\text{viii})^*,$$

and our best value of s will be determinable from $2s < \chi_0^2$, or s must be the greatest integer less than or the integer equal to $\frac{1}{2}\chi_0^2$.

Now the actual volume of the frequency surface inside the contour

$$\chi_0^2 = \frac{1}{1-r^2} \left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right)$$

is known to be $1 - e^{-\frac{1}{2}\chi_0^2}$, and it is thus easy to test the present generalised Tchebycheff limit as applied to this case.

* This result is almost at once extensible to any number of variates following the normal distribution, but as the actual value of the probability is known there is no value in writing down this limiting value.

TABLE II.

Generalised Tchebycheff Limit applied to the Probability that an association of two variables lies inside a given contour χ_0^2 of a normal frequency surface.

χ_0^2	Actual Probability	Minimum value of P
4	·8647	·5000 (I_1)
5	·9179	·6800 (I_2)
6	·9502	·7778 (I_2)
7	·9698	·8600 (I_3)
8	·9817	·9062 (I_3)
9	·9889	·9415 (I_4)
10	·9933	·9616 (I_4)
12	·9975	·9846 (I_5)
14	·9991	·9939 (I_6)
16	·9997	·9976 (I_7)
18	·9999	·9991 (I_8)
20	·99995	·99964 (I_9)

Here as in the case of a single variate the generalised Tchebycheff limit is not very useful for low values of χ_0^2 . But if in any particular type of observation we consider it desirable to look with suspicion on an observation which has occurred and yet the odds against which are greater than 50 to 1, the Tchebycheff limit may be of value. As illustration, suppose two variates are correlated with intensity ·7, what suspicion should we cast on an observation which gave the deviation of one variate 3·8 times its standard deviation and of the other 3·2 times? Here

$$\begin{aligned}
 \chi_0^2 &= \frac{1}{1-r^2} \left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right) \\
 &= \frac{1}{\cdot 51} \{ (3\cdot 8)^2 - 1\cdot 4 (3\cdot 8)(3\cdot 2) + (3\cdot 2)^2 \} \\
 &= 15\cdot 01, \text{ or say } 15.
 \end{aligned}$$

Then

$$P > 1 - \frac{2^7 (7!)}{15^7} > \cdot 9962,$$

or the odds are greater than 250 to 1 against it. Actually the probability of the occurrence of anything as unusual as or more unusual than this is ·9994, or the actual odds 1700 to 1 about. For many purposes the odds of 250 to 1 would amply suffice to mark suspicion, although of course in the case of *normal* frequency it would be as easy or even easier to calculate the real probability as the generalised Tchebycheff limit.

The chief interest of the investigation thus far is to show that unless we use an I_s of a high order the Tchebycheff limit is unlikely to be of very much service. We can obtain it in the case of material following a normal distribution, but then we know the exact result and do not need it!

I have considered very carefully the possibilities of deducing higher q 's from lower q 's for non-normal systems on various hypotheses as to the nature of the regression and the scedasticity. The simplest hypothesis is to suppose linearity of regression, homoscedasticity and homocliticity of both sets of arrays.

$$\text{Let } \beta_{28-2} = S(x^{28})/\sigma^{28}, \text{ and } \beta_{28-1}/\sqrt{\beta_1} = S(x^{28+1})/\sigma^{28+1}$$

as usual; let a single dash mark the β 's for the y variate, and double dashes the β 's for the y arrays of x 's and triple dashes the β 's for the x arrays of y 's. Then if $\bar{y}_x$ be the mean of the x -array of y 's,

$$\sqrt{\beta_1'} = \frac{1}{N} S(y^3)/\sigma_2^3 = \frac{1}{N} \Sigma S(\bar{y}_x + y')^3/\sigma_2^3,$$

where y' is measured from the mean of the array, S is the sum for all members of the array and Σ the sum for all arrays. Thus if $\bar{y}_x = \frac{r\sigma_2}{\sigma_1} x$ be the regression line,

$$\sqrt{\beta_1'} = \frac{1}{N} \left\{ r^3 \frac{\Sigma(n_x x^3)}{\sigma_1^3} + 3r^2 \frac{\Sigma(n_x x^2)}{\sigma_1^2} \frac{S(y')}{n_x \sigma_2} + 3r \frac{\Sigma(n_x x)}{\sigma_1} \frac{S(y'^2)}{n_x \sigma_2^2} + \frac{S(y'^3)}{n_x \sigma_2^3} \Sigma(n_x) \right\},$$

since $S \frac{(y'^2)}{n_x}, S \frac{(y'^3)}{n_x}$, is to be the same for every array. Thus

$$\sqrt{\beta_1'} = r^3 \sqrt{\beta_1} + \sqrt{\beta_1'''} (1 - r^2)^{\frac{3}{2}},$$

$$\text{or } \sqrt{\beta_1'''} = \frac{\sqrt{\beta_1'} - r^3 \sqrt{\beta_1}}{(1 - r^2)^{\frac{3}{2}}} \dots\dots\dots(\text{ix}).$$

$$\text{Similarly } \sqrt{\beta_1''} = \frac{\sqrt{\beta_1} - r^3 \sqrt{\beta_1'}}{(1 - r^2)^{\frac{3}{2}}} \dots\dots\dots(\text{x}).$$

Thus it is impossible in homoclitic systems for the skewness of the arrays to be equal to the skewness of the marginal totals if there be correlation*.

Again we have

$$\begin{aligned} \beta_2' &= \frac{1}{N} S(y^4)/\sigma_2^4 = \frac{1}{N} S\left(\frac{r\sigma_2}{\sigma_1} x + y'\right)^4 / \sigma_2^4 \\ &= r^4 \beta_2 + 6r^2 (1 - r^2) + \beta_2''' (1 - r^2)^2, \end{aligned}$$

$$\text{or } \beta_2''' = \frac{\beta_2' - r^4 \beta_2 - 6r^2 (1 - r^2)}{(1 - r^2)^2} \dots\dots\dots$$

$$\text{or, again, } \beta_2''' - 3 = \frac{\beta_2' - 3 - r^4 (\beta_2 - 3)}{(1 - r^2)^2} \dots\dots\dots(\text{xi}),$$

$$\text{and similarly, } \beta_2'' - 3 = \frac{\beta_2 - 3 - r^4 (\beta_2' - 3)}{(1 - r^2)^2} \dots\dots\dots(\text{xii}).$$

* We note that if the marginal totals be both without skewness, all the arrays will also be symmetrical. Equations (xi) and (xii) show us that if the marginal totals be mesokurtic the arrays will also be mesokurtic.

Now consider q_{22} ,

$$\begin{aligned} q_{22} &= \frac{1}{N} S(x^2 y^2) / (\sigma_1^2 \sigma_2^2) = \frac{1}{N} \Sigma S x^2 \left(r \frac{\sigma_2}{\sigma_1} x + y' \right)^2 / \sigma_1^2 \sigma_2^2 \\ &= \beta_2 r^2 + (1 - r^2) \left\{ \dots \dots \dots \right\} \dots \dots \dots (xiii) \\ &= \beta_2' r^2 + (1 - r^2) \left\{ \dots \dots \dots \right\} \dots \dots \dots \end{aligned}$$

by symmetry. Hence it follows that in linear homoscedastic systems $\beta_2 = \beta_2'$, and accordingly

$$\beta_2'' - 3 = \beta_2''' - 3 = (\beta_2 - 3) \frac{1 + r^2}{1 - r^2} \dots \dots \dots (xiv).$$

This is of interest as indicating that in linear homoscedastic systems the one with mesokurtic margins is the only one in which the kurtosis of the arrays can be the same as that of the margins.

$$\begin{aligned} \text{Again } q_{33} &= \frac{1}{N} S(x^3 y^3) / (\sigma_1^3 \sigma_2^3) = \frac{1}{N} \Sigma S x^3 \left(r \frac{\sigma_2}{\sigma_1} x + y' \right)^3 / \sigma_1^3 \sigma_2^3 \\ &= r^3 \beta_4 + \frac{3r}{N} \frac{\Sigma (n_x x^4)}{\sigma_1^4} S \left(\frac{y'^2}{n_x \sigma_2^2} \right) + \frac{1}{N} \Sigma \left(\frac{n_x x^3}{\sigma_1^3} \right) S \left(\frac{y'^3}{n_x \sigma_2^3} \right) \\ &= r^3 \beta_4 + 3r(1 - r^2) \beta_2 + \sqrt{\beta_1} \sqrt{\beta_1'} - r^3 \beta_1 \dots \dots \dots (xv) \\ &= r^3 \beta_4' + 3r(1 - r^2) \beta_2' + \sqrt{\beta_1'} \sqrt{\beta_1} - r^3 \beta_1' \dots \dots \dots (xvi) \end{aligned}$$

by symmetry.

It follows from (xv) and (xvi) that it is needful for

$$\beta_4 - \beta_1 = \beta_4' - \beta_1' \dots \dots \dots (xvii).$$

Finally we have

$$\begin{aligned} q_{44} &= \frac{1}{N} S(x^4 y^4) / (\sigma_1^4 \sigma_2^4) \\ &= \frac{1}{N} \Sigma S x^4 \left(r \frac{\sigma_2}{\sigma_1} x + y' \right)^4 / \sigma_1^4 \sigma_2^4 \\ &= r^4 \beta_6 + 6r^2(1 - r^2) \beta_4 - 4r^2 \beta_3 + 4\beta_3 \sqrt{\beta_1'} / \beta_1 + \beta_2 \beta_2''' (1 - r^2)^2, \end{aligned}$$

or

$$\begin{aligned} q_{44} &= r^4 \beta_6 + 6r^2(1 - r^2) \beta_4 - 4r^2 \beta_3 + 4\beta_3 \sqrt{\beta_1'} / \beta_1 - r^4 \beta_2^2 - 6r^2(1 - r^2) \beta_2 + \beta_2 \beta_2' \\ &= r^4 \beta_6' + 6r^2(1 - r^2) \beta_4' - 4r^2 \beta_3' + 4\beta_3' \sqrt{\beta_1} / \beta_1' - r^4 \beta_2'^2 - 6r^2(1 - r^2) \beta_2' + \beta_2' \beta_2 \\ &\dots \dots \dots (xviii), \end{aligned}$$

which again involves the complicated β -relation

$$\begin{aligned} r^4 (\beta_6 - \beta_6') + 6r^2(1 - r^2) (\beta_4 - \beta_4') - 4r^2 (\beta_3 - \beta_3') + 4 (\beta_3 \beta_1' - \beta_3' \beta_1) \sqrt{\beta_1 \beta_1'} &= 0 \\ \dots \dots \dots (xix). \end{aligned}$$

It is difficult to see how the form of variation of one character can be related by the correlation between that and another character to the form of variation of the second character as (xix) would indicate. If it were we should get into great difficulties in dealing with similar conditions to (xix) for a large number of characters

with different correlations. If as it appears to me (xix) would need to be satisfied independently of r , then we must have

$$\left. \begin{aligned} \beta_6 - 6\beta_4 &= \beta_6' - 6\beta_4' \\ 3\beta_4 - 2\beta_3 &= 3\beta_4' - 2\beta_3' \\ \beta_3/\beta_1 &= \beta_3'/\beta_1' \end{aligned} \right\} \dots\dots\dots (xx).$$

The second of (xx) by aid of (xvii) leads us to

$$3\beta_1 \left(1 - \frac{2}{3} \frac{\beta_3}{\beta_1}\right) = 3\beta_1' \left(1 - \frac{2}{3} \frac{\beta_3'}{\beta_1'}\right),$$

whence $\beta_1 = \beta_1'$, and as $\beta_2 = \beta_2'$ it follows that $\beta_3 = \beta_3'$, $\beta_4 = \beta_4'$, $\beta_6 = \beta_6'$, that is to say the total frequencies of the two correlated characters must possess variation practically of the same type.

Now I find this is very far from being the case in distributions which differ widely from the normal correlation surface. Thus it follows that the hypothesis of homoscedasticity, linear regression and homocliticity fails for such cases. I therefore modified the linear regression and adopted skew regression, homoscedasticity and homocliticity. I again got relations between the β 's, but of a much higher degree of complexity. These were tested by Mr A. W. Young and myself on the skew correlation surfaces of barometric data, but were found to fail. Direct investigation afterwards showed me that while the regression differed to some extent from linearity, it was the homoscedasticity which was in the first place the erroneous assumption. The arrays were very far from having the same standard deviations.

Until therefore some theoretical advance is made in the investigation of skew regression surfaces, especially for those which have linear or nearly linear regression combined with heteroscedasticity, it is unlikely that we shall have any adequate method of determining high product moment-coefficients from low ones. We are accordingly thrown back on direct determination of the high product moment-coefficients, if we wish to determine a Tchebycheff limit. The work of determining I_4 would involve a whole round of 8th order moment-coefficients and product moment-coefficients. It would then give us a limit of the order .95 for .99. Lower order I 's would hardly give values of much importance, and it may be questioned whether a rough limit of the kind required could not be better obtained by inserting the desired contour on a "scatter diagram" and simply counting the dots which fall outside it, or indeed by taking the best fitting normal surface to the actual distribution. The reader may question whether something better could not be achieved for skew correlation Tchebycheff limits by some contour other than the ellipse. This would undoubtedly be the case, if we knew the forms of the skew-correlation contours, for then we should undoubtedly choose this equi-probable locus for our boundary. But as we have only a knowledge of these empirically—experience shows them to be frequently pear or lemniscate loop shaped—we get little help for our present problem.

One other aspect of the matter may be briefly considered. We may find a limit to the probability that an event or individual will lie within a circle of radius R

round the origin. This corresponds to Schols' Problem*. It may be useful to have a Techebycheff limit for this case, although we have yet to meet the particular instance in practical statistics where it would be of marked advantage†.

We can best investigate this problem *de novo*.

$$\text{Let } I_s = \iint (x^2 + y^2)^s \phi(xy) dx dy.$$

Then if R be any radius round the origin,

$$I_s/R^{2s} = \iint \left(\frac{x^2 + y^2}{R^2} \right)^s \phi(xy) dx dy,$$

the integral being taken to include the whole volume of the probability surface $z = \phi(x, y)$. Now pick out those elements of the integral for which $x^2 + y^2$ is $> R^2$, then

$$I_s/R^{2s} > \iint \left(\frac{x^2 + y^2}{R^2} \right)^s \phi(xy) dx dy,$$

where the integration extends over the above-mentioned elements only, and is therefore

$$> \iint \phi(xy) dx dy,$$

but this integral is $1 - P$, where P is the probability that the individual falls within the distance R of the origin. Thus the Techebycheff limit is given by

$$P > 1 - \frac{I_s}{R^{2s}} \dots$$

Now clearly we have

$$\begin{aligned} I_s &= \iint (x^2 + y^2)^s \phi(xy) dx dy \\ &= p_{2s,3} + s p_{2s-2,2} + \frac{s(s-1)}{1 \cdot 2} p_{2s-4,4} + \dots \\ &= \sigma_1^{2s} q_{2s,0} + s \sigma_1^{2s-2} \sigma_2^2 q_{2s-2,2} + \frac{s(s-1)}{1 \cdot 2} \sigma_1^{2s-4} \sigma_2^4 q_{2s-4,4} + \dots \end{aligned}$$

Now write $R = \lambda \sqrt{\sigma_1^2 + \sigma_2^2}$, and further take $\tan \theta = \sigma_2/\sigma_1$. Then

$$\begin{aligned} \frac{I_s}{R^{2s}} &= \frac{1}{\lambda^{2s}} \left\{ \cos^{2s} \theta q_{2s,0} + s \cos^{2s-2} \theta \sin^2 \theta q_{2s-2,2} + \frac{s(s-1)}{1 \cdot 2} \cos^{2s-4} \theta \sin^4 \theta q_{2s-4,4} \right. \\ &\quad \left. + \frac{s(s-1)(s-2)}{1 \cdot 2 \cdot 3} \cos^{2s-6} \theta \sin^6 \theta q_{2s-6,6} + \dots \right\} \dots \end{aligned}$$

For the particular case in which $s = 1$,

$$\frac{I_1}{R^2} = \frac{1}{\lambda^2} (\cos^2 \theta + \sin^2 \theta) = \frac{1}{\lambda^2} \dots \dots$$

$$\text{For } s = 2, \quad \frac{I_2}{R^4} = \frac{1}{\lambda^4} (\cos^4 \theta \beta_2 + 2 \cos^2 \theta \sin^2 \theta q_{22} + \sin^4 \theta \beta_2').$$

* Over de Theorie der Fouten in Ruimte en in het platte Vlak, *Verhandlingen der K. Akademie van Wetenschappen*, Deel xv, pp. 1—68, Amsterdam, 1875. Translated into French in the *Annales de l'École polytechnique de Delft*, Tome II, pp. 123—178. Leide, 1886.

† It is conceivable that the solutions given might be serviceable in the case of testing machine guns against a target.

Now a good approximation to q_{22} by (xiii) must be $\frac{1}{2}(\beta_2 + \beta_2')r^2 + (1 - r^2)$; hence substituting

$$\frac{I_2}{R^4} = \frac{1}{\lambda^4} \{(\beta_2 - 3)(\cos^4 \theta + r^2 \sin^2 \theta) + (\beta_2' - 3)(\sin^4 \theta + r^2 \cos^2 \theta) + 3 - 4(1 - r^2) \cos^2 \theta \sin^2 \theta\} \dots\dots$$

For the special case of normal distribution, if we write $\kappa^2 = 4(1 - r^2) \cos^2 \theta \sin^2 \theta$,

$$\frac{I_2}{R^4} = \frac{1}{\lambda^4} (3 - \kappa^2) \dots\dots$$

Again

$$\frac{I_3}{R^6} = \frac{1}{\lambda^6} \{\cos^6 \theta \beta_6 + 3 \cos^2 \theta \sin^2 \theta (\cos^2 \theta q_{42} + \sin^2 \theta q_{24}) + \sin^6 \theta \beta_6'\} \dots\dots$$

and for a normal distribution,

$$\frac{I_3}{R^6} = \frac{1}{\lambda^6} (15 - 9\kappa^2) \dots\dots$$

Further general cases can be at once written down, but it will suffice to give here the leading values of I_s for a normal distribution :

$$\frac{I_1}{R^2} = \frac{1}{\lambda^2}, \quad \frac{I_2}{R^4} = \frac{1}{\lambda^4} (3 - \kappa^2), \quad \frac{I_3}{R^6} = \frac{1}{\lambda^6} (15 - 9\kappa^2),$$

$$\frac{I_4}{R^8} = \frac{1}{\lambda^8} (105 - 90\kappa^2 + 9\kappa^4), \quad \frac{I_5}{R^{10}} = \frac{1}{\lambda^{10}} (945 - 1050\kappa^2 + 225\kappa^4),$$

$$\frac{I_6}{R^{12}} = \frac{1}{\lambda^{12}} (10395 - 14175\kappa^2 + 4725\kappa^4 - 225\kappa^6),$$

$$\frac{I_7}{R^{14}} = \frac{1}{\lambda^{14}} (135,135 - 218,295\kappa^2 + 99,225\kappa^4 - 11,025\kappa^6),$$

$$\frac{I_8}{R^{16}} = \frac{1}{\lambda^{16}} (2,027,025 - 3,783,780\kappa^2 + 2,182,950\kappa^4 - 396,900\kappa^6 + 11,025\kappa^8) \dots\dots$$

The following table gives the maximum Tchebycheff limit for the probability of an individual falling within the circle $\lambda \sqrt{\sigma_1^2 + \sigma_2^2}$ for various values of

$$\kappa^2 = 4(1 - r^2) \sigma_1^2 \sigma_2^2 / (\sigma_1^2 + \sigma_2^2)^2.$$

(I_s) denotes the particular I from which the maximum limit is found. ($I_s ?$) denotes that the corresponding numerical value is a Tchebycheff limit found from I_s , but it is not known whether I_s would not give a higher value, I_s not having been tabled. The second part of the table provides the values of I_s from which the first part has been computed. They may be useful in the determination of the Tchebycheff limits for other values of λ .

I. *Generalised Tchebycheff Limit for Schols' Problem with a Normal Distribution.*

$$\text{Radius of circle} = \lambda \sqrt{\sigma_1^2 + \sigma_2^2}, \quad \kappa^2 = 4(1-r^2)\sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)^2.$$

Values of κ^2 .	$\lambda =$	1	1.25	1.5	2.0	2.5	3	3.5	4.0
	0.0	0 (I_1)	.36 (I_1)	.5556 (I_1)	.8125 (I_2)	.9386 (I_3)	.98400 ($I_4=I_5$)	.996924 (I_6)	.999528 (I_8)
	0.1	0 (I_1)	.36 (I_1)	.5556 (I_1)	.81875 (I_2)	.9422 (I_3)	.98574 (I_5)	.997329 (I_6)	.999611 (I_8)
	0.2	0 (I_1)	.36 (I_1)	.5556 (I_1)	.8275 (I_2)	.9459 (I_3)	.98740 (I_5)	.997707 (I_6)	.999685 (I_8 ?)
	0.3	0 (I_1)	.36 (I_1)	.5556 (I_1)	.83125 (I_2)	.9496 (I_3)	.98899 (I_5)	.998109 (I_7)	.999749 (I_8 ?)
	0.4	0 (I_1)	.36 (I_1)	.5556 (I_1)	.8375 (I_2)	.9538 (I_4)	.99099 (I_5)	.998478 (I_7)	.999805 (I_8 ?)
	0.5	0 (I_1)	.36 (I_1)	.5556 (I_1)	.84375 (I_2)	.9592 (I_4)	.99193 (I_5)	.998806 (I_7)	.999853 (I_8 ?)
	0.6	0 (I_1)	.36 (I_1)	.5556 (I_1)	.8500 ($I_2=I_3$)	.9645 (I_4)	.99333 (I_6)	.999096 (I_8)	.999893 (I_8 ?)
	0.7	0 (I_1)	.36 (I_1)	.5556 (I_1)	.8641 (I_3)	.9696 (I_4)	.99490 (I_6)	.999380 (I_8)	.999927 (I_8 ?)
	0.8	0 (I_1)	.36 (I_1)	.5654 (I_2)	.8781 (I_3)	.9746 (I_4)	.99631 (I_6)	.999609 (I_8)	.999954 (I_8 ?)
	0.9	0 (I_1)	.36 (I_1)	.5852 (I_2)	.8922 (I_3)	.9809 (I_5)	.99770 (I_7)	.999788 (I_8)	.999975 (I_8 ?)
	1.0	0 (I_1)	.36 (I_1)	.6049 (I_2)	.90625 ($I_3=I_4$)	.9879 (I_6)	.99895 (I_7)	.999920 (I_8 ?)	.999991 (I_8 ?)

II. *Values of the functions I_s forming the denominator of the Tchebycheff Limit to the probability that an Individual will fall for the case of Normal Bi-variate Frequency within a given circle of radius $\lambda \sqrt{\sigma_1^2 + \sigma_2^2}$.*

κ^2	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8
0.0	1	3.0	15.0	105.00	945.00	10,395.000	135,135.000	2,027,025.0000
0.1	1	2.9	14.1	96.09	842.25	9,024.525	114,286.725	1,670,080.7025
0.2	1	2.8	13.2	87.36	744.00	7,747.200	95,356.800	1,354,429.4400
0.3	1	2.7	12.3	78.81	650.25	6,561.675	78,279.075	1,077,729.5025
0.4	1	2.6	11.4	70.44	561.00	5,466.600	62,987.400	837,665.6400
0.5	1	2.5	10.5	62.25	476.25	4,460.625	49,415.625	631,949.0625
0.6	1	2.4	9.6	54.24	396.00	3,542.400	37,497.600	458,317.4400
0.7	1	2.3	8.7	46.41	320.25	2,710.575	27,167.175	314,534.9025
0.8	1	2.2	7.8	38.76	249.00	1,963.800	18,358.200	198,392.0400
0.9	1	2.1	6.9	31.29	182.25	1,300.725	11,004.525	107,705.9025
1.0	1	2.0	6.0	24.00	120.00	740.000	5,040.000	40,320.0000

The reader may be curious to know whether the Tchebycheff limit gives a better result for Schols' circles than for the elliptic contours. The actual probability of an individual falling within the circle of radius $\lambda \sqrt{\sigma_1^2 + \sigma_2^2}$ is given by

$$P = 1 - \frac{\kappa}{\pi} \int_0^\pi e^{-\frac{\lambda^2}{\kappa^2}(1 - \kappa' \cos \theta)} \frac{d\theta}{1 - \kappa' \cos \theta},$$

where $\kappa' = \sqrt{1 - \kappa^2}$ and $\kappa^2 = 4(1-r^2)\sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)^2$ as before.

I have not succeeded in finding any rapidly converging expansion for this expression*, and have been reduced to evaluating its argument and using a quadrature formula. Thus for $\lambda = 2$, $\kappa^2 = .4$, I find

$$P = .963,3694.$$

* Unfortunately Schols has not tabled P , but only gives the values of λ for ten values of κ' , which occur when $P=1/2$, i.e. radial values for generalised "probable errors."

The process is not as long as it might seem. Indeed if we only need four decimal places, it is quite adequate to integrate only through the first quadrant, the second contributes nothing of importance. The value given by the last Tchebycheff limit is

$$P > \cdot 8375.$$

This is of the same order of divergence as we found for the elliptic contour, i.e. for $\chi_0^2 = 7$, we had $P = \cdot 9698$, with a Tchebycheff limit $P > \cdot 8600$. Thus the measure of approach does not seem very close in this case until we reach higher values of λ .

On the whole we must express disappointment at the results of the Tchebycheff process. We had found Tchebycheff's own limit based only on the second moment of small practical value, although it is to be found occupying a prominent position in many continental works on probability. By extending it to higher moments and product-moments we have reached results which are great improvements on the original Tchebycheff limit, but the method still lacks the degree of approximation (except for probabilities over $\cdot 99$, say) which would make the result of real value in practical statistics. It is, however, conceivable that some more ingenious application of Tchebycheff's idea may lead to a limit more close to the actual value of the probability.



CHARLES BUCKMAN GORING,

from a sketch by R. Arai in 1912.

CHARLES B. GORING, 1870—1919.

"His work won full recognition from those who value scientific research. But it is a strange commentary on the Civil Service, that, when so pressing a problem as prison reform still confronts us, so fine a worker and so human a man should have been given but the (medical) administration of a great prison instead of being called in to deal with a work for which all his gifts supremely fitted him." *The Nation*.

The late Charles B. Goring, M.D., was a distinguished student of University College, London, and afterwards a Fellow of that College. During his career his studies were far from confined to medicine: he was much interested in literature and philosophy, being awarded the John Stuart Mill Studentship in Philosophy of Mind and Logic in 1893, probably the only occasion on which that studentship has fallen to a medical exhibitioner*. It was not therefore surprising to those who knew something of the remarkable powers of sympathy, the width of interests and the facility of expression which characterised Goring to find that he would write a blue-book, as no blue-book has been written since the time of Matthew Arnold. He would handle facts, but at the same time he would appeal by his imagination and gift of language not only to the sociologist but to every man who is fascinated by the human spirit in all its diverse phases. Goring lived with his criminals, and studied them in and out of prison as the naturalist studies life in the field, and as the humanist studies mankind in its thronged resorts. Ask Goring what a convict's mind was like and he replied unhesitatingly: Like yours and mine. The same delicate spirit of sympathy that went out to his friends in both the joy and the sorrow of life, drew the criminal to him, and the link often grew so close that the prison medical officer became the father-confessor: the psychology of the criminal mind was laid bare, and thus Goring's insight into criminality, its source and its motives, grew deeper and more and more coordinated as the years of service increased. Yet he never hesitated to exhibit the same tender sympathy alike to each new sojourner and to each oft returning old prison inmate, while his own nature widened and strengthened under an environment which appears to dull the mentality of so many men in the prison service. Only last Christmas the present writer discussed with him the possibility of a series of essays on the psychology of crime to be based indeed on facts acquired by scientific study, but to exhibit a structure from which the scaffolding should have been stript, and which should convince the beholder of the fitness of its purpose solely by the beauty and truth of its lines. The path to truth is an arduous one, but when we have reached the

* Goring was awarded the Weldon Medal and premium by the University of Oxford in 1914 and never will a more fitting award of that medal be made; his work "The English Convict" was undoubtedly the finest contribution to biometry of its quinquennium.

summit we know by the width of our prospect into all neighbouring spheres that we have attained it:

“Qui veram habet ideam, simul scit se veram habere ideam, nec de rei veritate potest dubitare.”

We may now have to await that work for generations until another prison medical officer arises with Goring's scientific knowledge, discriminative sympathy and fine power of expression. Battling with a gaol epidemic of influenza, when he should himself have been in bed, Goring fell an easy prey to pneumonia, which a strong will coupled with a spare and delicate frame cannot resist as their combination so often does many of Death's onsets. Goring died as he himself and his friends would have wished, doing his duty to the last at his post. His work was uncompleted as good men's work so often must be. He was studying at the time the influence of the war on the nature and frequency of crime—a subject on which much will no doubt be said, but most probably with small scientific basis. How shall we estimate his work, now that he has left us? We pass by the criticisms of men inside and outside the prison service, for they will leave neither in their own productions nor in their criticisms anything that will remain of permanent value to the new science of criminology as Goring outlined it; those who have had like experience lack either his insight, or his logical mentality, or his power of expression. They were not trained in the same school, nor had they the *penetralia mentis*, or rather what “the Romans called *ingenium*,” which through its very innateness carries mankind onward a step, assured, not doubtful or to be retraced. The contest between mediocrity and inspiration is as old as history and the creator, the poet, wins, if not in life, yet thereafter. The world has yet to realise that achievement in every field is the product of trained imagination alone. Truth in science as in art is not the product of mere computation or careful observation, but of these guided by fertility of imagination. The creative mind has the potentiality of poet, artist and scientist within its grasp, and Goring's friends were never very certain in which category to place him. Perhaps the specification was as difficult and would be as unprofitable as it must ever be in the case of the Florentine, the master spirit of this type of mind.

To the present writer fell the good fortune to be in close touch with Goring (and his keen co-worker, H. E. Soper) for that long period of two and a half years during which “The English Convict” was in process of creation. He observed Goring in times of difficulty when the intertwined skein would not unravel, and in times of achievement when the tangle loosened as by magic. He realised the quiet persistency with which Goring grappled with the most intricate problems and the gentle satisfaction he exhibited when assimilating and recording a new and striking point. When finally the great manuscript had gone to press, we who had been working alongside him at our own tasks knew one and all that while we were losing a cherished daily intimacy, we had still individually gained a life-long friend. We felt that had the world been rightly organised—which it ever fails to be—a post in our midst would have been found available for Charles Goring, for no

man was better fitted than he to "study those agencies under social control that may improve or impair the racial qualities of future generations, either physically or mentally"; none we had come across was so well suited to make knowledge reached by scientific research a factor of social progress. He knew how to clothe scientific results in a garb which captivated the mental eye of him who listened to his spoken or read his written words. Goring was intended by nature for a master-craftsman of exposition. His sceptical spirit demanding a rigid foundation for truth was combined with an unlimited enthusiasm that truth when known should be proclaimed to the many. Yet in his own life, "Thrones, powers, dominions blocked the view, with episodes and underlings."

What then is the outcome of Goring's work? Has he decreased crime or bettered the lot of the criminal? Not directly, the solitary individual can achieve little in this sense; he has moved stones from the path of the outcast, and we can picture many a criminal who would have wished to stand by his graveside. Has he pointed out the lines upon which the state in future should deal with its defaulters? Again not directly, but only indirectly. What then has he achieved? He has given us a portrait of the criminal as he really exists; he has painted in the nature of his physique, he has indicated his facial and underlying mental traits, his hereditary tendencies and his home associations. And he has made for ever atypical the criminal of current drama and novelistic literature. Here it is that literature owes a deep debt to Goring. It cannot survive without its villains but the individual writer will never be as intimate as Goring was with poisoner, murderer and spy. Yet if that writer approaches with intuition not the masses of statistical data, but the text of Goring's life-work, even in its recently issued abridgement*, he will learn to see the criminal as Goring saw him, he will learn to know the real man and his attitude to crime. He will learn that Goring was a creator in the literary sense†, and with imagination stirred he will feel the impulse to adopt and adapt that realistic portrait of the criminal as only true art can do. Through literature the world at large will know at last what crime and the criminal really are. Not only will literature profit, but the world which easily grasps truth when depicted by art will understand and gain something of the spirit of the man whose life's work alas! is embraced within the livid wrappers of a government publication.

"En mands gerning er hans sjæl, og sin gerning skal blive ved at leve på jorden."—The work of a man is his soul, and on earth his work shall not perish.

* "The English Convict" (Abridgement), Wymans & Co., 1915.

† The present writer has many sins to atone for, but perhaps none he regrets now more than the stringency with which he docked the original MS. of Charles Goring of many of its literary qualities as unsuited to a scientific and government publication.

K. P.

APPRECIATIONS OF CHARLES GORING.

To the readers of *Biometrika* the following sympathetic accounts of the personality of Charles Goring will appeal as they do to the Editor, who deeply values the privilege of being allowed to publish these very intimate characterisations. The first is by Mr E. V. Lucas, a college friend of Goring's; they both belonged to one of those periods of keen intellectual activity which arise occasionally in college life owing partly to the action of waves of external thought, but more often to the presence internally of one or two original minds. Outwardly the period in question was marked by the foundation of the Students' Union and the meteoric brilliancy of *The Privateer*—a college journal that one did not grudge purchasing. It was for Goring the moulding time,—the golden days, when there was leisure to think, interpolated between an uncongenial office experience and the wider but none the less toilsome experience of a medical officer on a hospital ship during the South African War.

The second appreciation is the oration bravely spoken over his grave by his widow. I have not ventured to leave out a sentence of it. Round the grave were gathered the friends of his creative period, the friends of his youth, the friends of his prison calling, from prison commissioner to warder, and a scattering of humbler friends unknown to most of us, but none the less there out of love to one of the finer spirits of this life. That brilliant June day, with its unique ritual, when we paid the last respects to Charles Goring, will remain in the memory of those present as unique as the nature of the man, who in leaving us reduces still further that little school of trained biometricians, who value humanism as well as science.

I. CHARLES B. GORING AS A STUDENT.

I have been asked to write a few words about Charles Goring, and I have tried, because I respect the asker; but they will be incomplete because I have seen Goring of late so little and hardly knew him in maturity at all: as a husband, and a father, and an intellectual force with all his powers at their richest. But of the Charles whom, in the eighteen nineties, we knew, the Charles whom we loved, my impressions are fresh and will always be. His personality provided for that.

I say "whom we loved," but I think we did more than love. I think that if it were possible, if it were conceivable, that any harm should be coming to him, there is nothing we would not have done to interpose our own inferior bodies between him and it. For he inspired not only affection but protectiveness. We felt that we were his guardians: his—in a very peculiar sense—owners. Not that he lacked any qualities of self-defence. Far from it. His mind was crystal clear, his attitude to life and its problems was fearless; but he had an unworldliness, a childlike radiance, that seemed to demand from his friends a contribution of cotton wool. Let me say again that he did not *need* this, but we all wanted to

provide it. I have said that his attitude to life and its problems was fearless. But it was more than that: it was challenging and ardent. Had there been nothing to probe and inquire into, he would not have been the happy man he was; for he was a born inquirer—inquisitor even—and mistrusted all traditional face-values.

Exactly how I came to be admitted to Goring's circle I never understood then, and cannot now fathom. Because where he and his friends brought to their discussions and disputations knowledge and seriousness, I had nothing but instinct and impatience. But they suffered me, and I was permitted to sit on the outskirts and listen, and now and then to interrupt. What I chiefly remember of those evenings—at all kinds of places—at Highgate, at Hampstead, in rooms near the Museum, on the boat to Margate, on the Broads,—what I chiefly remember is Charles in argument: eager, stimulating, vivid, humorous, always gently reasonable and never losing sight of the main proposition. I suppose he was the honestest and most understandingly tolerant man that ever lived. He never trimmed; he rarely condemned; and he had no fear. No fact was too stark and naked for him; indeed, what he wanted was stark and naked facts. We would all have our say—some of us solid and some of us fluid—and then he would deal with us, with quiet Socratic questionings; and all the while we would see, burning within his beautiful workmanlike brain, the soft steady flame of that lamp of enthusiasm which was never to be dimmed until a few weeks ago it was all too soon extinguished: enthusiasm for the truth, wherever found.

Of what dark passages that lamp was to illumine it is not for me to speak. There are others who have authority. But that no sweeter nature was ever allied to a passion for scientific investigation I feel myself to have the right to affirm.

E. V. LUCAS.

June 17, 1919.

II. CHARLES GORING AS HUMANIST.

In asking you all to come here today, I have done what seems to me a right thing to do, and a beautiful one: for, with your presence, I have made a circle round my husband's spirit of those minds and hearts most intimate with his, and most valued by him....You all loved him; he loved everyone of you. With each one of you he had a separate and private friendship....It seems to me that I can do him no greater honour on this day than to give him what you have let me give him by coming here—your undivided thought of him, your clear memory, and the warm and poignant tenderness that I well know possesses each heart here at the very mention of his name—Charles Goring.

I must ask you to forgive me if I read from this paper what I have to say, instead of speaking it, in a more natural manner. I should not find any difficulty in speaking to each one of you separately. It seems absurd that, simply because you are all here together, in a number, that I should find it difficult. Yet so it is.

And, therefore, for this reason, and also because on this occasion I can trust neither my memory, nor my self-control, I hope you will forget—won't even see—this bit of paper between us.

I want to say, first, why I am speaking at all. There are two reasons. One is that I want to say something about my husband which may, perhaps, for a few instants, trace an outline of him upon the air, for you as well as for me—which may, for a moment, mark out his features for us, give us a glimmer of himself. That is one reason. The other is that I want to make, at his grave-side, and in the knowledge of death, certain affirmations.

I have great difficulty in expressing myself here. I will ask you for your generosity with your tolerance. I ask it the more particularly because I know there is at least one amongst you—and probably there are more than one—who will find my attitude and desire foreign to his own.

To this person, who has my respect, affection, gratitude, as I hope he knows, I want to say that, though I understand his inability to speak to us here today about my husband—and, in a way, I love him for that inability—yet I do regret it; and also I do not accept his point of view.

My regrets are for the fact that his silence deprives us of a criticism, an appreciation of my husband—of my husband's scientific mind and work especially—which no one else could give with equal authority, sincerity and eloquence. So there is room enough for regret I think.... And then also, as I said, I do not accept my friend's point of view, though I can salute it for its dignity.

His view is, I understand, that reticence, and silence, and solitude best suit the great occasions of human experience—those of grief and loss, particularly. *I feel—I more than feel: I believe—the opposite. I believe in Voltaire's saying: "Le but de l'homme c'est l'action." Action means words as well as deeds. I believe that for whatever other purposes we may also possess life, there is a secret injunction upon us—within us—to express things: to do, to make, to show. And it seems to me—it is more than feeling: it is a sort of moral urging—that when the great emotional experiences come to us, we ought to give them some outward, visible sign: Form: form, in accordance with that law that, as I said just now, seems to me to impose action upon us during our humanity: form that is beautiful.*

I have felt, then, in the great experience which has just come to me—the greatest I shall ever know—that unless I am to be false to my own instincts, and a coward to my own truth, I must testify by some outer form, and beauty of symbol, to the quality of my husband's spirit, and the sacredness of his memory, at the hour of the burial of his body.

Feeling, and believing this, I realise the disadvantage at which we stand—we who are Freethinkers—when, for our great occasions, we need a ceremonial, dignified, harmonious, simple. There, all the Churches, who have had time to grow old and beautiful, have the advantage of us. Their poets have had time to shape inarticulate cries and struggling aspirations into pathetic and stately ritual. Their artists have had time to bring colour, and line and music to the spaces set aside

for those who suffer, and those who conquer themselves. We here today—those among us who are Freethinkers—would not give up for these achievements, fine though they are, for our own best: the very essence that makes us Freethinkers. Nevertheless, the Churches, in this way, have the advantage of us. And when a person like myself wants, as I do today, to mark by outer form and beautiful symbol, the great spiritual experience that has come to me, there is no ceremonial—the legacy of the genius of ages—waiting for me: and I am at a loss. So disconcerting might this position have been, that it would have been easy to yield to the temptation that has for the last week assailed me: the temptation to do nothing; to give way to difficulty; to accept despair. All the time that I have felt it urgent within me to do honour, somehow, to my husband, on the day of this burial—all that time I have also felt an unworthy fear of the effort: and I have very often nearly decided to make none, but to have the ashes of his body buried without a sign, and myself alone as witness.

I am glad I have cheated neither the memory of my husband, nor my own instincts, by doing that. I am glad that, by your presence, by the mysterious sense of the unity there is upon us in these moments—by the singing of these boys, whose music he loved, by these flowers, by the good fortune of an exquisite day of sunshine and warmth—I am glad that outward forms acknowledge the inward grace: I am glad that the influence of a lovely spirit is abroad in the air above this grave.

In speaking of my husband himself, I shall have to choose one quality only of him, I suppose, if I am to be clear. You will all know of others. And I shall not speak at all of his special intellectual gifts....I think, perhaps, his rarest and most endearing quality was his particular kind of humaneness. I say "his particular kind of humaneness" because it was not in the least like what is called "humanitarianism." He had no sentimentality. And he was never in the least taken in by humbug. But his humaneness enabled him to know, and to like, the humanity even behind the humbug. There was in him at once a complete lack of prudery and a perfect personal rectitude. Charlie was as incapable of being shocking himself as he was of being shocked at another person's shockingness....The fact is that, apart from cruelty, he did not take what is called "evil" very literally. He thought that nearly all people were intensely likeable when you got to know them. So that his charity—of which everyone speaks who knows him—was far less forgiveness than it was sympathy; and his kindness was always *loving-kindness*.

If you will let me, I will tell you one or two things about him that may, perhaps, trace that silvery outline of which I spoke....I think of a certain day, some years ago, when we had a really wonderful walk together. It was one of those fortunate days—those gift-days—when everything turns out successfully; when the unexpected leaps up; when there is adventure through it all. I won't give you the whole history of the walk, but only these points—to show you Charlie.

We had just come down Villiers Street from the Strand, and were near the Embankment Gardens, when he pulled up suddenly with a look of intense

alarm. He told me that one of his old convicts, discharged from Parkhurst, had taken to newspaper-selling outside the Embankment Station. "He talks for hours," said Charlie desperately: "He has the eyes of a lynx. He spots me amongst thousands. He'll spot me in a minute. He always does. And that'll mean interminable conversation, and half-a-crown. Let's get into the Gardens while there's still a chance." So we dived for the Gardens, and were just through the gate, when he again pulled up. "After all," he said, "I have managed to give him the slip three times lately. It seems rather unfair to cheat the poor old boy again, so soon. Let's go by the station." So we went by the station; and he was duly pounced upon, and a lengthy, amicable gossip ensued; and the half-crown passed from one pocket to the other....Now that seems to me very like Charlie: that pang of conscience, that sense of fellowship which made him feel that, by evading him *too* often, he was, what he called, "cheating the old boy."

After this, we got out on to the Embankment. It was a wonderful day, I remember, in early autumn. The river was stiffly rippled; the plane trees were brilliant in colour and movement; rapid clouds were passing in the blue sky; bright traffic was flashing and humming by in the broad roadway: it was a delight to swing along the pavement, in the keen air, arm in arm—and quarrelling all the time, as we mostly did! Presently we reached the Temple, and passed upwards through the narrow passages and dark archways, and across the smiling silence of the Courtyards: and then out again, into Fleet Street; and up through Chancery Lane to Holborn; and so to the left, towards Oxford Street. And when we were in New Oxford Street, I suddenly became aware of an astounding apparition on the opposite side of the road. Charlie observed people very acutely when he was in close contact with them, but he didn't notice things in crowds. He didn't notice this man; and he continued to arguefy at my side, while I continued to amaze myself at the man.

This person seemed to take up the whole street. It was not so much that he was so large, as that he was so blatant. His clothes were the most astounding things in vulgarity and newness that could be conceived. He wore a buttonhole that was an insult. And the way his boots shone, and his hat shone, and his walking-stick shone as he twirled it—the way he simply glared and revolved in glory, as it were, with Oxford Street as a mere margin for him—simply took one's breath away.

I hadn't time to pull my husband's arm, and stop his Infinites and Indefinites, before the whole bulk of this being was descending upon us across the road, and clasping Charlie with a fervent hand. I left my poor man stuttering in his grasp; and went and looked into the Cameo Shop window. It was perfectly clear that he hadn't the remotest idea who the man was, though he was pretending he knew him! And presently the volubilities broke down, and I heard this: "I don't believe you know me, Doctor? I am....." I didn't catch the rest; but Charlie's voice cleared up in relief: "Oh, of course; of course!" and then proceeded to rapid and friendliest conversation.

At last, with some terrific laughter, and a perfect blaze of complacency, I heard the man exclaim: "Well, its all A.1—A.1, that's what it is. A bit of All Right. Everything's going swimmingly. We're off to the Rhine for a few weeks' holiday. Hope to see you again, Doctor. So long!"—and he was away, with a flourish of his hat, down the street towards Holborn; while my husband came up to me, convulsed with merriment, saying: "Will you believe it? *That's another of them!*" He was, in fact, another convict. Also from Parkhurst. A very bad case of fraud, I believe. Quite unpardonable. Still, there he *was*: free again: at large: enjoying every moment of his regained existence. And, as we watched him disappear towards Holborn in his outrageous radiancy, the spectacle didn't merely amuse Charlie or stagger him,—it didn't shock him, and it didn't sentimentalise him: but it made him rejoice at the thing in the man that could so rejoice in liberty; that could swagger so in the sun; that could be so little of a snob, and so free from the Past, that it could actually come bounding, in 'camaraderie,' to an official of the Prison in which its convict days had been spent! *That* was all right in him, whatever else might be wrong: and it caught Charlie's affection: he *liked* the man.

I hope this also may give you a touch of your friend. It is so difficult, with heavy words, to convey an intangible thing. But I hope your own knowledge of him may give you the feeling of his quality in all this.

There is one thing more I want to tell you about him. It was last March, and we were in Manchester. We were having rather a rough time of it. We had no servant, and most of the rooms of the house were shut up, to reduce work and fires. The kitchen was our children's playroom, and it was our dining-room as well. We had breakfast there, one morning, and were, as usual, distinctly late! and my husband had to hurry off immediately after into the Prison. It was a bitter morning: there was a perfect blizzard of snow and rain. Five minutes after he had gone, I heard his latch-key in the door, and he rushed back to the kitchen in a tremor of excitement and pity. He said he had found a woman in the street who was so ill she could hardly move. She was coughing herself to pieces; and he thought she had consumption, or some virulent form of influenza. He had brought her back with him. She was in the hall.

And here I have a confession to make. I must make it, because if I do not, I cannot show you what Charlie was. I was angry with him. I was angry because I was in deadly terror of the children catching influenza. It seemed to me terrible, at the moment, to bring that poor, infected creature into the room where the children were—the one room where there was a fire.

Well, it is the *look* he gave me when I was angry with him that I want to tell you about—a look in which there were not so much reproach and surprise (though these were there) as a kind of lovely guilt: a baffled look: a look pleading for pardon, and saying in desperation: "Yes, I know, I know. But, in God's name, what was I to do?" All this was in that look, which was the very essence of Charlie: and, without a word between us, I bundled the children upstairs, and we fetched the poor thing from the hall into the kitchen.

I can see him now, settling her by the fire, bringing her a footstool, taking her poor, dripping shawl off her shoulders, and hanging it up to dry. We thought for a time she was going to die; but she got better in a little while, and sat, uncomplainingly, coughing; and, when she was not coughing, smiling at the fire. He had to tear off to the Prison, as soon as he could leave her, through the frightful storm, promising to bring help for her as soon as he could leave his work.

He returned later in the morning, with an ambulance, and an order for a hospital: and again now, I can see him leading her carefully through the hall, lifting her into the carriage, nodding at her affectionately through the doorway, as the carriage drove off; and then coming back to me for a moment, before he returned to his work, with that same muteness, that same look of an angel's apology in his eyes....

This was Charlie absolutely—this passion of pity for suffering. In his last two days on earth, during the height of his delirium, one memory recurred, and haunted him over and over again: the memory of two little children whose case had been tried at the Assizes, and whose bodies he had had to examine, and had found marked and mutilated by the fiendish cruelty of their parents. These children he could not forget: he mourned and lamented them, seeing them before him in his fever, and calling, and calling upon us to take them, and save them....

If I had not told you this, I could not have shown you all that I meant by my husband's humaneness: but I do not want the last impression that I, at any rate, leave with you, to be one of sadness. I want it to be one of happiness: because he was really an extraordinarily happy man. He was happy chiefly because of his nature and character, of course; but, also, he was fortunate. He had got the things he most wanted in life. He never had any worldly ambitions at all. He had always wanted three things: first, freedom to live a life of the intellect—of observation, and of criticism; and this he was able very largely to do, in spite of the fact that he also had to earn our living. And, secondly, he wanted Friendship: and he had Friendship. And, thirdly, he wanted Romantic Love: and he had Romantic Love. The things he wanted and hoped for when he was young, he found, and still wanted when he was middle-aged. And when he died at forty-nine, he took with him enthusiasms as eager as they were when he was twenty-five.

I will say no more except to read you the inscription that I shall be putting over the place where his ashes will lie.

For a great many years, I have had in my mind a line of words whose music and meaning I very much liked. I only vaguely knew where it came from. It corresponded to the Christian triad: "Faith, Hope, and Charity"; and it ran thus: "Love, Pity, and Equanimity."

During the last few days, when I was wanting to find something beautiful, and expressive of him, to put in words above my husband's grave, I thought of this line again: and I have found out that it comes from a Buddhist Sutta. I am not very clear what a "Sutta" is? but I think it means a "Gospel." This particular Sutta,

from which I have got my line, describes the being who has attained the perfect life—that is to say, the life of self-conquest. The passages from which I have made my extracts are these :

(1) “And he lets his mind pervade one quarter of the world with thoughts of Love, and so the second, and so the third, and so the fourth. And thus the whole wide world, above, below, around, and everywhere, does he continue to pervade with heart of Love, far-reaching, grown great, and beyond measure.”

(2) “Just as a mighty trumpeter makes himself heard—and that without difficulty—in all the four directions ; even so, of all things that have shape or life, there is not one that he passes by or leaves aside, but regards them all with mind set free, and deep-felt love, pity and equanimity.”

The inscription as I shall put it above the grave will be this :

“Here lie, in Sacredness and Honour,
the Ashes of the Body
of

CHARLES BUCKMAN GORING,
Doctor of Medicine, Bachelor of Science,
Fellow of University College, London,
and

Medical Officer in Chief of Strangeways Prison, Manchester.

Born January 31st, 1870 ; Died May 5th, 1919.”

And underneath I shall put this :

“.....Of all things that have shape or life there is not one that he passes by or leaves aside, but regards them all with mind set free and deep-felt love, pity, and equanimity.”

KATIE MACDONALD GORING.

ON THE NEST AND EGGS OF THE COMMON TERN (*S. FLUVIATILIS*). A COOPERATIVE STUDY.

W. ROWAN, E. WOLFF, AND THE LATE P. L. SULMAN, Fieldworkers.

K. PEARSON, Reporter.

E. ISAACS, E. M. ELDERTON, AND M. TILDESLEY,
Tabulators and Computers.

(1) *Origin of the Material and Method of Measurement.*

This paper may be looked upon as a continuation of that published in *Biometrika*, Vol. x. pp. 144—168. It is based upon a census of the eggs made July 3rd—20th, 1914, and contained in Rowan's *Fifth MS. Report on the Faunistics of Blakeney Point*, the Field Station under Professor F. W. Oliver's direction on the Norfolk coast. The year was a record year for the common tern, a marked contrast to 1913, the young were abundant as well as the eggs, and many of the birds were still laying. Some peculiar nests were found: (a) one entirely of seaweed, (b) another of large wood shavings, (c) one of selected small pebbles, (d) a very large nest—the largest yet met with. Some of the nests are illustrated in Plate II and will suffice to indicate the considerable differences between their make up and environment*. The range of ground colour with extent and distribution of mottling are indicated in Plate III, which should be taken in conjunction with Plate VIII of the earlier paper. There is every reason to believe that the two clutches, each of three eggs, were in both cases due to a single bird; the seventh egg, from a one-egg clutch, represents a peculiar egg found in the examination of this year's material. In all 515 clutches were recorded as against 203 in 1913. In that year there were 13 clutches with 3 eggs each; in 1914 there were 198, and many of those with one or two eggs at the time had also one or two newly hatched chicks, bringing the total up to three. Even the nests with one egg (122 as compared with 119 in 1913) were actually nests with the first egg only of the clutch, for the birds were still laying, while most of the one-egg clutches in 1913 were either deserted or the egg addled.

Plate IV gives some further photographs taken of the Ternery. Fig. *a* is an attempt to catch the bird alighting in order to indicate the great length of the

*. The following illustrates a method of nest building, that of nest (d) above. "A common tern laid close to the observation tent. At first there was no material whatever. But on the same day a few of the *Psamma* leaves from the tent were taken and deposited round the egg. The next day another egg was laid and more stuff was added. None of the *Psamma* had then been broken and the leaves radiated from the centre in all directions. On the second day the first few were broken and tucked neatly in all round. Then a third egg was deposited. More pieces of *Psamma* were added and the nest then had a very ragged appearance. It took two more days before the nest was completed and tidied up."

wings. Fig. *b* is the only four-egg clutch observed. Plates V and VI show birds sitting, the camera being about 18 inches from the bird.

The characters observed were identical with those of 1913, namely :

1. Length (*L*); 2. Breadth (*B*); 3. Longitudinal Girth (*G_l*); 4. Transverse Girth (*G_b*); 5. Tone or Ground Colour; 6. Mottling; 7. Type of Nest.

The tone or ground colour was in 1914, however, divided into browns and greens. The scale of browns was that of the Colour Value Scale of Plate VIII of the first paper, and the green values were judged on a similar scale divided into corresponding classes *a, b, c, d, e, f, g, h, i, k*. These classes are distinguished by the subscripts 1 and 2 for brown and green values respectively. Two eggs only had to be excluded from these colour value observations; one was blue and the other slaty gray in ground colour. These reduced the total number of eggs available for colour value reduction from 1110 to 1108. The classification of mottling follows Plate IX of the earlier paper. Types of Nest were divided into three categories, *t*₁ = no hole in the ground and no materials, *t*₂ = a hole but no materials, *t*₃ = both hole and materials. As only one nest (with a three-egg clutch) occurred in type *t*₁, we have grouped *t*₁ with *t*₂, so that the distinction is really of unelaborated and elaborated nests.

Of the characters dealt with, the transverse girth (*G_b*) was really taken as a check on the general accuracy of measurements. We should have

$$\pi = \text{Mean Transverse Girth} / \text{Mean Breadth}$$

or rather π is equal to this ratio multiplied by the factor $(1 - r_{G_b B} v_{G_b} v_B + v_B^2)$ where $r_{G_b B}$ is the correlation of the transverse girth with the breadth and v_{G_b} and v_B equal $\frac{1}{100}$ of the coefficients of variation of transverse girth and breadth respectively. This factor was .99990 in the previous set of observations and is 1.00006 now. Hence its influence on $\pi = \bar{G}_b / \bar{B}$ is insensible for our purposes. We find $\pi = 3.2071$ against 3.2237 of the earlier series. Thus although the value of π is bettered, we still find the transverse girth is somewhat exaggerated, i.e. π is about 2% in error when thus deduced. It might at first sight suggest itself that the transverse section of the egg may not be truly circular. Suppose it an ellipse of eccentricity *e*. Then if we agree that it is equally likely that the breadth of the egg may be measured in any meridian we find

$$\frac{\text{Transverse Girth}}{\text{Mean Breadth}} = \pi (1 + \frac{1}{15} e^4),$$

if *e* be small. If, however, we put in the values found, i.e. $\bar{G}_b / \bar{B} = 3.2071$, we have

$$e^4 = .3320,$$

leading to $b = .6510a$ for the relation between the semi-axes of the ellipse—a quite impossible value. It may be suggested that our chance of taking every breadth is not equal and that we are most likely to take the *minimum* breadth. In this case we should have

$$\bar{G}_b / \bar{B} = \pi (1 + \frac{1}{4} e^2),$$

and with our numbers $e^2 = .0832$, leading to $b = .9576a$ —an improbable but not so impossible a relation as the former. It could hardly, however, escape observation, as even slightly distorted eggs are easily recognised. It seems, therefore, probable that the exaggeration of the girth in the transverse sense is due to the difficulty of adjusting the tape to the true maximum transverse section—the temptation being to bring the reading edge of the tape into contact with itself *with the scale facing outwards*. If we suppose the celluloid scales to be 0.5 mm. thick this would account for the deviation. Probably the longitudinal girth is exaggerated in like manner.

Unfortunately it did not apparently seem possible for the fieldworkers to adopt a more elaborate system of classification for the mottling than was used in 1913 and accordingly no further light is obtainable with regard to the difficulties suggested on p. 146 of the first paper.

The question of possible pressure on the surface of the egg as it passes through the oviduct influencing the amount of pigment deposited was again investigated by considering the *broader* egg in each pair from the same clutch (see l.c. p. 146).

The broader egg in every possible clutch pair has:

Greater mottling in 189 cases				More dense ground colour in 223 cases			
The same	"	"	312	"	"	"	290
Less	"	"	234	"	"	"	224

Thus our 735 pairs confirm the previous result (on about 100 pairs) as far as the mottling is concerned, but not the density of ground colour. There is no distinction in ground colour on the average between eggs of different breadths from the same hen, but the broader egg does appear to have less marked mottling. We shall consider later whether this result for eggs of the same clutch holds for the general population.

(2) *Change of Type of Egg with Season.*

We have:

TABLE I.

Character	Mean	
	Season 1913	Season 1914
Length L	$4.14 \pm .007$	$4.21 \pm .004$
Breadth B	$2.98 \pm .004$	$3.01 \pm .002$
Longitudinal Girth G_l ...	$11.39 \pm .015$	$11.56 \pm .007$
Transverse Girth G_b ...	$9.59 \pm .014$	$9.66 \pm .006$
Index $100 B/L$	$72.04 \pm .136$	$71.75 \pm .070$
Index of Ovality O ...	$56.35 \pm .171$	$55.81 \pm .088$

It is clear from this table that the eggs of 1914 were significantly larger than those of 1913. As the fieldworkers remarked before the eggs were tabled and

reduced, 1914 was a splendid contrast to 1913; never were so many birds seen and the young were as abundant as the eggs. At first sight it seemed strange to find such a flourishing colony after the comparative failure of the previous year, but in the summer of 1914 the channel was phosphorescent at night with Plankton, and probably as a result of this the channel was also swarming with myriads of "Whitebait," which in their turn attracted the Terns. The suggestion is thus thrown out that a plentiful food supply increases the size of the eggs. It must, however, be borne in mind that possibly only the stronger and bigger birds survived the previous bad season. There may have been fewer very young or very old birds and thus the eggs larger.

We may now consider the variabilities of the two years.

TABLE II.

Character	Standard Deviation		Coefficient of Variation	
	1913	1914	1913	1914
Length L	$180 \pm .035$	$185 \pm .003$	$4.34 \pm .12$	$4.39 \pm .006$
Breadth B	$.099 \pm .010$	$.099 \pm .001$	$3.33 \pm .09$	$3.28 \pm .005$
Longitudinal Girth G_l ...	$376 \pm .010$	$350 \pm .005$	$3.30 \pm .09$	$3.03 \pm .005$
Transverse Girth G_b ...	$347 \pm .010$	$300 \pm .004$	$3.62 \pm .10$	$3.10 \pm .005$
Index 100 B/L	$3.449 \pm .096$	$3.479 \pm .050$	$[4.79 \pm .13]^*$	$[4.84 \pm .069]^*$
Index of Ovality O ...	$4.334 \pm .121$	$4.326 \pm .062$	$[7.69 \pm .22]$	$[7.75 \pm .111]$

The table indicates that the material for 1914 is slightly less variable than that of 1913 taken as a whole. This is possibly due as we have suggested to the bad season of 1913 reducing the number of very young or very old birds and so the small eggs in 1914. But most of the differences are insignificant except those in the two girths. We anticipate that a good deal of interest from the evolutionary standpoint might be reached by secular observations on the eggs of this tern colony, taken in conjunction with records of the food supply and climate both in the nesting season and after. It would be of interest also to mark certain birds and record if possible their return.

(3) *Associations of Nest and Egg Pattern.*

It is of great interest to discover whether there is any protective action in the colouring and mottling of the egg. In an egg which varies in itself so largely as the tern's this question must be considered not so much in regard to the general nesting habits of the species, but in regard to the nest and environment of each individual bird. The occasional and possibly habitual practice (see our *ftn.* p. 308) of laying and nest building simultaneously may indeed suggest that the birds adapt the immediate environment and material of the nest to the actual

* See remarks, footnote †, p. 147 of previous paper.

character of their eggs. If the egg in shape, colour-value and mottling be related to the individual nest, it is hardly conceivable that a hen, especially when a young bird, can *à priori* appreciate what the type of her egg is likely to be and prepare the corresponding protective nest accordingly. Such an instinct would be conceivable in the case of a species with more uniform eggs and building a specific type of nest; it is hard to conceive it possible in the case of such a wide colouring and mottling range as we find in the common tern. The alternative is to suppose a considerable variety of tern *gentes*, who like the suggested cuckoo *gentes* select a particular environment for their eggs. Such a suggestion is not without difficulty; it involves mating within the *gens*, or a transmission of the egg colouring mechanism through the female only. To accept the latter is not consonant with our experience that sexual characters of the female are transmitted through the male, i.e. the fertility of the mare and the character of a cow's milk are correlated with the like characteristics in their *paternal* grandmothers. It is conceivable that the pigmentation may vary to some extent with the immediate food supply. In this case green and brown eggs *of the same shape and size* within the same clutch might be more readily accounted for than by the hypothesis of two hens of different *gentes* using the same nest*. It might also admit of the hen having some inkling of the character of her forthcoming eggs, if the nest be made beforehand. Besides this it would free us from any hypothesis as to tern *gentes*. Thus far we have written as if the protective colouring of eggs was a demonstrated phenomenon. It is highly probable in the case of many species building specific nests in specific environments. Can it be asserted of the common tern? If not, elaborate and most varied colouring and mottling would appear to be physiological, and originate before they attain protective character. In other words egg patterns have been specially selected for protective purposes, but did not originate in the survival of the better protected.

It will be remembered that we have divided our nests into the unelaborated nests, i.e. nests with no material, and with no hole, or merely a hole in the ground, and elaborated nests or nests formed by a hole and with accumulated material. We shall denote these by *S* and *C*, i.e. simple and complex†. We will consider first absolute size as measured by the longitudinal girth, G_L .

The following table gives the data. The mean of the *S*-nest eggs is 11.556 as against 11.373 for the total population. The correlation found by the biserial *r* method was

$$r = +.0685 \pm .0322.$$

* Clutch *a*, figured in Plate III, shows three eggs practically identical in shape and size yet of very different ground colour. Since the size is quite abnormal—being the smallest found in 1914—one can hardly believe that three birds laid three such eggs in one and the same nest! Again in the *Psamma* nest referred to in the ftn. p. 308, the three eggs were laid on three successive days; two eggs were alike in colour, but the third completely different.

† Actually of course every degree of elaboration can occur with a hole and every degree of accumulation of material. Thus although we have only two categories these cover practically continuous grades of elaboration and justify the use of biserial *r* method of determining the association.

TABLE III.
Correlation of Nest Type and Longitudinal Girth of Egg.

Longitudinal Girth.

Type of Nest	Totals	10-295*	10-395	10-495	10-595	10-695	10-795	10-895	10-995	11-095	11-195	11-295	11-395	11-495	11-595	11-695	11-795	11-895	11-995	12-095	12-195	12-295	12-395	12-495	12-595	12-695	Totals
S ...	143	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	143
C ...	967	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	967
Totals	1110	1	0	4	2	2	11	22	17	30	76	75	122	132	142	109	77	94	78	43	36	23	7	6	1	1	1110

* 10-295— signifies all values from 10-295 to 10-395 cms., the measurements being taken to two decimal places only. Thus all values from 10-30 to 10-39 fell in this group.

TABLE IV.
Correlation of Nest Type and Length of Egg.

Length of Egg.

Type of Nest	Totals	3-245*	3-295	3-345	3-395	3-445	3-495	3-545	3-595	3-645	3-695	3-745	3-795	3-845	3-895	3-945	3-995	4-045	4-095	4-145	4-195	4-245	4-295	4-345	4-395	4-445	4-495	4-545	4-595	4-645	4-695	4-745	4-795	4-845	4-895	4-945	4-995	Totals						
S ...	143	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	143					
C ...	967	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	967					
Totals	1110	1	1	1	0	4	4	4	6	6	18	33	33	67	97	146	118	155	75	118	59	50	50	45	5	8	14	14	104	51	45	47	26	4	3	4	2	2	9	3	2	1	2	1110

* 3-245— denotes all the values from 3-245 to 3-295, i.e. all the recorded values to two decimals from 3-25 to 3-29.

This relationship is hardly significant, and if significant only of very small intensity. It would indicate that the eggs of greater longitudinal girth were on the whole deposited in the more elaborated nests.

To investigate the matter more closely we now correlated the length and breadth of the egg with the nature of the nest, obtaining Tables IV and V.

Here the mean of the egg lengths in the simple nests is 4.177 cms. and for the total population 4.206 cms. while the correlation is given by

$$r = +.0953 \pm .0321.$$

This is probably just significant although only slightly larger than that for the longitudinal girth.

TABLE V.

Correlation of Nest Type and Breadth of Egg.

Breadth of Egg.

Type of Nest	* 2.595—	— 2.645	— 2.695	— 2.745	— 2.795	— 2.845	— 2.895	— 2.945	— 2.995	— 3.045	— 3.095	— 3.145	— 3.195	— 3.245	— 3.295	Totals
<i>S</i>	—	—	1	—	4	6	8	26	34	34	21	6	3	—	—	143
<i>C</i>	1	3	6	19	32	54	120	154	200	185	126	45	18	3	1	967
Totals	1	3	7	19	36	60	128	180	234	219	147	51	21	3	1	1110

* 2.595— denotes all values from 2.595 to 2.645, i.e. all the recorded values to two decimals from 2.60 to 2.64.

Here the mean of the egg breadths for the simple nests is 3.028, while for the total population it is 3.013. We have

$$r = -.0952 \pm .0321,$$

or the broader eggs are on the whole in the less-elaborated nests. Thus far then the rough nests appear associated with a short broad egg, although the correlations are only slight.

With a view of analysing this point further we now investigate the correlation of the index with the type of nest.

TABLE VI.

Correlation of Nest Type and Egg Index B/L.

Values of Index.

Type of Nest	* 57.95—	— 59.95	— 61.95	— 63.95	— 65.95	— 67.95	— 69.95	— 71.95	— 73.95	— 75.95	— 77.95	— 79.95	— 81.95	— 83.95	— 85.95	— 87.95	— 89.95	— 91.95	Totals
<i>S</i>	—	—	—	3	8	14	31	40	33	10	4	—	—	—	—	—	—	—	143
<i>C</i>	2	1	17	33	71	168	229	213	142	59	23	4	—	3	1	—	—	1	967
Totals	2	1	17	36	79	182	260	253	175	69	27	4	0	3	1	0	0	1	1110

* 57.95— denotes all values from 57.95 to 59.95, i.e. all recorded values from 58.0 to 59.9, the indices being recorded to one decimal place.

The mean index for the rougher nest was 72.530 and for the general population 71.752. We find for the correlation :

$$r = -.1372 \pm .0319,$$

a value greater than in the case of either length or breadth, the less elaborated nests having the rounder egg.

We now took $C = B^2 \times L$ as a rough measure of the volume of the egg and found :

$$r = -.0223 \pm .0322,$$

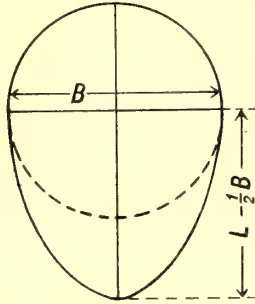
or r is sensibly zero. In other words there is no relation of volume of egg to the type of the nest. Since we might suppose the younger bird to lay smaller eggs, or at any rate less broad eggs, the solution of the simple nests being due to young birds finds no confirmation in our analysis; it is the shape of the egg rather than its size which is associated with its environment. In order to test this further the lower portion of the axis $L - \frac{1}{2}B$ and the Second Index of Ovality*, $100(L - \frac{1}{2}B)/B$, were correlated with the type of nest.

They gave respectively :

$$r = +.1233 \pm .0319,$$

and

$$r = +.1492 \pm .0318.$$



In other words the greater the extension below the hemisphere and the greater the ovality the more likely the nest to be elaborated. Thus we see that the rotund egg is more characteristic of the careless nest. It is conceivable that the rounder the egg the less likely it is to catch the eye when laid amid small pebbles and shingle. We next turn to investigate the association of colour and mottling with type of nest. First we inquire as to the simple relation of green and brown to the nest. Here we cannot go further than a fourfold table :

* The relative advantage of $O_2 = 100(L - \frac{1}{2}B)/B$ and $O_1 = 100B/(L - \frac{1}{2}B)$ consists solely in the ovaloid character of the egg increasing as O_2 increases, while it decreases as O_1 increases. Either may really be used indifferently if this be borne in mind.

TABLE VII.

Type of Nest and Colour.

Type of Nest.			
Colour		<i>S</i>	<i>C</i>
			Totals
	Brown ...	63	376
	Green. ...	79	590
Totals		142	966

One 'slatey grey' egg and one 'blue' egg had to be omitted from this table.

We find for tetrachoric r

$$r = +.0745 \pm .0409.$$

This cannot in itself be considered significant. The sign indicates that green egg-layers make the more elaborate nests. No stress can, however, be laid on the result.

We now take mottling and type of nest using the arrangement below as the best order we could devise of decreasing mottling.

TABLE VIII.

Mottling and Type of Nest.

Categories of Mottling.

Type of Nest	<i>d</i>	<i>e</i>	<i>g</i>	<i>a + b</i>	<i>c</i>	<i>h</i>	<i>f</i>	<i>i</i>	Totals
<i>S</i> ...	1	39	10	7	58	5	20	3	143
<i>C</i> ...	15	198	71	61	430	24	135	31	965
Totals	16	237	81	68	488	29	155	34	1108

The method adopted was that of 'biserial η ' with class index correction for the mottling categories. We find, the class index correlation being .9534,

$$\text{Correlation} = +.1141 \pm .0325,$$

the sign indicating that the finer blotches are associated with the more elaborate nests.

* The preponderance of green eggs over brown in the ternery at Blakeney Point deserves consideration because it has not always been recognised. H. Seebohm, *Eggs of British Birds*, London, 1896, writes that the eggs "vary in ground colour from pale greyish-buff to brownish-buff, occasionally with a tinge of green" (p. 102). F. O. Morris, *Natural History of the Nests and Eggs of British Birds*, London, 1892, gives a wider range of colours, "pale blue, pale yellow, green, brown, white or light dull yellowish or stone colour" (Vol. III. p. 136), which certainly does not emphasise the broad alternative categories brown or green, with a fractional percentage of blue or grey.

Lastly we turn to the intensity of the ground colour and the type of nest. Here we have worked independently brown and green eggs and the results are given in Tables IX and X.

TABLE IX.

Type of Nest and Density of Colour in Brown Eggs.

Density of Colour.

Type of Nest	A_1	B_1	C_1	D_1	E_1	$F_1 + G_1$	$H_1 + I_1 + K_1$	Totals
<i>S</i> ...	15	11	5	12	8	6	6	63
<i>C</i> ...	39	65	58	83	43	38	50	376
Totals	54	76	63	95	51	44	56	439

Again we use the 'biserial η ' method and correction for class index correlation (.9785); we find

$$\text{Correlation} = + \cdot 2189 \pm \cdot 0481.$$

Thus there is significant, if only still very moderate, correlation, the relationship being between denser brown ground colour and the simpler nests, i.e. holes in the ground.

TABLE X.

Type of Nest and Density of Colour in Green Eggs.

Density of Colour.

Type of Nest	A_2	B_2	C_2	D_2	E_2	F_2	G_2	$H_2 + I_2 + K_2$	Totals
<i>S</i> ...	3	5	14	10	12	6	15	14	79
<i>C</i> ...	31	46	57	123	73	79	54	127	590
Totals	34	51	71	133	85	85	69	141	669

Using the same method as before (class index correction .9860), but with one more category as F_2 and G_2 could be separated as their total was more considerable, we have

$$\text{Correlation} = - \cdot 2366 \pm \cdot 0407.$$

Thus the dark tones of green are on the whole more frequently associated with the nests to which material is brought.

Accordingly in the case of both ground colours, although we cannot definitely assert that either brown or green egg-layers are the more elaborate nest builders, we can assert that the denser brown and lighter greens are somewhat more usual when the nest is a mere hole in the shingle, and that the lighter brown and darker green eggs are associated with more elaborately constructed nests. Again the larger blotches are in somewhat greater proportion to be associated with unelaborated nests and the finer mottling with the elaborate nests. There is no

reason to believe in any appreciable difference in volume of the simple nest and the complex nest eggs, but the former differ somewhat in shape from the latter, being broader and shorter, i.e. the eggs in mere holes are more rotund and in the elaborate nests more ovaloid.

Although none of these characters appear to be *highly* correlated with the type of nest as determined by the simple alternative categories adopted by the field-workers, yet they are of a nature which more or less lend themselves to explanation on the basis of a protective colouring. It is not possible to determine whether the great variety of colouring and mottling in the common tern's egg is a vestige of an elaborate system once developed for protective purposes, and now falling into disuse, or, as a product of physiological causes, it is now being slowly adapted to protective purposes. The problem is a very interesting one and further light we think might be thrown on it, if a fuller record were in future to be made of the immediate colouring of the nest,—the colour of the materials out of which it is made, and in the case of holes the colour of the ground, shape and nature and colour of the adjacent pebbles or shingle. It would mean much additional labour, but considerable information bearing on the points discussed above might arise from such data.

(4) *The Problem of the Mixed Colour Clutches.*

We propose in this section to discuss the problem of the mixed colour clutches. The following are the data to be analysed :

TABLE XI.
Colour Composition of Clutches.

Size of Clutch	Number	Colour Composition	Number of Eggs
1	138	74 B + 63 G + 1 SG	138
2	178	67 B ² + 19 BG + 92 G ²	356
3	204	62 B ³ + 8 B ² G + 14 BG ² + 119 G ³ + 1 BL	612
4	1	0 B ⁴ + 0 B ³ G + 0 B ² G ² + 0 BG ³ + 1 G ⁴	4
Totals	521	203 B only, 41 composite, 275 G only, 2 anomalous	1110

$B^n = n$ brown, $G^m = m$ green, SG = slatey-grey, BL = blue eggs*.

Putting aside the two anomalous eggs, we have 41 clutches out of 519 wherein brown and green eggs are mixed. Putting aside the clutch with 4 green eggs we see that as a whole there are

443 brown eggs to 659 green eggs,

* The blue egg may be accounted for by the oxidisation of a green egg—a phenomenon observed by Newton (Art. 'Birds' Eggs,' *Encycl. Brit.*); the origin of the oxidisation being unrecognised in this case. Newton also states that the individuals of some few species of birds do not always lay eggs of the same ground colour, but the source indicated by him, i.e. change with age of bird, would not apply to our case.

but the proportions vary with the size of the clutch; for we have

74 brown to 63 green eggs in the clutches of 1,
153 brown to 203 green eggs in the clutches of 2,
216 brown to 393 green eggs in the clutches of 3,

or 100 to 85, 100 to 133, 100 to 184 brown to green eggs respectively. In other words the proportion of green to brown eggs increases with the size of the clutch. Those readers who will examine Plate VIII in the first memoir* will see how distinct the brown and green ground colours are, and will understand how necessary it is to find some explanation for the change in proportions of colour as the clutch increases in size, and for the mixture of colours in the same nest. The field-workers appear to be confident that the same bird can lay different coloured eggs, basing their statement apparently on diversity of colour appearing in clutches of eggs having the same size or shape.

The hypotheses that suggest themselves are:

(i) That the common terns consist of two *gentes* one of which lays brown and the other green eggs. The mixture of colour arises from the existence of 'cuckoo' terns who lay in other hen's nests.

We cannot ascertain the number of brown egg-laying tern 'cuckoos' who lay in brown egg nests or of green egg-laying tern 'cuckoos' who lay in green egg nests. But if the 19 *BG* arise from cuckoo-terns, we must originally have had $74 + 63 + 19$ single egg nests and in these 156 nests 19 tern 'cuckoos' of opposite colour laid. The chance therefore of a tern 'cuckoo' of opposite egg colour laying in the 1 egg nests is $\cdot 1218$. Treating the 2 egg nest in the same manner, we have $67 + 92 + 8 + 14 = 181$ of them and in 22 we have occurring the egg of the tern cuckoo of opposite colour, or the chance is $\cdot 1215$; this number is substantially the same as we reached before and the coincidence is remarkable. But it collapses when we go a stage further. We have $62 + 119$ whole colour clutches of 3, we should therefore expect 25 clutches of 4 with composite colours, i.e. $25/(62 + 119 + 25) = \cdot 122$ nearly. Now only a single 4 clutch nest was found and this had all green eggs. With a chance of about 1 in 8 that a cuckoo-tern will lay in any nest, it is hard to believe that it missed at least 181 nests. It appears that three eggs is the practical limit to the size of the clutch laid by one hen, but it seems hard to believe that the cuckoo-tern would avoid all nests which already had three eggs†, i.e. the cuckoo-tern hypothesis seems to involve a considerable percentage of composite four egg clutches, which do not appear. This argument seems sufficient to render the hypothesis very improbable.

(ii) There is only one *gens* of the common tern which can lay both brown and green eggs. Since, however, the number of green eggs increases with the size of the clutch, it is not possible to consider the chance of laying a brown,

* *Biometrika*, Vol. x. p. 146.

† Or that the rightful owner having laid two eggs would refrain from laying the third because the 'cuckoo' tern had already laid it.

respectively a green egg as the same in successive layings. The change of pigment in successive layings may be a physiological exhaustive process as a change from a melanin to a lipochrome. This hypothesis does not assume that any given bird may or may not lay a green or brown egg according to a given law of chance, but that physiologically there is a tendency with successive laying to alter the nature of the pigment in the glands or on the surface of the oviduct. For example the hen, as the incubation period approaches, may change the quantity or character of her food.

It is probable, however, that the changes will not be the same for small and large layers, we shall therefore give generality to the problem by supposing the probability of laying a brown egg to vary not only with the number of eggs laid but with each egg.

We have then the following system of notation: $p'_s, p''_s, p'''_s, \dots$ = chance of laying a brown egg in the 1st, 2nd, 3rd, ... laying of a hen who lays a clutch of s eggs. The corresponding chances of laying green eggs will be $q'_s = 1 - p'_s, q''_s = 1 - p''_s, q'''_s = 1 - p'''_s, \dots$. Let there be N_s s -clutch common tern hens.

Then our data are to be provided by the equations:

$$\begin{aligned} N_1 p'_1 + N_1 q'_1 &= 74 + 63 \quad (N_1 = 137), \\ N_2 p'_2 p''_2 + N_2 (p'_2 q''_2 + q'_2 p''_2) + N_2 q'_2 q''_2 &= 67 + 19 + 92 \quad (N_2 = 178), \\ N_3 p'_3 p''_3 p'''_3 + N_3 (p'_3 p''_3 q'''_3 + p'_3 p''_3 q'''_3 + p'_3 p''_3 q'''_3) \\ &+ N_3 (p'_3 q'''_3 q'''_3 + p'_3 q'''_3 q'''_3 + p'_3 q'''_3 q'''_3) + N_3 q'_3 q''_3 q'''_3 \\ &= 62 + 8 + 14 + 119 \quad (N_3 = 203). \end{aligned}$$

Dividing out by the totals in each case and equating corresponding terms we have the following system of equations to solve:

$$\begin{aligned} p'_1 &= .540,1460, \quad q'_1 = .459,8540 \quad \dots\dots\dots(i), \\ p'_2 p''_2 &= .376,4045, \quad p'_2 q''_2 + p'_2 q''_2 = .106,7416, \quad q'_2 q''_2 = .516,8539 \quad \dots\dots(ii), \\ p'_3 p''_3 p'''_3 &= .305,4187, \quad p'_3 p''_3 q'''_3 + p'_3 p''_3 q'''_3 + p'_3 p''_3 q'''_3 = .039,4089, \\ p'_3 q'''_3 q'''_3 + p'_3 q'''_3 q'''_3 + p'_3 q'''_3 q'''_3 &= .068,9655, \quad q'_3 q''_3 q'''_3 = .586,2069 \dots(iii). \end{aligned}$$

(i) is solved as it stands. But it is clearly impossible to take $q'_2 = q'_1$ for this would involve q'_2 being greater than unity, an impossible value. Similarly q'_3 and q'_3 cannot be equal to q'_2 and q'_2 respectively, or we should have $q'_3 > 1$. Thus it is needful that the probability of laying a green egg should increase with successive eggs or be a function of the fertility. Assuming this change of probability, we may write the first equations of (ii), the third is not independent:

$$p'_2 p''_2 = .376,4045, \quad p'_2 (1 - p''_2) + p'_2 (1 - p''_2) = .106,7416,$$

which gives us $p'_2 + p'_2 = .859,5506$, or p'_2, p'_2 are roots of the quadratic

$$p_2^2 - .859,5506 p_2 + .376,4045 = 0.$$

These roots are *imaginary*.

Turning now to (iii) we find from the first three equations:

$$p_3' p_3'' p_3''' = 305,4187,$$

$$p_3'' p_3''' + p_3''' p_3' + p_3' p_3'' = 955,6650,$$

$$p_3' + p_3'' + p_3''' = 1.064,0394.$$

These lead to the cubic for p_3 ,

$$p_3^3 - 1.064,0394 p_3^2 + 955,6650 p_3 - 305,4187 = 0.$$

One root of this cubic is $p_3 = .449,5251$, which gives on dividing the factor $p_3 - .449,5251$ out

$$p_3^2 - .614,5143 p_3 + .679,4254 = 0.$$

The roots of this quadratic are both *imaginary*.

Accordingly neither the records for the nests with two eggs nor those for the nests with three eggs are consistent with a single *gens* the hens of which lay brown eggs with a tendency to lay green increasing with greater fertility. This hypothesis has therefore to be discarded.

(iii) As a last hypothesis we will assume that there are two *gentes* or types of females, one of which lays brown eggs (p_1) with a small chance of laying green ($q_1 = 1 - p_1$), and the other of which lays green eggs (p_2) with a slight chance of laying brown ($q_2 = 1 - p_2$). Let $N_s \nu_s$, $N_s (1 - \nu_s)$ be the number of brown and green laying hens in the group N_s which lays s eggs in the clutch. We suppose p_1 and p_2 to be independent of the fertility of the hen, until this assumption is shown to be inadequate.

Clutches of 1 egg.

$$N_1 \nu_1 p_1 + N_1 (1 - \nu_1) q_2 = \text{number of brown eggs} = N_1 \epsilon_1', \text{ say,}$$

$$N_1 \nu_1 q_1 + N_1 (1 - \nu_1) p_2 = \text{number of green eggs} = N_1 \epsilon_2', \text{ say.}$$

For our special case:

$$\nu_1 p_1 + (1 - \nu_1) q_2 = .540,1460,$$

$$\nu_1 q_1 + (1 - \nu_1) p_2 = .459,8540.$$

These equations are not, however, independent and only suffice to determine ν_1 from

$$\nu_1 = (\epsilon_1' - q_2)/(p_1 - q_2) \dots\dots\dots(\text{iv}),$$

or the proportions of brown and green egg layers in clutches of one, when p_1 and q_2 have been found.

Clutches of 2 eggs.

If the distribution of clutches be $N_2 (\epsilon_1'' + \epsilon_2'' + \epsilon_3'')$

$$\nu_2 p_1^2 + (1 - \nu_2) q_2^2 = \epsilon_1'',$$

$$\nu_2 p_1 q_1 + (1 - \nu_2) q_2 p_2 = \frac{1}{2} \epsilon_2'',$$

$$\nu_2 q_1^2 + (1 - \nu_2) p_2^2 = \epsilon_3''.$$

Only two of these equations are independent and it is convenient to write them in the form :

$$\left. \begin{aligned} \nu_2 p_1 + (1 - \nu_2) q_2 &= \epsilon_1'' + \frac{1}{2} \epsilon_2'' \\ \nu_2 p_1^2 + (1 - \nu_2) q_2^2 &= \epsilon_1'' \end{aligned} \right\} \dots\dots\dots (v).$$

These will give, if p_1 and q_2 are known, one equation for the determination of ν_2 and one equation of condition.

Clutches of 3 eggs.

If the distribution of clutches be $N(\epsilon_1''' + \epsilon_2''' + \epsilon_3''' + \epsilon_4''')$ we have :

$$\begin{aligned} \nu_3 p_1^3 + (1 - \nu_3) q_2^3 &= \epsilon_1''' \\ \nu_3 p_1^2 q_1 + (1 - \nu_3) q_2^2 p_2 &= \frac{1}{3} \epsilon_2''' \\ \nu_3 p_1 q_1^2 + (1 - \nu_3) q_2 p_2^2 &= \frac{1}{3} \epsilon_3''' \\ \nu_3 q_1^3 + (1 - \nu_3) p_2^3 &= \epsilon_4''' \end{aligned}$$

Only three of these equations are independent and these may be written :

$$\left. \begin{aligned} \nu_3 p_1^3 + (1 - \nu_3) q_2^3 &= \epsilon_1''' \\ \nu_3 p_1^2 + (1 - \nu_3) q_2^2 &= \epsilon_1''' + \frac{1}{3} \epsilon_2''' \\ \nu_3 p_1 + (1 - \nu_3) q_2 &= \epsilon_1''' + \frac{2}{3} \epsilon_2''' + \frac{1}{3} \epsilon_3''' \end{aligned} \right\} \dots\dots\dots (vi).$$

These suffice to determine, ν_3 , p_1 and q_2 .

Uniting the right-hand sides of (vi) f_1''' , f_2''' , f_3''' respectively we find :

$$\nu_3 = (f_1''' - q_2)/(p_1 - q_2) \dots\dots\dots (vii),$$

which suffices to find ν_3 when p_1 and q_2 are found, and

$$\left. \begin{aligned} p_1 &= (f_2''' - f_1''' q_2)/(f_1''' - q_2) \\ p_1^2 &= (f_3''' - f_2''' q_2)/(f_1''' - q_2) \end{aligned} \right\} \dots\dots\dots (viii),$$

which lead to the quadratic for q_2

$$q_2^2 (f_2''' - f_1'''^2) - q_2 (f_3''' - f_2''' f_1''') + f_1''' f_3''' - f_2'''^2 = 0 \dots\dots\dots (ix).$$

We could therefore solve (ix) and choose the appropriate root for q_2 , find the corresponding p_1 from (viii), determine ν_3 from (vii), ν_2 from the first of (v) and ν_1 from (iv). We might then use the second equation of (v) as an equation of condition. But clearly this would not be satisfactory as all our quantities are subject to considerable sampling errors. The correct method would be to determine ν_1 , ν_2 , ν_3 , p_1 and q_2 from the *six* equations (iv), (v) and (vi) so as to get the best values of these variables. But this would be a very laborious process. We propose therefore to determine p_1 and q_2 by the method of least squares from the three equations

$$\left. \begin{aligned} p_1 &= (f_2''' - f_1''' q_2)/(f_1''' - q_2)^* \\ p_1 &= (f_2''' - f_1''' q_2)/(f_1''' - q_2) \\ p_1^2 &= (f_3''' - f_2''' q_2)/(f_1''' - q_2) \end{aligned} \right\} \dots\dots\dots (x),$$

* Obtained by writing $f_1'' = \epsilon_1'' + \frac{1}{2} \epsilon_2''$, $f_2'' = \epsilon_1''$ and eliminating ν_2 between two equations of (v).

using to obtain linearity $q_2 = \bar{q}_2 + \eta$, where $\bar{q}_2$ is the value given by the quadratic (ix) and η is supposed a small quantity with negligible square. The values of p_1 and q_2 found from (x) will be good, if not the best. Our system for $\nu_1, \nu_2, \nu_3, p_1, q_2$ will not be the *optimum possibile*, but if our system is probable, that will be still more probable and the hypothesis of the two *gentes* of tern hens will not be contradicted by the data.

Our system of ϵ 's is:

$$\begin{aligned}\epsilon_1' &= .540,1460, & \epsilon_1'' &= .376,4045, & \epsilon_2'' &= .106,7416, \\ \epsilon_1''' &= .305,4187, & \epsilon_2''' &= .039,4089, & \epsilon_3''' &= .068,9655,\end{aligned}$$

leading to:

$$\begin{aligned}f_1'' &= .429,7753, & f_2'' &= .376,4045, \\ f_1''' &= .354,6798, & f_2''' &= .318,5550, & f_3''' &= .305,4187.\end{aligned}$$

(ix) now becomes:

$$.192,7573q_2^2 - .192,4337q_2 + .006,8485 = 0,$$

giving the *small* value $\bar{q}_2 = .036,9571$ for the chance of a green *gens* hen laying a brown egg.

We now return to (x) substituting the f 's and $.036,9571 + \eta$ for q_2 . Expanding and neglecting η^2 we obtain, on extracting the root of p_1^2 in the third equation

$$\begin{aligned}p_1 &= .917,7816 + 1.242,3209 \eta, \\ p_1 &= .961,3638 + 1.909,4759 \eta, \\ p_1 &= .961,3638 + .991,4412 \eta.\end{aligned}$$

Solved by least squares these equations give for type equations:

$$\begin{aligned}p_1 &= .946,8364 + 1.381,0793 \eta, \\ p_1 &= .948,2960 + 1.489,7513 \eta,\end{aligned}$$

leading to:

$$\begin{aligned}p_1 &= .928,2876, & q_1 &= .071,7124, \\ p_2 &= .976,4736, & q_2 &= .023,5264.\end{aligned}$$

Whence from (vii), the first of (v) and (iv),

$$\begin{aligned}\nu_3 &= .366,0119, & 1 - \nu_3 &= .633,9881, \\ \nu_2 &= .449,0123, & 1 - \nu_2 &= .550,9877, \\ \nu_1 &= .571,0011, & 1 - \nu_1 &= .428,9989.\end{aligned}$$

Thus about 7% of the eggs laid by hens of the brown-laying *gens* will be green, and only about 2% of the eggs laid by hens of the green-laying *gens* will be brown. Further the green-laying *gens* is far more fertile than the brown-laying *gens*, the proportion of brown to green layers falling from 57 to 43 in the single clutches to 37 to 63 in the triple clutches. The following is our analysis on this basis.

*Single Egg Nests.*Observed $B\ 74, G\ 63.$ Theoretical $B\ 74, G\ 63.$

Number of brown egg layers 78.23.

" green " 58.77.

Number of brown egg layers with brown eggs 72.62.

" " " with green eggs 5.61.

Number of green egg layers with green eggs 57.39.

" " " with brown eggs 1.38.

Two Egg Nests.

	B^2	BG	G^2
Observed	67	19	92

Theoretical	68.92	15.15	93.93
-------------	-------	-------	-------

Number of brown egg layers 79.92.

" green " 98.08.

Number of brown egg layers who lay both eggs brown 68.87.

" green " " " 0.05.

Number of brown egg layers who lay one brown, one green 10.64.

" green " " " 4.51.

Number of brown egg layers who lay both eggs green 0.41.

" green " " " 93.52.

Three Egg Nests.

	B^3	B^2G	BG^2	G^3
Observed	62	8	14	119
Theoretical	59.43	13.98	9.72	119.87

Number of brown egg layers 74.30.

" green " 128.70.

Number of brown egg layers who lay 3 brown eggs 59.43.

" green " " " 0.00.

Number of brown egg layers who lay 2 brown and 1 green 13.77.

" green " " " 0.21.

Number of brown egg layers who lay 1 brown and 2 green 1.06.

" green " " " 8.66.

Number of brown egg layers who lay 3 green eggs 0.03.

" green " " " 119.84.

It will be noted that the theory gives for the B^2G and BG^2 about inverted proportions. It also falls short in the BG group. These would very probably have been bettered with a more general solution of our six equations. But are

the existing frequencies inconsistent with our observations and beyond the limits of random sampling? Summing up our results we have:

	<i>B</i>	<i>G</i>	<i>B</i> ²	<i>BG</i>	<i>G</i> ²	<i>B</i> ³	<i>B</i> ² <i>G</i>	<i>BG</i> ²	<i>G</i> ³
Observed	74	63	67	19	92	62	8	14	119
Calculated	74	63	68.92	15.15	93.93	59.43	13.98	9.72	119.87

From these we find $\chi^2 = 5.631$, giving $P = .688$, or in 69 trials out of 100 the sample would be more discordant from the calculated than the actual observations. There is accordingly nothing to be said against the theory on the ground of its statistical improbability.

Again of the two hypotheses involved, (i) the greater fertility of the green egg layers, (ii) the fixed small probability that a hen of one *gens* will lay occasionally an egg of the colour of the other *gens*, the first seems not unreasonable; the second gives merely a quantitative measure of the assumption made by a number of ornithologists that birds can lay eggs of two colours. It assumes, however, that as a rule they do not. Clearly we need to know more of the mechanism of egg coloration before we can settle how it happens that a bird usually staining its egg brown will stain it green on a few occasions. If it be a result of type of food, we have to assume that our two *gentes* feed as a rule differently, which is not easily to be admitted. Will this feeding habit then be hereditary and if so are the male birds also divided into two *gentes* and is the mating assortative? Granted on the other hand that it is not due to food, but to differences of pigmentation mechanism, we are compelled to ask whether this mechanism is inherited only through the female. If not, then are the matings within the *gens*, or what is the pigmentation mechanism of heterozygote hens? If we could establish the existence of the two *gentes* each with its rule and its fixed exception to rule; if further the pigmentation mechanism as one must decidedly expect from the eggs of many species is markedly hereditary, then it is possible that in these clutches of composite colour lies the solvent of some difficulties which the Mendelian explanation meets with when the product of two protogene zygotes instead of being protogene is in rare cases found to be allogene.

(5) *The Organic Correlations.*

We devote this section to a consideration of the degree of relationship between size, shape and colour characters of the same egg, and their relative values in the seasons 1913 and 1914.

(i) *Mottling and Breadth, Length and Index of Egg.*

The value of the correlation of mottling and breadth in the 1913 census was .1803, but unfortunately the sign of it was possibly wrongly given, as may be seen from the Table p. 150 of the former paper (*Biometrika*, Vol. x.). We have taken occasion already to refer to the difficulties in the mottling scale used, but after

much consideration we are unable to substantially modify the assumed order of mottling of the previous paper. In broad lines we have :

TABLE XII.

Mottling	Mean Breadth	
	1913 census (291 eggs)	1914 census (1108 eggs)
Confluent blotches, $d+e+g$	2.97	3.03
Transition Forms, $a+b \dots$	2.97	2.95
Discrete, Copious, $c \dots$	2.99	3.02
Discrete, Sparse, $h+f+i$	2.96	3.01

The value of polyserial η corrected for class index correlation of mottling is .1753 for the census of 1914. It is therefore certainly within the probable error of the difference. Now in both cases the confluent mottling gives a greater breadth than the discrete and sparse mottling, but the transition forms $a+b$, and c , are anomalous. The correlation ratio η in both cases is significant and shows a relation, not very intense, between mottling and breadth, but in the present stage of the mottling classification it is certainly not possible to unravel the relationship. The 1914 returns undoubtedly seem to indicate that not only the confluent but the discretely mottled eggs have the greater breadths, the lesser breadths being found in the transition forms. It should be noted that the returns for 1914 being nearly four times as numerous are worth twice as much.

If we could really lay any stress on the sign to be given to the association, we should have to assert that in the species at large the rule is opposite to that for the individual hen. In her case the broader egg has less mottling, while in the species the broader egg has the greater or at least the more confluent mottling. The former relation overrides any result to be obtained from the species as a whole, and seems to oppose any theory that greater pressure during transition through the oviduct is the source of greater mottling*.

We have further worked out the association† of Index and Length to the Mottling. We have

	Census 1913	Census 1914
Mottling and Breadth	.1803	.1753
Mottling and Length	—	.0937
Mottling and Index	.1550	.1598

$$\left. \begin{array}{l} .1753 \\ .0937 \\ .1598 \end{array} \right\} \eta_0 = .0850 \pm .0203 \ddagger.$$

Since the probable error is of the order .02 we see that the value is insignificant for length. On the other hand the order of mottling classes in the three

* The time of transition through the oviduct may conceivably be a factor of greater importance.

† Obtained from polychoric η with correction for number of arrays and the class index correction for mottling.

‡ η_0 is the mean value of the correlation ratio on the assumption of no association.

cases does not appear interpretable. The following table gives the means for each class of mottling as specified in Plate VIII of the first memoir.

TABLE XIII.
Mottling and Size and Shape of Egg.

Type of Mottling.	Index	Breadth	Length
<i>b</i>	69.52	2.954	4.256
<i>i</i>	70.60	2.998	4.255
<i>a</i>	70.68	2.947	4.170
<i>e</i>	71.71	3.025	4.228
<i>c</i>	71.86	3.016	4.200
<i>f</i>	71.90	3.001	4.184
<i>h</i>	72.12	3.042	4.220
<i>g</i>	72.65	3.032	4.184
<i>d</i>	73.08	3.017	4.133

The series for index in ascending order corresponds roughly to a series in ascending order for breadth and descending order for length, but the system does not correspond to any easily appreciated mottling order. It appears as if the fieldworkers might have been influenced by shape of egg, instead of merely comparing the nature of the mottling in selecting type. At any rate in this section no final conclusions can be drawn, and it seems very desirable that more elaborate descriptions of mottling should in future be carried out.

(ii) *Ground Colour and Breadth, Length and Index of Egg.*

The following scheme gives our results. The first value of η is the uncorrected η' , the second the value when corrected for number of arrays and class index correlation, which is .9785 for brown and .9860 for green eggs.

TABLE XIV.

	Index		Breadth		Length	
	Brown	Green	Brown	Green	Brown	Green
η'	.1747	.1733	.1348	.1385	.2061	.1432
η	.1011	.1313	imaginary†	.0773	.1530	.0857
η_0^*	{ .1432 { $\pm$.0322	{ .1160 { $\pm$.0261	{ .1432 { $\pm$.0322	{ .1160 { $\pm$.0261	{ .1432 { $\pm$.0322	{ .1160 { $\pm$.0261

It will be fairly obvious from this table that there is no association of ground

* Mean value of η supposing no association.

† This signifies that if η_0^2 be taken from η'^2 the difference is negative, i.e. η' is less than the mean value of η for zero association.

colour, whether green or brown, with either size or shape of egg*. This does not appear at all unreasonable if we assume the ground colour to be deposited before the egg enters the oviduct or the shell becomes finally hardened.

The general conclusion therefore to be drawn from the present investigation is that intensity of ground colour, whether green or brown, has no relation to egg size and shape, but that breadth of egg, whether considered directly or through the index, is more probably related though not intensely to mottling, but the nature of the relationship must be obscure until a more elaborated classification of mottling has been adopted.

(iii) *Relation of Mottling to Ground Colour.*

The data are given in Table G^a at the end of this memoir, where we have separated brown from green eggs, because it is conceivable that the relationships, if any, for the two categories might be different. If $\bar{C}_2$ denote the mean contingency when there is zero association we have :

For the Brown Eggs : $\bar{C}_2 = .2830 \pm .0323$.

Uncorrected Contingency : $C_2 = .2030$.

For the Green Eggs : $\bar{C}_2 = .2118 \pm .0261$.

Uncorrected Contingency : $C_2 = .2557$.

Thus for the brown eggs there is no significance in C_2 , it being less than the mean value of the contingency, when there is no association. For the green eggs C_2 is greater than $\bar{C}_2$ but the difference is less than twice the probable error of $\bar{C}_2$; we cannot therefore assert any real relation to exist between mottling and intensity of ground colour†. Under the circumstances of the above relation of $\bar{C}_2$ to C_2 , it did not seem necessary to correct C_2 , as such correction would not alter the conclusion of no significant association. Although the intensity of ground colour may have no relation to mottling, it is conceivable that the colour of the egg may itself have relation to mottling or indeed to intensity of ground colour, i.e. a brown egg may have deeper tones of ground colour and denser mottling than a green egg.

We have the following biserial tables to illustrate these cases.

TABLE XV.
Mottling and Colour of Egg.
Mottling Categories.

Colour of Egg	$g+d$	e	$a+b$	$c+h$	f	i	Totals
Brown ...	36	89	29	215	57	11	437
Green ...	61	148	39	300	98	23	669
Totals ...	97	237	68	515	155	34	1106

* This statement is not really contradicted by the $\eta = .1506$ of p. 148 of the previous memoir, for with the small number 291 eggs of that census $\eta_0 = .1655 \pm .0395$!, so that η is less than the value for zero association.

† Examined in the same manner the result for 1913 appears not to have the significance we attributed to it. We have $\bar{C}_2 = .2813 \pm .0395$, while the corrected contingency is only $C_2 = .2260$. Thus C_2 is actually less than the mean value when there is no contingency.

The order of mottling categories seems to correspond as closely as we can determine from the plate to the order of relative amount of mottling.

We find for the mean η when there is no association :

$$\bar{\eta}^2 = \cdot 004521.$$

Hence η' corrected for number of arrays but not for class-index is given by

$$\eta'^2 = \frac{\cdot 004,702 - \cdot 004,521}{\cdot 996,383} = \cdot 0001817,$$

or

$$\eta' = \cdot 0135.$$

This is insignificant and therefore we need not trouble to find the class-index correction. It would not appear therefore that the brown eggs are more densely mottled than the green eggs.

We now pass to intensity of ground colour. It will be remembered that two scales were formed of 'values' giving as far as possible equal values by the same letters for both green and brown colours.

TABLE XVI.

Colour and Value.

Ground Colour Values.

Colour of Egg	A	B	C	D	E	F+G	H+I+K	Totals
Brown ...	52	76	63	95	51	44	56	437
Green ...	34	51	71	133	85	154	141	669
Totals ...	86	127	134	228	136	198	197	1106

It is clear on the face of this table that the percentage of high values in the brown series is far greater than in the green series, which has a much greater percentage of low-colour values. To get an appreciation of this association we use biserial η . We have for zero association

$$\bar{\eta}^2 = \cdot 005,425,$$

while uncorrected $\eta'^2 = \cdot 113,734$.

Accordingly corrected for a number of arrays

$$\eta''^2 = \frac{\cdot 113,734 - \cdot 005,425}{\cdot 995,479} = \cdot 108,8009,$$

leading to

$$\eta'' = \cdot 3299.$$

Calculating the class-index correlation, we find it $\cdot 9674$ and thus finally corrected

$$\eta = \cdot 3410 \pm \cdot 0197.$$

This is a significant and fairly substantial correlation between colour and colour value.

It would appear as if absence of Sorby's oozhodeine pigment* also involved less copious pigment material in general.

(iv) *Organic Relations in Shape and Size.*

The fundamental tables are Tables G to L at the end of the paper. The correlations are as follows:

TABLE XVII.
Organic Correlations in two Seasons.

Character Pair	Symbols	Correlation, 1914 (c. 1110)	Correlation, 1913 (c. 294)
Length and Breadth	L, B	$\cdot 2104 \pm \cdot 0193$	$\cdot 2220 \pm \cdot 0374$
Longitudinal and Equatorial Girths	G_l, G_b	$\cdot 5139 \pm \cdot 0149$	$\cdot 5297 \pm \cdot 0284$
Length and Longitudinal Girth ...	L, G_l	$\cdot 8515 \pm \cdot 0055$	$\cdot 8804 \pm \cdot 0088$
Breadth and Longitudinal Girth ...	B, G_l	$\cdot 4840 \pm \cdot 0155$	$\cdot 5216 \pm \cdot 0286$
Index and Length	I, L	$-\cdot 7577 \pm \cdot 0086$	$-\cdot 7284 \pm \cdot 0185$
Index and Breadth	I, B	$\cdot 4537 \pm \cdot 0161$	$\cdot 5033 \pm \cdot 0294$
Index and Longitudinal Girth ...	I, G_l	$-\cdot 4496 \pm \cdot 0161$	$-\cdot 3832 \pm \cdot 0336$

The following table contains the seasonal difference and its probable error.

TABLE XVIII.
Seasonal Change in Correlation.

Character Pair	$\Delta = 1914 - 1913$	Probable error of Δ
L and B	$-\cdot 0116$	$\pm \cdot 0421$
G_l and G_b	$-\cdot 0158$	$\pm \cdot 0321$
L and G_l	$-\cdot 0289$	$\pm \cdot 0104$
B and G_l	$-\cdot 0376$	$\pm \cdot 0325$
I and L	$-\cdot 0293$	$\pm \cdot 0204$
I and B	$-\cdot 0496$	$\pm \cdot 0335$
I and G_l	$-\cdot 0664$	$\pm \cdot 0373$

With the exception of the correlation of Length and Longitudinal Girth none of these differences has a significant relation to their probable errors. In the case mentioned, however, such a deviation would occur in excess 3 times in 100 trials and in defect 3 times in 100 trials, or as we have made 7 trials the odds against it are only 52 to 48. We cannot therefore lay much stress on it, and conclude that no seasonal change in the organic correlations is to be observed between 1913 and 1914. As there were considerable changes in the means (see our p. 310) this result confirms

* "On the Colouring-matters of the Shells of Birds' Eggs," *Zoological Society's Proceedings*, 1875, p. 359.

the general conclusion that except for very skew distributions, change in means does not involve change in correlation. Change in variability does usually denote change in correlation, but as we have indicated (p. 311) the changes in variability are not significant except in the girths, and this may be the source of such modification as we find in the correlation of Length and Longitudinal Girth. To test this we note that if Longitudinal Girth only be changed the regression coefficient of the Length on the Longitudinal Girth ought not to be changed within the limits of random sampling. For 1914 this coefficient of regression is $.4501 \pm .0056$ and for 1913 is $.4215 \pm .0089$. Hence the difference is $.0286 \pm .0105$. Thus the difference in the regression coefficients is just as significant as it was in the correlation coefficients, or is not explicable on the basis of increased variability in the Longitudinal Girth. If it is, which we doubt, to be considered significant it must depend on something else than a more variable Longitudinal Girth.

We may consider in this place what changes have taken place in the formula connecting Longitudinal Girth with Length and Breadth. For 1914 we have :

$$G_l = 1.1273 B + 1.4840 L + 1.9180,$$

while for 1913 we had :

$$G_l = 1.2701 B + 1.6415 L + .8224.$$

The changes in the coefficients look more considerable than the changes that will be found in the values for G_l calculated from either formula for eggs which are not extreme variants. At the same time the differences rather tend to emphasise the suggestion given by the correlation of G_l with L , that there may have been a seasonal change in the organic relationship between these characters.

(6) *The Homotypic Correlations.*

The results for the 1914 season are of a very startling character ; they demonstrate that while the organic correlations remain nearly constant the homotypic correlations can suffer a very considerable seasonable modification. In other words the birds laid eggs very much more alike in 1914 than in 1913. The reader will remember that 1913 was a bad season for the birds, many young perished and there were few nests. On the other hand 1914 was a good season ; there was plenty of food, numerous and possibly stronger birds. The eggs in the clutches were more alike in 1914 than in 1913.

We proceeded to investigate in the first place whether the greater intensity of homotyposis was due to there being a far larger proportion of three-egg clutches. Accordingly we took only the 1st and 2nd eggs in the clutches and obtained the homotypic correlation for Equatorial Girth. It was .7535, for 383 pairs of eggs. When we took all possible pairs out of all the clutches we had 796 pairs, and the correlation instead of rising, fell, but insignificantly to .7469. The difference between 1913 and 1914 cannot therefore be due to a far larger number of clutches providing three pairs in the latter than in the former year.

TABLE XIX.

Direct Homotyposis in Size and Shape.

Characters	Symbols	Season 1913	Season 1914
Lengths of Eggs ...	L, L	$\cdot 4643 \pm \cdot 0346$	$\cdot 6056 \pm \cdot 0107$
Breadths of Eggs ...	B, B	$\cdot 5176 \pm \cdot 0326$	$\cdot 7327 \pm \cdot 0078$
Longitudinal Girths ...	G_1, G_1	$\cdot 5076 \pm \cdot 0327$	$\cdot 6689 \pm \cdot 0093$
Equatorial Girths ...	G_b, G_b	$\cdot 4621 \pm \cdot 0350$	$\cdot 7469 \pm \cdot 0075$
Index ...	$100 B/L, 100 B/L$	$\cdot 5537 \pm \cdot 0308$	$\cdot 5327 \pm \cdot 0120$

It must at once be admitted that this result is of a very startling character. Only the homotyposis of the Index has remained without any significant change, i.e. the degree of likeness in shape does not exhibit a seasonal change; in all four cases of absolute size there are most substantial and of course significant changes in the homotyposis. The mean size homotyposis has risen from $\cdot 4879$ to $\cdot 6885$, i.e. by about 40%! It is difficult to offer a demonstrable explanation of this great change. The factor we are seeking for must be one which modifies so to speak the individuality of the bird between its successive egg layings. For example, a change in the climatic condition or in the food supply occurring in 1913 somewhere during the egg-laying period. Such a factor, however, would lead us to suppose that the high values of 1914 were the normal homotypic values, whereas they appear to us from the comparative standpoint to be the abnormal. If we suppose only the stronger birds survived to the season 1914 and that there was a plentiful food supply, it would seem that the community as a whole should have exhibited *less* individuality in size and not more,—the weaker birds obtaining less food supply would not appear. There is, however, so little change of type and variability of the eggs in the two seasons that it is hard to believe that selection of the birds is the source of the change. Further if anything the variability of the eggs is less in 1914 than 1913, and such reduction of variability would tend to reduce rather than increase correlation. If we suggest that 1913 killed off many of the old birds and that there was a larger proportion of young birds in 1914, so that there was a more heterogeneous community in 1914, we are pulled up by the fact that the eggs were on the average very slightly larger in 1914, which is, perhaps, not what we should anticipate with a larger proportion of first layers. It would seem as if we had to take refuge in some very vague statement that the seasonal environment for 1914 interfered less with individuality than that of 1913. But this does not really help us and leaves us with the greater difficulty, that it suggests that 'individuality' is an indefinite quantity from the statistical side and might result under favourable environmental conditions in all the eggs of a clutch being perfectly alike! The persistency in the Index value seems in itself to point to a limitation in individuality, and it seems wisest at present to await further material before

speculating on the source of this marked seasonal change in size homotyposis. One point, however, we can investigate, namely, whether pigmentation homotyposis has or has not kept pace with size homotyposis.

With this aim in view the direct homotyposis has been worked out between mottling of one egg and mottling of a second in the same clutch, and between ground colour of one egg and ground colour of a second in the same clutch. Further the cross-homotyposis has been determined between the mottling of one egg and the ground colour of a second in the clutch. The fundamental difficulty here lies in the treatment of the 'values' of the ground colour. We cannot separate green eggs from brown, because of the occasional appearance of mixed colour clutches. Nor would it be reasonable to work with contingency on a 20×20 category table. We have accordingly been compelled to pool green and brown eggs, when they have the same 'value' on our colour scale. This at any rate renders our present results comparable with those of 1913. But until we know more of the mechanism of egg pigmentation it is impossible to assert that equal 'values' in brown and green ground colours are what we should anticipate as a result of individuality working occasionally with one and occasionally with another pigment. The homotyposis pigmentation tables are given as Tables R, S, and T at the end of this paper. In actually determining the contingency we have clubbed *d* and *e* in the mottling together, and A_1 and A_2 , B_1 and B_2 , C_1 and C_2 , etc. in the value of the ground colour, thus reaching 8×8 , 10×10 and 10×8 contingency tables. These have then been corrected for number of cells and for class-index correction. The class-index correction for mottling is .9531, and for value of ground colour .9848.

We consider first the cross-homotyposis of ground colour and mottling. The coefficient of mean square contingency on the supposition that there is no association between value of ground colour in one egg and mottling in a second would be

$$\bar{C}_2 = .1992 \pm .0169.$$

The corrected actual coefficient of mean square contingency is

$$C_2 = .1479,$$

which is less than the mean square contingency coefficient for no association. Accordingly there is no cross-homotyposis between mottling and ground colour, and there should not be if our view be correct that the organic relationship in the same egg is zero (see p. 328).

The value found for the 1913 data was

$$C_2 = .3989 \pm .0379,$$

and was spoken for as significant. But the fact was overlooked that

$$\bar{C}_2 = .3169 \pm .0451,$$

so that C_2 is less than twice the probable error greater than $\bar{C}_2$, and may well not be significant. This conclusion is confirmed by the consideration that the organic correlation of mottling and ground colour was really insignificant in 1913, and

thus it is exceedingly improbable that the cross-homotypis could be significant*. Direct homotypis provides the results of the following table:

TABLE XX.

Direct Homotypis in Mottling and Ground Colour Value.

Character	Season 1913	Season 1914
Mottling of Eggs in same Clutch	·3500	·6267
Ground Colour Value of Eggs in same Clutch	·5709	·7480

The probable errors of the 1913 values are well below ·045 and of the 1914 values well below ·017. Accordingly the differences are markedly significant, or in the nature of pigmentation the resemblance of eggs in the same clutch is much more intense in 1914 than in 1913. Thus the results for size and shape of egg are confirmed by those for pigmentation. We have therefore this very remarkable fact—a fact which it seems to us may be of some consequence—namely that the season can affect the extent to which the female bird impresses her individuality on the *external* characters of the egg. It does not follow from this that seasonal differences can affect in the like marked manner the individuality of the *internal* characters of the egg. But it does raise the suggestion that it would be well worth inquiring whether the degree of resemblance of offspring born in one season can differ sensibly from the degree of resemblance of those born in another season. Should such a difference be established, it would indicate that heredity—in other words the nature of the germ plasm—could be more readily influenced by seasonal differences than has yet been anticipated. We ourselves should be very unwilling to admit this, but we must at the same time confess that we see no obvious explanation of these significant changes in homotypis. If individuality impressed in the ovary and in the oviduct on the form and colouring of eggs can be increased or decreased by seasonal differences, it is not a very long step to believe that other physiological processes of this region which impress individuality on the internal characters of the ovum can be modified by the nature of the season.

We now turn to the cross-homotypis in size and shape of the tern's egg:

TABLE XXI.

Cross-Homotypis in Size Characters.

Characters of the two Eggs	Season 1913	Season 1914
Length and Breadth... ..	·0922 ± ·0441	·2621 ± ·0157
Longitudinal and Transverse Girths	·2603 ± ·0413	·4546 ± ·0134
Length and Longitudinal Girth ...	·4229 ± ·0362	·5854 ± ·0111
Breadth and Longitudinal Girth ...	·2530 ± ·0416	·4162 ± ·0140

* See above our second footnote on p. 328. In 1913 we had not fully realised how high $\bar{C}_2$ could be for such short samples as a couple of hundred. Hence the source of the error.

We are thus again faced with the fact that the cross-homotypis of the eggs of 1914 is substantially higher than that of 1913. We still see the markedly emphasised individuality of the female birds.

We have next to enquire whether, the organic relations being practically constant, the cross-homotypis has increased in proportion or not to the direct homotypis. We can test this by Pearson's suggested relationship*, namely Cross-Homotypic Correlation of x and $y = \frac{1}{2} \{ \text{correlation of } x \text{ with } x + \text{correlation of } y \text{ with } y \} \times \{ \text{the organic correlation of } x \text{ with } y \}$. The following table gives the calculated and observed cross-homotypic correlations for the seasons 1913 and 1914.

TABLE XXII.

Cross-Homotypic Correlations as Calculated and Observed.

Character Pair of two Eggs	Season 1913		Season 1914	
	Calculated	Observed	Calculated	Observed
Length and Breadth... ..	·1090	·0922	·1408	·2621
Longitudinal and Transverse Girths	·2568	·2603	·3638	·4546
Length and Longitudinal Girth ...	·4278	·4229	·5426	·5854
Breadth and Longitudinal Girth ...	·2674	·2530	·3392	·4162

Thus while the calculated values were in excellent accordance with the observed in 1913, they are very inadequate to express the increased individuality in 1914. In other words the cross-homotypis appears increased even at a greater rate than the direct homotypis which we have shown in itself to be markedly emphasised.

What we are accordingly confronted with in the season 1914 is an exuberance of individuality and the possibilities which such a variation of individuality suggests. It may be confined to the externals of the egg, but the physiological factors which determine those externals must at least be in close proximity and may, perhaps, be affiliated with others which affect matters much more important. The approximate constancy of type, variability and organic correlation for these two seasons coupled with the marked change in homotypis is a problem which demands further observations and much hard thinking.

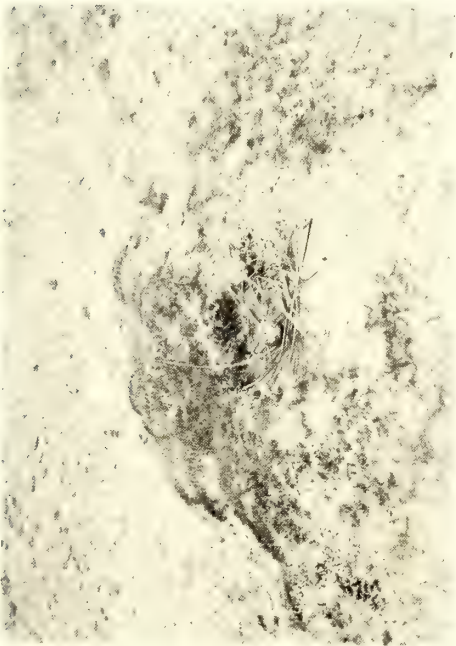
* *Phil. Trans.* Vol. 197 A, p. 290.

LIST OF PLATES BELONGING TO THIS PAPER.

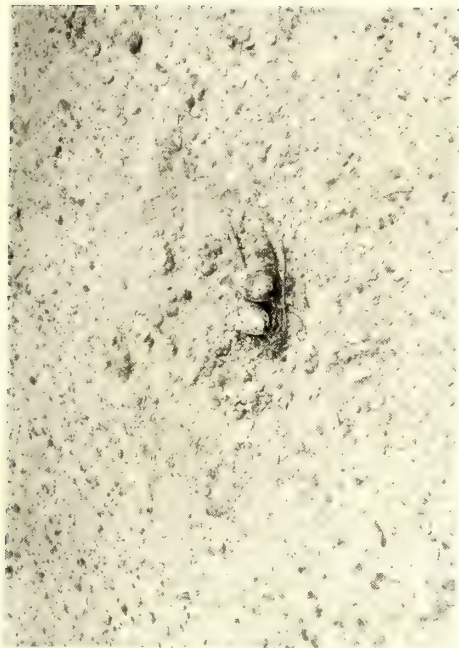
- Plate II. Simple and complex nests; *a* and *b* less elaborated, *c* and *d* more elaborated nests.
- Plate III. *a* Clutch of three eggs; *b* clutch of three eggs; note agreement of size and shape with diversity of colour and mottling; *c* noteworthy individual pigmentation in a one-egg clutch.
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" E ^a .	" "	" Brown " " Length.
" E ^b .	" "	" Green " " "
" F ^a .	" "	" Brown " " Index.
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" G ^b .	" "	Length and Breadth.
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" W.	" "	Length and Longitudinal Girth.
" X.	" "	Breadth " "



(a)



(b)



(c)



(d)

Nests of the Common Tern indicating the range of material used and character of immediate environment.

Photographs by W. Rowan



Eggs of the Common Tern, painted by William Rowan.
a, a, a, Eggs of one clutch. *b, b, b*, Eggs of a second clutch. These eggs illustrate diversity within the clutch.
c, Atypical egg, found only on one occasion.



(a)

Common Tern just alighting, to indicate great length of wings.



(b)

The four-egg Clutch : see p. 318.

Photographs by W. Rowan



(a)



(b)



(c)



(d)

Common Tern arranging her eggs and settling down on them. Camera about 18 inches from the bird.



(a)



(b)



(c)



(d)

Common Tern sitting. The bird is angry at the operations of the photographer. Camera about 18 inches away.

Photographs by W. Rowan

TABLE A. *Organic Correlation. Mottling and Breadth.*
Mottling*.

Breadth	a	b	c	d	e	f	g	h	i	Totals
2·60—†	—	1	—	—	—	—	—	—	—	1
2·65—	1	—	1	—	—	1	—	—	—	3
2·70—	—	—	4	—	1	1	—	—	1	7
2·75—	—	1	7	—	3	5	2	—	1	19
2·80—	3	4	22	—	3	2	1	—	1	36
2·85—	2	7	28	—	9	9	4	—	1	60
2·90—	5	9	45	3	30	20	7	4	5	128
2·95—	4	5	81	2	25	33	15	6	9	180
3·00—	3	10	98	4	62	28	18	5	4	232
3·05—	1	8	96	7	54	30	11	7	5	219
3·10—	3	—	65	—	40	21	11	2	5	147
3·15—	—	—	27	—	5	5	9	4	1	51
3·20—	—	1	10	—	5	—	3	1	1	21
3·25—	—	—	3	—	—	—	—	—	—	3
3·30—	—	—	1	—	—	—	—	—	—	1
Totals	22	46	488	16	237	155	81	29	34	1108

* Two eggs with no recorded mottling.

† 2·60— contains all breadths from 2·595 to 2·645.

TABLE B. *Organic Correlation. Mottling and Length.*
Mottling*.

Length	a	b	c	d	e	f	g	h	i	Totals
3·25—†	—	—	—	—	1	—	—	—	—	1
3·30—	—	—	—	—	—	—	1	—	—	1
3·35—	—	—	1	—	—	—	—	—	—	1
3·40—	—	—	—	—	—	—	—	—	—	0
3·45—	—	—	—	—	—	—	—	—	—	0
3·50—	1	—	—	—	—	—	—	—	—	1
3·55—	—	—	—	—	1	—	—	—	—	1
3·60—	—	—	—	—	—	—	—	—	—	0
3·65—	—	1	—	1	—	2	—	—	—	4
3·70—	—	—	—	—	1	3	—	—	—	4
3·75—	1	—	2	—	—	1	2	—	—	6
3·80—	—	—	3	—	1	2	—	—	—	6
3·85—	—	—	10	—	3	2	3	—	—	18
3·90—	1	—	14	1	7	6	2	1	1	33
3·95—	1	1	18	—	3	4	3	1	2	33
4·00—	2	2	31	—	17	9	3	3	—	67
4·05—	1	3	47	2	18	12	10	1	3	97
4·10—	2	6	66	4	26	24	12	3	3	146
4·15—	3	8	57	1	21	14	4	5	5	118
4·20—	3	4	65	4	34	25	13	3	4	155
4·25—	—	2	32	2	19	10	7	2	1	75
4·30—	2	4	50	1	32	13	6	4	5	117
4·35—	—	4	25	—	12	11	3	3	1	59
4·40—	1	4	20	—	10	6	5	1	3	50
4·45—	2	4	23	—	12	3	3	—	3	50
4·50—	2	—	12	—	8	2	2	1	2	29
4·55—	—	1	10	—	1	3	1	—	1	17
4·60—	—	1	1	—	6	2	—	1	—	11
4·65—	—	—	1	—	2	—	—	—	—	3
4·70—	—	—	—	—	—	—	—	—	—	0
4·75—	—	—	—	—	1	1	—	—	—	2
4·80—	—	—	—	—	1	—	—	—	—	1
4·85—	—	1	—	—	—	—	1	—	—	2
Totals	22	46	488	16	237	155	81	29	34	1108

* No mottling is recorded in the case of two eggs.

† 3·25— embraces all lengths from 3·245 to 3·295.

TABLE C.

Organic Correlation. Mottling and Index.

Mottling*.

Index	a	b	c	d	e	f	g	h	i	Totals
58—†	—	—	—	—	1	—	1	—	—	2
60—	—	—	1	—	—	—	—	—	—	1
62—	1	3	5	—	3	5	—	—	—	17
64—	2	6	12	—	7	4	2	—	3	36
66—	2	4	34	—	20	10	4	3	1	78
68—	5	13	82	1	29	24	8	6	13	181
70—	5	8	110	4	71	34	16	4	8	260
72—	4	6	119	6	51	35	25	5	2	253
74—	1	6	77	4	36	24	12	10	5	175
76—	1	—	36	—	10	12	8	—	2	69
78—	—	—	9	1	6	6	4	1	—	27
80—	—	—	2	—	2	—	—	—	—	4
82—	—	—	—	—	—	—	—	—	—	0
84—	1	—	1	—	—	1	—	—	—	3
86—	—	—	—	—	—	—	1	—	—	1
88—	—	—	—	—	—	—	—	—	—	0
90—	—	—	—	—	—	—	—	—	—	0
92—	—	—	—	—	1	—	—	—	—	1
Totals	22	46	488	16	237	155	81	29	34	1108

* Two eggs with no recorded mottling.

† 58— contains all eggs 57·95—59·95.

TABLE D^a.*Organic Correlation. Value of Brown Ground Colour and Breadth.*

Value of Ground Colour.

Breadth	A ₁	B ₁	C ₁	D ₁	E ₁	F ₁	G ₁	H ₁	I ₁	K ₁	Totals
2·60—	—	—	—	—	—	—	—	—	—	—	0
2·65—	—	—	—	—	—	—	1	—	—	—	1
2·70—	1	1	—	—	—	—	1	—	—	—	3
2·75—	—	—	4	2	1	2	—	1	—	—	10
2·80—	2	1	3	1	5	—	—	1	—	—	13
2·85—	4	5	2	4	3	1	—	2	—	2	23
2·90—	7	6	6	9	4	2	3	4	2	3	46
2·95—	9	13	8	16	6	4	6	1	8	4	75
3·00—	12	15	18	27	9	4	1	4	4	3	97
3·05—	9	14	8	18	13	4	3	5	4	2	80
3·10—	8	11	9	10	8	3	7	1	1	2	60
3·15—	1	6	4	4	1	1	—	—	—	2	19
3·20—	1	4	1	3	—	—	—	—	—	—	9
3·25—	—	—	—	1	—	—	1	—	—	—	2
3·30—	—	—	—	—	1	—	—	—	—	—	1
Totals	54	76	63	95	51	21	23	19	19	18	439

TABLE D^b. *Organic Correlation. Value of Green Ground Colour and Breadth.*

Value of Ground Colour.

Breadth	A ₂	B ₂	C ₂	D ₂	E ₂	F ₂	G ₂	H ₂	I ₂	K ₂	Totals
2·60—	—	—	1	—	—	—	—	—	—	—	1
2·65—	—	—	1	—	1	—	—	—	—	—	2
2·70—	—	—	2	—	—	1	1	—	—	—	4
2·75—	1	—	1	3	2	—	1	1	—	—	9
2·80—	1	1	1	1	2	6	3	2	3	2	22
2·85—	—	2	7	5	5	4	5	4	3	2	37
2·90—	3	6	13	20	10	9	3	6	9	3	82
2·95—	5	16	7	24	16	12	10	4	9	2	105
3·00—	9	10	15	26	22	16	11	13	11	4	137
3·05—	6	11	10	32	17	17	17	6	13	9	138
3·10—	7	4	10	18	5	11	14	7	8	3	87
3·15—	2	—	2	3	3	6	2	4	5	5	32
3·20—	—	1	—	1	2	3	2	—	1	2	12
3·25—	—	—	1	—	—	—	—	—	—	—	1
Totals	34	51	71	133	85	85	69	47	62	32	669

TABLE E^a. *Organic Correlation. Value of Brown Ground Colour and Length.*

Value of Ground Colour.

Length	A ₁	B ₁	C ₁	D ₁	E ₁	F ₁	G ₁	H ₁	I ₁	K ₁	Totals
3·25—	—	—	—	—	—	—	—	1	—	—	1
3·30—	—	—	—	—	—	—	—	—	—	—	0
3·35—	—	—	—	—	—	—	—	—	—	—	0
3·40—	—	—	—	—	—	—	—	—	—	—	0
3·45—	—	—	—	—	—	—	—	—	—	—	0
3·50—	1	—	—	—	—	—	—	—	—	—	1
3·55—	—	—	—	—	—	—	—	—	—	—	0
3·60—	—	—	—	—	—	—	—	—	—	—	0
3·65—	1	—	—	—	—	1	—	—	—	—	2
3·70—	—	—	—	—	—	—	—	—	—	—	0
3·75—	—	—	—	2	—	—	—	—	—	—	2
3·80—	—	—	1	1	—	—	1	—	—	—	3
3·85—	—	1	1	1	—	—	3	—	—	1	7
3·90—	4	2	1	2	1	1	1	—	2	—	14
3·95—	1	2	3	5	2	—	3	1	—	—	17
4·00—	4	5	3	5	3	1	1	1	1	—	24
4·05—	3	6	7	6	2	2	2	3	3	4	38
4·10—	10	8	4	18	11	3	3	4	2	1	64
4·15—	4	10	8	9	4	1	3	3	2	3	47
4·20—	5	7	9	14	2	2	—	1	4	6	50
4·25—	3	7	10	5	5	2	1	2	3	1	39
4·30—	6	11	5	14	7	—	3	1	—	—	47
4·35—	2	2	2	5	7	1	1	1	1	—	22
4·40—	4	7	3	3	1	3	—	—	1	1	23
4·45—	1	3	2	2	3	—	1	1	—	1	14
4·50—	4	2	2	1	3	1	—	—	—	—	13
4·55—	—	1	1	1	—	1	—	—	—	—	4
4·60—	1	—	—	1	—	1	—	—	—	—	3
4·65—	—	—	1	—	—	1	—	—	—	—	2
4·70—	—	—	—	—	—	—	—	—	—	—	0
4·75—	—	1	—	—	—	—	—	—	—	—	1
4·80—	—	—	—	—	—	—	—	—	—	—	0
4·85—	—	1	—	—	—	—	—	—	—	—	1
Totals	54	76	63	95	51	21	23	19	19	18	439

TABLE E^b. *Organic Correlation. Value of Green Ground Colour and Length.*
Value of Ground Colour.

Length	A ₂	B ₂	C ₂	D ₂	E ₂	F ₂	G ₂	H ₂	I ₂	K ₂	Totals
3.25—	—	—	—	—	—	—	—	—	—	—	0
3.30—	—	—	—	—	—	—	1	—	—	—	1
3.35—	—	—	—	1	—	—	—	—	—	—	1
3.40—	—	—	—	—	—	—	—	—	—	—	0
3.45—	—	—	—	—	—	—	—	—	—	—	0
3.50—	—	—	—	—	—	—	—	—	—	—	0
3.55—	—	—	—	—	1	—	—	—	—	—	1
3.60—	—	—	—	—	—	—	—	—	—	—	0
3.65—	—	—	1	—	—	—	—	—	1	—	2
3.70—	—	1	—	2	—	—	1	—	—	—	4
3.75—	—	—	1	1	—	—	1	1	—	—	4
3.80—	—	—	—	1	1	—	1	—	—	—	3
3.85—	1	1	2	3	1	2	1	—	—	—	11
3.90—	1	1	4	6	1	—	2	1	1	2	19
3.95—	1	—	3	3	—	—	2	4	3	—	16
4.00—	4	3	4	6	7	10	3	3	2	1	43
4.05—	3	6	5	9	11	8	8	3	6	—	59
4.10—	4	5	7	20	10	9	8	7	7	3	80
4.15—	6	5	8	12	11	8	6	4	7	4	71
4.20—	8	8	12	24	13	17	8	5	8	2	105
4.25—	—	2	3	5	5	7	5	2	4	3	36
4.30—	1	5	9	17	7	8	12	4	5	3	71
4.35—	1	—	4	5	7	4	2	4	5	5	37
4.40—	1	3	1	6	2	2	4	1	5	2	27
4.45—	2	5	4	3	4	5	1	2	4	6	36
4.50—	—	3	2	3	3	1	1	1	3	—	17
4.55—	1	3	—	2	—	3	1	3	—	—	13
4.60—	—	—	1	2	—	1	1	1	1	1	8
4.65—	—	—	—	1	—	—	—	—	—	—	1
4.70—	—	—	—	—	—	—	—	—	—	—	0
4.75—	—	—	—	—	1	—	—	—	—	—	1
4.80—	—	—	—	—	—	—	—	1	—	—	1
4.85—	—	—	—	1	—	—	—	—	—	—	1
Totals	34	51	71	133	85	85	69	47	62	32	669

TABLE F^a. *Organic Correlation. Value of Brown Ground Colour and Index.*
Value of Ground Colour.

Index	A ₁	B ₁	C ₁	D ₁	E ₁	F ₁	G ₁	H ₁	I ₁	K ₁	Totals
58—	—	—	1	—	—	—	—	—	—	—	1
60—	—	—	—	—	—	—	—	—	—	—	0
62—	—	1	2	—	2	2	—	—	—	—	7
64—	4	4	3	2	—	2	—	2	—	—	17
66—	5	6	2	8	6	—	2	1	2	—	32
68—	8	13	10	12	10	4	2	3	2	4	68
70—	9	14	18	27	13	4	1	2	5	6	99
72—	15	19	9	16	11	4	8	7	3	4	96
74—	6	13	12	18	6	5	4	2	4	3	73
76—	5	3	5	6	1	—	5	1	2	1	29
78—	—	3	1	4	2	—	1	—	1	—	12
80—	—	—	—	2	—	—	—	—	—	—	2
82—	—	—	—	—	—	—	—	—	—	—	0
84—	2	—	—	—	—	—	—	—	—	—	2
86—	—	—	—	—	—	—	—	—	—	—	0
88—	—	—	—	—	—	—	—	—	—	—	0
90—	—	—	—	—	—	—	—	—	—	—	0
92—	—	—	—	—	—	—	—	1	—	—	1
Totals	54	76	63	95	51	21	23	19	19	18	439

TABLE F^b.*Organic Correlation. Value of Green Ground Colour and Index.*

Value of Ground Colour.

Index	A ₂	B ₂	C ₂	D ₂	E ₂	F ₂	G ₂	H ₂	I ₂	K ₂	Totals
58—	—	—	—	1	—	—	—	—	—	—	1
60—	—	—	—	—	—	1	—	—	—	—	1
62—	—	—	2	2	3	—	1	2	—	—	10
64—	—	6	1	5	1	1	—	3	1	1	19
66—	1	2	5	4	5	7	5	5	11	2	47
68—	5	11	13	18	16	13	12	9	7	9	113
70—	8	11	20	36	21	24	10	6	16	9	161
72—	11	12	19	31	20	21	17	7	13	6	157
74—	4	6	6	23	12	12	17	10	8	3	101
76—	5	2	3	7	5	6	4	3	4	1	40
78—	—	1	2	5	1	—	2	1	2	1	15
80—	—	—	—	—	1	—	—	1	—	—	2
82—	—	—	—	—	—	—	—	—	—	—	0
84—	—	—	—	1	—	—	—	—	—	—	1
86—	—	—	—	—	—	—	1	—	—	—	1
88—	—	—	—	—	—	—	—	—	—	—	0
90—	—	—	—	—	—	—	—	—	—	—	0
92—	—	—	—	—	—	—	—	—	—	—	0
Totals	34	51	71	133	85	85	69	47	62	32	669

TABLE G^a.*Organic Correlation. Mottling and Ground Colour Value.*

Value of Ground Colour*.

Mottling†	A ₁	A ₂	B ₁	B ₂	C ₁	C ₂	D ₁	D ₂	E ₁	E ₂	F ₁	F ₂	G ₁	G ₂	H ₁	H ₂	I ₁	I ₂	K ₁	K ₂	Totals
a	2	2	3	2	1	3	—	—	1	2	—	1	—	—	—	2	1	2	—	—	22
b	3	—	2	2	4	1	5	4	2	3	2	—	—	3	1	2	—	8	2	2	46
c	20	12	34	27	33	31	48	54	21	38	9	38	11	31	8	20	8	24	9	11	487
d	1	—	1	—	1	—	—	5	2	2	1	—	—	1	1	—	—	1	—	—	16
e	11	8	15	6	15	15	19	26	13	21	5	24	2	15	4	9	3	16	2	8	237
f	7	2	9	9	3	11	9	24	7	9	3	17	7	9	3	6	6	6	3	5	155
g	3	9	7	3	5	7	6	10	2	5	—	3	3	5	2	5	1	2	—	3	81
h	3	1	4	1	—	1	4	1	2	4	—	1	—	2	—	2	—	1	1	—	28
i	2	—	1	1	1	2	4	9	1	1	1	1	—	3	—	1	—	2	1	3	34
Totals	52	34	76	51	63	71	95	133	51	85	21	85	23	69	19	47	19	62	18	32	1106

* Two eggs, one given as 'slatey grey' and the other as blue, have no ground colour value recorded.

† Two eggs have no mottling recorded.

TABLE G^b.
Organic Correlation of Length and Breadth.

Length.

Breadth	3.25—	3.30—	3.35—	3.40—	3.45—	3.50—	3.55—	3.60—	3.65—	3.70—	3.75—	3.80—	3.85—	3.90—	3.95—	4.00—	4.05—	4.10—	4.15—	4.20—	4.25—	4.30—	4.35—	4.40—	4.45—	4.50—	4.55—	4.60—	4.65—	4.70—	4.75—	4.80—	4.85—	Totals	
2.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2.65—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
2.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
2.75—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
2.80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
2.85—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	60
2.90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	128
2.95—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	180
3.00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	234
3.05—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	219
3.10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	147
3.15—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
3.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
3.25—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
3.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	1	1	0	0	1	1	0	4	4	6	6	18	33	33	67	97	146	118	155	75	118	59	50	50	30	17	11	3	0	2	1	2	1110	

TABLE H.
Organic Correlation of Longitudinal and Transverse Girths.

Longitudinal Girth*.

Trans-verse Girth†	10-30	10-40	10-50	10-60	10-70	10-80	10-90	11-00	11-10	11-20	11-30	11-40	11-50	11-60	11-70	11-80	11-90	12-00	12-10	12-20	12-30	12-40	12-50	12-60	Totals
8-60—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
8-70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
8-80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
8-90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
9-00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
9-10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	27
9-20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	59
9-30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	70
9-40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	95
9-50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	145
9-60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	145
9-70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	136
9-80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	140
9-90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	109
10-00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	79
10-10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34
10-20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
10-30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
10-40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
10-50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
10-60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10-70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	0	4	2	11	22	17	30	76	75	122	132	142	109	77	94	78	43	36	23	7	6	1	1	1110

* 10-30— signifies the longitudinal girths from 10-295 to 10-395.

† 8-60— signifies the transverse girths from 8-595 to 8-695.

TABLE J. Organic Correlation. Breadth and Longitudinal Girth.
Longitudinal Girth.

Breadth	10-30	10-40	10-50	10-60	10-70	10-80	10-90	11-00	11-10	11-20	11-30	11-40	11-50	11-60	11-70	11-80	11-90	12-00	12-10	12-20	12-30	12-40	12-50	12-60	12-70	Totals
2-60--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2-65--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
2-70--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
2-75--	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
2-80--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
2-85--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	60
2-90--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	128
2-95--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	180
3-00--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	234
3-05--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	219
3-10--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	147
3-15--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
3-20--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
3-25--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
3-30--	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	0	4	2	11	22	17	30	76	75	122	132	142	109	77	94	78	43	36	23	7	6	1	1	1	1110

TABLE K. Organic Correlation of Length and Index.
Length.

Index	5-25	3-30	5-35	3-40	5-45	3-50	5-55	3-60	5-65	3-70	5-75	3-80	5-85	3-90	5-95	4-00	4-05	4-10	4-15	4-20	4-25	4-30	4-35	4-40	4-45	4-50	4-55	4-60	4-65	4-70	4-75	4-80	4-85	Totals	
58	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2
60	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	17
62	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	36
64	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	79
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	182
68	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	260
70	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	175
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	69
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	27	4
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
78	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
80	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
82	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
86	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
88	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
90	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
92	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	1	1	1	0	1	1	0	4	4	6	6	18	33	33	67	97	146	118	155	75	118	59	50	50	30	17	11	3	0	2	1	2	1110	

TABLE L^a.*Organic Correlation of Breadth and Index.*

Breadth.

Index	2'60—	2'65—	2'70—	2'75—	2'80—	2'85—	2'90—	2'95—	3'00—	3'05—	3'10—	3'15—	3'20—	3'25—	3'30—	Totals
58—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	2
60—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
62—	—	1	1	4	2	2	3	1	2	1	—	—	—	—	—	17
64—	—	—	3	3	2	8	8	10	1	1	—	—	—	—	—	36
66—	—	—	1	—	9	6	14	16	21	12	—	—	—	—	—	79
68—	—	1	—	4	12	20	34	36	27	26	20	2	—	—	—	182
70—	1	1	1	4	6	5	35	58	67	42	28	11	1	—	—	260
72—	—	—	1	2	2	11	22	35	74	54	37	10	5	—	—	253
74—	—	—	—	—	1	3	10	11	36	56	27	15	12	2	1	175
76—	—	—	—	—	—	2	1	9	3	15	27	10	2	—	—	69
78—	—	—	—	—	—	—	1	4	—	12	7	3	—	—	—	27
80—	—	—	—	—	—	1	—	—	1	—	—	—	1	1	—	4
82—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
84—	—	—	—	—	1	—	—	—	1	—	1	—	—	—	—	3
86—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
88—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
92—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	1	3	7	19	36	60	128	180	234	219	147	51	21	3	1	1110

TABLE L^b.*Organic Correlation of Longitudinal Girth and Index.*

Longitudinal Girth.

Index	10'30—	10'40—	10'50—	10'60—	10'70—	10'80—	10'90—	11'00—	11'10—	11'20—	11'30—	11'40—	11'50—	11'60—	11'70—	11'80—	11'90—	12'00—	12'10—	12'20—	12'30—	12'40—	12'50—	12'60—	12'70—	Totals
58—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
62—	—	—	—	—	—	—	—	1	—	—	—	—	1	5	—	—	4	1	—	2	—	—	—	—	—	18
64—	—	—	—	—	—	—	—	1	5	—	2	—	1	3	6	3	3	3	5	1	2	—	—	1	—	3
66—	—	—	—	—	1	—	—	—	1	2	6	3	7	6	5	8	13	14	3	6	1	3	—	—	—	7
68—	—	—	—	—	—	1	4	2	7	7	20	21	21	17	8	21	16	11	13	8	3	2	—	—	—	26
70—	—	—	—	—	2	4	—	4	13	15	25	29	39	33	26	27	21	6	12	4	—	—	—	—	—	25
72—	—	—	—	1	4	2	5	7	19	24	28	36	38	27	20	25	10	6	1	—	—	—	—	—	—	17
74—	1	—	—	—	1	6	4	6	18	14	26	31	22	12	11	9	9	2	1	1	1	—	—	—	—	6
76—	—	—	3	—	—	3	4	2	5	7	15	9	12	6	1	1	—	—	—	—	—	—	—	—	—	2
78—	—	—	1	—	3	—	—	6	7	6	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—
80—	—	—	—	—	—	2	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—
82—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
84—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
86—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
88—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
92—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	1	0	4	2	11	22	17	30	76	75	122	132	142	109	77	94	78	43	36	23	7	6	1	1	1	1110

TABLE M.

Direct Homotyposis. Lengths.

Length of First Egg.

[illegible]

TABLE N.

Direct Homotyposis. Breadths.

Breadth of First Egg.

Breadth of Second Egg	2·60—	2·65—	2·70—	2·75—	2·80—	2·85—	2·90—	2·95—	3·00—	3·05—	3·10—	3·15—	3·20—	3·25—	3·30—	Totals
2·60—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2·65—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	4
2·70—	—	—	—	4	3	1	2	1	—	—	—	—	—	—	—	11
2·75—	—	—	4	10	6	3	3	—	—	—	—	—	—	—	—	26
2·80—	—	—	3	6	16	13	5	3	3	1	1	—	—	—	—	51
2·85—	—	—	1	3	13	16	25	19	5	—	—	—	—	—	—	82
2·90—	—	—	2	3	5	25	48	46	38	8	4	2	1	—	—	182
2·95—	—	—	1	—	3	19	46	56	67	35	10	—	2	—	—	239
3·00—	—	—	—	—	3	5	38	67	98	81	37	7	3	1	—	340
3·05—	—	—	—	—	1	—	8	35	81	108	70	12	5	—	—	320
3·10—	—	—	—	—	1	—	4	10	37	70	64	27	8	—	—	221
3·15—	—	—	—	—	—	—	2	—	7	12	27	22	5	1	1	77
3·20—	—	—	—	—	—	—	1	2	3	5	8	5	8	—	—	32
3·25—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	1	3
3·30—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2
Totals	2	4	11	26	51	82	182	239	340	320	221	77	32	3	2	1592

TABLE O.

Direct Homotyposis. Longitudinal Girths.

Longitudinal Girth of First Egg.

Longitudi- nal Girth of Second Egg	10·50—	10·60—	10·70—	10·80—	10·90—	11·00—	11·10—	11·20—	11·30—	11·40—	11·50—	11·60—	11·70—	11·80—	11·90—	12·00—	12·10—	12·20—	12·30—	12·40—	12·50—	12·60—	12·70—	Total
10·50—	2	1	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
10·60—	1	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
10·70—	2	1	—	1	2	2	2	4	2	—	1	—	1	—	—	—	—	—	—	—	—	—	—	18
10·80—	3	1	1	4	2	2	3	6	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	25
10·90—	—	—	2	2	4	2	7	1	5	—	—	1	—	—	—	—	—	—	—	—	—	—	—	24
11·00—	—	1	2	2	2	2	7	10	7	3	2	1	—	1	2	1	—	—	—	—	—	—	—	43
11·10—	—	—	2	3	7	7	16	18	20	8	14	7	3	—	3	2	1	—	—	—	—	—	—	111
11·20—	—	—	4	6	1	10	18	22	17	12	12	2	4	1	3	—	—	—	—	—	—	—	—	112
11·30—	—	—	2	—	5	7	20	17	40	31	19	19	7	2	4	3	1	—	—	—	—	—	—	177
11·40—	—	—	—	1	—	3	8	12	31	42	31	27	7	9	5	3	—	—	—	—	—	1	—	180
11·50—	—	—	1	1	—	2	14	12	19	31	52	25	27	17	7	3	1	—	—	—	—	—	—	212
11·60—	—	—	—	—	1	1	7	2	19	27	25	14	9	22	10	7	1	4	—	—	—	—	—	149
11·70—	—	—	1	—	—	—	3	4	7	7	27	9	10	13	13	7	3	2	1	—	—	—	—	107
11·80—	—	—	—	—	—	1	—	1	2	9	17	22	13	32	20	10	5	5	3	2	—	—	—	142
11·90—	—	—	—	1	—	2	3	3	4	5	7	10	13	20	22	9	10	7	1	1	—	—	—	118
12·00—	—	—	—	—	1	2	—	3	3	3	3	7	7	10	9	10	6	2	—	—	—	—	—	63
12·10—	—	—	—	—	—	—	1	—	1	—	1	3	5	10	6	10	7	3	1	—	—	—	—	49
12·20—	—	—	—	—	—	—	—	—	—	—	—	4	2	5	7	2	7	4	—	2	—	—	—	33
12·30—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	1	—	3	—	—	—	—	—	8
12·40—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	2	—	—	—	—	—	7
12·50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
12·60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
12·70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
Totals	8	4	18	25	24	43	111	112	177	180	212	149	107	142	118	63	49	33	8	7	0	1	1	1592

TABLE P.

Direct Homotyposis. Transverse Girths.

Transverse Girth of First Egg.

Transverse Girth of Second Egg	8·70—	8·80—	8·90—	9·00—	9·10—	9·20—	9·30—	9·40—	9·50—	9·60—	9·70—	9·80—	9·90—	10·00—	10·10—	10·20—	10·30—	10·40—	10·50—	10·60—	10·70—	Totals
8·70—	—	1	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
8·80—	1	2	—	2	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	8
8·90—	—	—	4	2	8	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	16
9·00—	1	2	2	4	4	9	2	2	—	—	2	—	—	—	—	—	—	—	—	—	—	28
9·10—	1	—	8	4	4	10	6	4	1	2	—	—	—	—	—	—	—	—	—	—	—	40
9·20—	1	1	—	9	10	18	12	11	10	5	2	2	—	—	—	—	—	—	—	—	—	81
9·30—	—	1	1	2	6	12	10	21	16	13	8	—	2	1	1	—	—	—	—	—	—	94
9·40—	—	—	1	2	4	11	21	22	21	21	21	5	4	—	—	—	—	—	—	—	—	134
9·50—	—	—	—	—	1	10	16	22	56	37	23	19	8	1	1	—	—	—	—	—	—	194
9·60—	—	—	—	—	2	5	13	21	37	58	34	21	13	7	—	—	2	—	—	—	—	213
9·70—	—	1	—	2	—	2	8	21	23	34	36	47	23	10	1	—	—	—	—	—	—	208
9·80—	—	—	—	—	—	2	—	5	19	21	47	48	32	22	8	3	1	—	—	—	—	208
9·90—	—	—	—	—	—	—	2	4	8	13	23	32	36	23	9	3	2	—	—	—	—	155
10·00—	—	—	—	—	—	—	1	—	1	7	10	22	23	34	10	7	—	1	—	—	—	116
10·10—	—	—	—	—	—	—	1	—	1	—	1	8	9	10	8	8	2	1	—	—	1	50
10·20—	—	—	—	—	—	—	—	—	—	—	—	3	3	7	8	—	1	1	—	—	1	24
10·30—	—	—	—	—	—	—	—	—	—	2	—	1	2	—	2	1	2	1	—	—	—	11
10·40—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	—	—	—	1	5
10·50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10·70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	—	3
Totals	4	8	16	28	40	81	94	134	194	213	208	208	155	116	50	24	11	5	0	0	3	1592

TABLE Q.

Direct Homotyposis. Indices.

Index of First Egg.

Index of Second Egg	58—	60—	62—	64—	66—	68—	70—	72—	74—	76—	78—	80—	82—	84—	86—	88—	90—	92—	Totals
58—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	3
60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
62—	—	—	4	6	3	8	3	2	2	—	—	—	—	—	—	—	—	—	28
64—	1	—	6	4	11	14	9	3	2	—	—	—	—	—	—	—	—	—	50
66—	—	—	3	11	32	29	26	9	2	—	—	—	—	—	—	—	—	—	112
68—	1	—	8	14	29	50	82	48	12	6	—	—	—	1	—	—	—	—	251
70—	1	—	3	9	26	82	110	88	27	18	4	1	—	—	—	—	—	—	369
72—	—	—	2	3	9	48	88	104	79	34	7	3	—	—	—	—	—	1	378
74—	—	—	2	2	2	12	27	79	88	29	8	—	—	1	—	—	—	—	250
76—	—	—	—	—	—	6	18	34	29	12	10	—	—	1	—	—	—	—	110
78—	—	—	—	—	—	—	4	7	8	10	6	1	—	1	1	—	—	—	38
80—	—	—	—	—	—	—	1	3	—	—	1	—	—	—	1	—	—	—	6
82—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
84—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	4
86—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	2
88—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
92—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
Totals	3	0	28	50	112	251	369	378	250	110	38	6	0	4	2	0	0	1	1592

TABLE R.

Direct Homotyposis. Mottling with Mottling.*

Mottling of First Egg.

Mottling of Second Egg.		<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	Totals
	<i>a</i>	4	6	4	1	7	6	3	1	—	32
	<i>b</i>	6	16	15	1	11	8	5	2	1	65
	<i>c</i>	4	15	388	4	148	64	29	21	14	687
	<i>d</i>	1	1	4	8	3	3	4	—	—	24
	<i>e</i>	7	11	148	3	106	42	24	6	3	350
	<i>f</i>	6	8	64	3	42	78	4	4	5	214
	<i>g</i>	3	5	29	4	24	4	60	2	2	133
	<i>h</i>	1	2	21	—	6	4	2	10	—	46
	<i>i</i>	—	1	14	—	3	5	2	—	14	39
Totals		32	65	687	24	350	214	133	46	39	1590

* In one two-egg clutch no mottling categories were provided.

TABLE S.

Direct Homotyposis. Values of Ground Colour†.

Ground Colour Value in First Egg.

Ground Colour Value in Second Egg.		<i>A</i> ₁	<i>A</i> ₂	<i>B</i> ₁	<i>B</i> ₂	<i>C</i> ₁	<i>C</i> ₂	<i>D</i> ₁	<i>D</i> ₂	<i>E</i> ₁	<i>E</i> ₂	<i>F</i> ₁	<i>F</i> ₂	<i>G</i> ₁	<i>G</i> ₂	<i>H</i> ₁	<i>H</i> ₂	<i>I</i> ₁	<i>I</i> ₂	<i>K</i> ₁	<i>K</i> ₂	Totals
	<i>A</i> ₁	30	1	13	2	6	1	4	1	3	2	—	1	1	1	—	—	1	—	1	—	68
	<i>A</i> ₂	1	30	—	6	—	4	1	2	—	—	—	1	—	3	—	—	—	—	—	—	48
	<i>B</i> ₁	13	—	32	—	14	2	15	4	6	5	2	3	1	4	2	4	—	—	—	—	107
	<i>B</i> ₂	2	6	—	24	—	14	—	11	—	1	—	7	—	1	—	3	1	—	—	1	71
	<i>C</i> ₁	6	—	14	—	22	—	22	3	8	—	2	3	1	3	—	—	2	1	—	—	88
	<i>C</i> ₂	1	4	2	14	—	34	—	23	—	15	—	5	—	2	—	3	—	2	—	1	106
	<i>D</i> ₁	4	1	15	—	22	—	38	4	10	1	5	1	8	—	1	4	4	1	2	1	122
	<i>D</i> ₂	1	2	4	11	3	23	4	92	—	14	—	22	—	8	—	4	—	5	—	4	197
	<i>E</i> ₁	3	—	6	—	8	—	10	—	26	—	6	2	7	—	5	—	1	—	1	—	75
	<i>E</i> ₂	2	—	5	1	—	15	1	14	—	52	—	14	—	11	1	4	—	3	—	4	127
	<i>F</i> ₁	—	—	2	—	2	—	5	—	6	—	6	—	2	—	—	—	—	—	2	—	25
	<i>F</i> ₂	1	1	3	7	3	5	1	22	2	14	—	50	—	16	—	2	—	2	—	2	131
	<i>G</i> ₁	1	—	1	—	1	—	8	—	7	—	2	—	6	—	2	3	3	1	2	—	37
	<i>G</i> ₂	1	3	4	1	3	2	—	8	—	11	—	16	—	42	—	12	—	5	—	3	111
	<i>H</i> ₁	—	—	2	—	—	—	1	—	5	1	—	—	2	—	8	—	3	—	2	—	24
	<i>H</i> ₂	—	—	4	3	—	3	4	4	—	4	—	2	3	12	—	26	—	6	—	1	72
	<i>I</i> ₁	1	—	—	1	2	—	4	—	1	—	—	—	3	—	3	—	2	—	4	—	21
	<i>I</i> ₂	—	—	—	—	1	2	1	5	—	3	—	2	1	5	—	6	—	60	—	9	95
	<i>K</i> ₁	—	—	—	—	—	—	2	—	1	—	2	—	2	—	2	—	4	—	—	—	14
	<i>K</i> ₂	—	—	—	1	1	1	1	4	—	4	—	2	—	3	—	1	—	9	—	22	49
Totals		68	48	107	71	88	106	122	197	75	127	25	131	37	111	24	72	21	95	14	49	1588

† The total number of pairs of eggs was 1592 ; from this table are omitted the four pairs which arise from one blue egg in a three-egg clutch.

TABLE T. *Cross-Homotypis. Mottling and Ground Colour*.*
Mottling of First Egg.

	a	b	c	d	e	f	g	h	i	Totals
A ₁	2	2	25	—	17	9	4	6	1	66
A ₂	1	—	22	1	7	4	12	1	—	48
B ₁	2	1	46	4	22	12	11	7	2	107
B ₂	2	1	38	2	9	9	8	1	1	71
C ₁	2	7	47	2	16	7	6	1	—	88
C ₂	2	3	42	—	20	20	12	4	3	106
D ₁	1	6	45	1	34	9	14	6	6	122
D ₂	1	1	77	2	46	43	13	5	9	197
E ₁	4	6	34	4	17	7	3	—	—	75
E ₂	4	6	55	4	26	18	13	1	—	128
F ₁	—	1	11	—	6	5	1	—	1	25
F ₂	1	4	67	1	33	17	4	2	3	132
G ₁	—	6	12	—	8	9	6	1	1	37
G ₂	2	3	49	1	26	13	5	5	4	111
H ₁	1	3	8	2	4	2	3	—	2	24
H ₂	3	2	34	—	15	5	7	4	—	72
I ₁	—	2	10	—	4	3	2	—	—	21
I ₂	1	10	36	—	24	16	5	2	1	95
K ₁	1	—	9	—	1	2	1	—	—	14
K ₂	2	4	19	—	14	4	3	—	3	49
Totals	32	65	686	24	349	214	133	46	39	1588

Ground Colour Value of Second Egg.

* In one two-egg clutch no mottling characters were provided, and in one three-egg clutch one egg was returned as 'blue' and therefore could not be used for ground colour in this table for cross-homotypis.

TABLE U. *Cross-Homotypis. Length and Breadth.*
Length of First Egg.

Breadth of Second Egg	3.25—	3.30—	3.35—	3.40—	3.45—	3.50—	3.55—	3.60—	3.65—	3.70—	3.75—	3.80—	3.85—	3.90—	3.95—	4.00—	4.05—	4.10—	4.15—	4.20—	4.25—	4.30—	4.35—	4.40—	4.45—	4.50—	4.55—	4.60—	4.65—	4.70—	4.75—	4.80—	Totals	
2.60—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	2
2.65—	—	—	—	—	—	—	—	—	2	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	4
2.70—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	—	—	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	11
2.75—	—	—	—	—	—	—	—	—	—	1	1	—	2	—	—	—	—	3	6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26
2.80—	—	—	—	—	—	—	—	—	—	1	2	—	1	—	—	—	—	6	12	7	—	—	—	—	—	—	—	—	—	—	—	—	—	51
2.85—	—	—	—	—	—	—	—	—	—	1	5	—	2	—	—	—	—	14	24	4	—	—	—	—	—	—	—	—	—	—	—	—	—	82
2.90—	—	—	—	—	—	—	—	—	—	1	3	—	7	—	—	—	—	12	25	6	—	—	—	—	—	—	—	—	—	—	—	—	—	182
2.95—	—	—	—	—	—	—	—	—	—	2	3	—	3	—	—	—	—	25	39	25	15	—	—	—	—	—	—	—	—	—	—	—	—	239
3.00—	—	—	—	—	—	—	—	—	—	—	2	2	2	—	—	—	—	37	47	36	16	21	—	—	—	—	—	—	—	—	—	—	—	340
3.05—	—	—	—	—	—	—	—	—	—	—	—	2	2	—	—	—	—	47	36	57	25	38	—	—	—	—	—	—	—	—	—	—	—	320
3.10—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	—	—	37	36	47	26	34	—	—	—	—	—	—	—	—	—	—	—	221
3.15—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	15	29	16	32	18	—	—	—	—	—	—	—	—	—	—	—	77
3.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	5	7	11	13	7	—	—	—	—	—	—	—	—	—	—	—	32
3.25—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	7	2	—	—	—	—	—	—	—	—	—	—	—	—	3
3.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	1	2	2	0	0	1	2	0	5	7	12	6	23	41	47	101	133	210	173	227	113	165	97	71	63	37	24	19	5	0	1	2	2	1592

TABLE V.
Cross-Homotyposis. Longitudinal and Transverse Girths.
Longitudinal Girth.

Transverse Girth	10.50	10.60	10.70	10.80	10.90	11.00	11.10	11.20	11.30	11.40	11.50	11.60	11.70	11.80	11.90	12.00	12.10	12.20	12.30	12.40	12.50	12.60	12.70	Totals
8.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
8.80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
8.90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
9.00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
9.10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	40
9.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	81
9.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	94
9.40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	134
9.50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	194
9.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	213
9.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	208
9.80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	155
9.90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	116
10.00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	50
10.10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
10.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
10.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
10.40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10.50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
10.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
Totals	8	4	18	25	24	43	111	112	177	180	212	149	107	142	118	63	49	33	8	7	0	1	1	1592

TABLE W.
Cross-Homotyposis. Length and Longitudinal Girth.

Length of First Egg.

Longitudi- nal Girth of Second Egg	3.25—	3.30—	3.35—	3.40—	3.45—	3.50—	3.55—	3.60—	3.65—	3.70—	3.75—	3.80—	3.85—	3.90—	3.95—	4.00—	4.05—	4.10—	4.15—	4.20—	4.25—	4.30—	4.35—	4.40—	4.45—	4.50—	4.55—	4.60—	4.65—	4.70—	4.75—	4.80—	4.85—	Totals	
10.50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
10.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
10.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
10.80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
10.90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
11.00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	43
11.10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	111
11.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	112
11.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	177
11.40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	180
11.50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	212
11.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	149
11.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	107
11.80—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	142
11.90—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	118
12.00—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
12.10—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	49
12.20—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33
12.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
12.40—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
12.50—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
12.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
12.70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	2	2	0	0	1	2	0	5	7	12	6	23	41	47	101	133	210	173	227	113	165	97	71	63	37	24	19	5	0	1	2	2	1592	

TABLE X.
Cross-Homotypis. Breadth and Longitudinal Girth.

Longitudinal Girth of First Egg.

Breadth of Second Egg	10.50—	10.60—	10.70—	10.80—	10.90—	11.00—	11.10—	11.20—	11.30—	11.40—	11.50—	11.60—	11.70—	11.80—	11.90—	12.00—	12.10—	12.20—	12.30—	12.40—	12.50—	12.60—	12.70—	Totals
2.60—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	2
2.65—	—	—	—	—	1	2	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	4
2.70—	—	—	—	1	3	1	5	—	—	1	5	—	—	—	1	—	—	—	—	—	—	—	—	11
2.75—	—	1	2	6	—	3	1	—	—	1	2	2	3	—	6	—	—	2	—	—	—	—	—	26
2.80—	—	1	3	3	4	5	6	—	13	10	8	3	6	2	3	1	1	—	1	—	—	—	—	51
2.85—	6	1	3	3	6	9	23	22	26	24	16	14	6	5	3	5	3	3	1	—	—	—	—	82
2.90—	1	1	3	5	6	11	25	23	29	33	43	39	16	17	7	4	5	1	1	1	—	—	—	182
2.95—	—	—	—	3	5	7	25	22	29	34	46	44	24	30	20	12	7	5	1	—	—	—	—	239
3.00—	—	—	—	3	5	5	21	21	40	42	58	51	27	38	30	20	16	6	1	2	—	—	—	340
3.05—	—	—	—	4	1	3	16	14	33	38	51	30	17	27	26	16	12	7	1	3	—	—	—	320
3.10—	—	—	—	1	—	4	7	8	21	22	28	16	5	16	26	24	12	8	3	1	—	—	—	221
3.15—	—	—	—	—	—	2	1	—	2	4	1	12	3	6	14	5	4	1	—	—	—	—	—	77
3.20—	—	—	—	—	—	—	3	—	—	—	—	3	—	—	8	3	—	1	1	—	—	—	—	32
3.25—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1	—	—	—	—	—	—	3
3.30—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	2
Totals	8	4	18	25	24	43	111	112	177	180	212	149	107	142	118	63	49	33	8	7	0	1	1	1592

ON THE DEGREE OF PERFECTION OF HIERARCHICAL ORDER AMONG CORRELATION COEFFICIENTS.

BY GODFREY H. THOMSON, D.Sc., Armstrong College,
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(1) Introduction.

When several mental tests are applied to a group of subjects, and the correlations between the tests (taken in pairs) are worked out, the coefficients are as a rule found not to be arranged entirely in haphazard order, but to show a certain degree of what has become known as hierarchical order. This means that if the total correlation of each test with all the others is found by adding together its coefficients, and if the tests are then arranged in sequence according to the order of magnitude of this total correlation, they are found to be also in sequence, or nearly so, according to the order of magnitude of their correlations with any one of their number.

If the correlation coefficients are set out, as is convenient, in a square table such as the following, the letters x_1, x_2 , etc. being the names of certain mental tests, and the quantities r_{12}, r_{13} , etc. the correlations between the marks scored in these tests, then hierarchical order shows itself in the fact that each coefficient is smaller than that on its right or than that below it, provided the tests have been arranged in sequence according to the magnitude of the total correlation of each with all the others.

•	x_1	x_2	x_3	x_4	•	•
x_1	•	r_{12}	r_{13}	r_{14}	•	•
x_2	r_{12}	•	r_{23}	r_{24}	•	•
x_3	r_{13}	r_{23}	•	r_{34}	•	•
x_4	r_{14}	r_{24}	r_{34}	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•

The observed numbers in an actual experiment naturally do not in any case come out in perfect hierarchical order, and it becomes important to have a measure

of the degree of perfection present, and some means of estimating from what "true" correlations the observed numbers are most probably derived, and the degree of hierarchical order among these "true" correlations. The importance of this matter arises in the Theory of General Ability which has been proposed by Professor Spearman, for that theory can only be considered proved if the correlations are derived from an absolutely perfect hierarchy. A merely high degree of hierarchical order can be attained without any General Factor whatever, by the random selection of Group Factors. The very difficult question therefore arises of deciding (if possible) whether the hierarchies actually observed in experimental psychology are more probably derived from perfect hierarchies such as are postulated in the Theory of General Ability, or from the good but not perfect hierarchies which arise in the Theory of Group Abilities*.

A criterion which, it was hoped, would give such a measure of the perfection of the true hierarchy from which the observed numbers were derived by experiment, and which has been widely adopted for this purpose, was worked out by Dr Bernard Hart and Professor C. Spearman in the *British Journal of Psychology* for March, 1912. The object of the present paper is to inquire into the accuracy of that criterion.

(2) *A Criterion for Hierarchical Order.*

The underlying idea was that if the above square table of correlation coefficients shows hierarchical order in any degree, there will be correlation between the columns of that table taken in pairs, and that when the hierarchical order is perfect the columnar correlation R will rise to unity, except in so far as it is blurred by the sampling errors, which obviously cannot increase an already perfect correlation, but can only decrease it. Let us write dashed letters throughout for the true values of the various quantities, which in ordinary experiment are unknown, reserving undashed letters for their measured values. We then have:

$$\begin{aligned} r' &= \text{true correlation coefficient,} \\ e &= \text{its sampling error on one occasion, so that} \\ r &= r' + e, \\ \bar{r'} &= \text{mean of the column of true values } r', \\ \bar{r} &= \text{mean of the column of observed values } r. \end{aligned}$$

In finding these means, that coefficient is omitted which has no partner in the column with which correlation is being found. Write also

$$\begin{aligned} \rho' &= r' \text{ measured from the mean of the true column, i.e.} \\ &= r' - \bar{r'}, \text{ and similarly} \\ \rho &= r \text{ measured from the mean of the observed column, i.e.} \\ &= r - \bar{r}, \\ \epsilon &= \rho - \rho', = e - \bar{e}, \end{aligned}$$

where $\bar{e}$ is the mean of the column of e 's.

* See G. H. Thomson, "The Hierarchy of Abilities," *Brit. Journ. Psychol.* 1919, ix. p. 337 and "The Cause of Hierarchical Order among the Correlation Coefficients of a Number of Variates taken in Pairs," *Roy. Soc. Proc. A*, xcv. p. 400 (April 1st, 1919).

Then for two columns a and b , the true columnar correlation which we desire to know is

$$R'_{ab} = \frac{S(\rho'_{xa}\rho'_{xb})}{\sqrt{\{S(\rho'^2_{xa})S(\rho'^2_{xb})\}}} \dots\dots\dots(1),$$

by the Bravais-Pearson product-moment formula, S indicating summation over the various values of x , i.e. summation up the column.

This can be written

$$R'_{ab} = \frac{S(\rho_{xa}\rho_{xb}) - S(\epsilon_{xa}\epsilon_{xb}) - S(\rho'_{xb}\epsilon_{xa}) - S(\rho'_{xa}\epsilon_{xb})}{\sqrt{\{S(\rho_{xa}\rho_{xa}) - S(\epsilon_{xa}\epsilon_{xa}) - 2S(\rho'_{xa}\epsilon_{xa})\}} \sqrt{\{S(\rho_{xb}\rho_{xb}) - S(\epsilon_{xb}\epsilon_{xb}) - 2S(\rho'_{xb}\epsilon_{xb})\}}} \dots\dots\dots(2).$$

In this expression, the three quantities of the form $S(\rho\rho)$ are known. The three quantities of the form $S(\epsilon\epsilon)$ are not known, but an attempt can be made to estimate their probable values from the known standard deviations of the correlation coefficients. The four quantities of the form $S(\rho'\epsilon)$ are treated by Dr Hart and Professor Spearman, in their paper, as negligible, on the ground that ρ' will not in general be correlated with ϵ . It is the object of the next section of this paper to examine the nature of the correlation of these two quantities.

(3) *The Relationship between the Correlation Coefficients and their Sampling Errors, in the Case of Correlation between a Number of Variates taken in Pairs.*

Consider the formula for the standard deviation of a correlation coefficient, viz.

$$\sigma_r = \frac{1 - r^2}{\sqrt{N}} \dots\dots\dots(3),$$

where N is the number in the sample. It follows from this that the larger correlation coefficients will probably have the smaller sampling errors e , disregarding the sign of e for the moment.

But these signs of the quantities e are not likely to be indiscriminately positive and negative. On the contrary, they will have a tendency to be either all positive or all negative, if, as is the case in most of the columns of coefficients considered by Professor Spearman, the correlations in the square table are mainly positive. The errors in the correlation of a variate x_1 with a variate a are themselves correlated with the errors in the correlation of the variate a with another variate x_2 , according to the formula

$$R_{r_{ax_1} r_{ax_2}} = r_{x_1 x_2} - \frac{r_{x_1 a} r_{x_2 a} (1 - r^2_{x_1 x_2} - r^2_{x_2 a} - r^2_{x_1 a} + 2r_{x_1 x_2} r_{x_2 a} r_{x_1 a})}{2(1 - r^2_{x_1 a})(1 - r^2_{x_2 a})} \dots\dots(4)*.$$

That is, the correlation of the sampling errors of $r_{x_1 a}$ with the sampling errors of $r_{x_2 a}$ depends chiefly upon $r_{x_1 x_2}$. To illustrate, let us take three correlations from an experiment in psychology, carried out by Mr Wyatt†.

* Karl Pearson and L. N. G. Filon, "On the Probable Errors of Frequency Constants," *Phil. Trans. of the Royal Soc.* 1898, cxci. A. p. 259.

† Stanley Wyatt, "The Quantitative Investigation of Higher Mental Processes," *Brit. Journ. Psychol.* 1913, vi. p. 131.

If we let

x_1	be the mental test	"Rearranged Letters,"
x_2	"	"Missing Digits,"
a	"	"Analogies,"

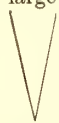
the values there found were

$$r_{x_1 a} = 0.63,$$

$$r_{x_2 a} = 0.61.$$

Then by the above formula the correlation of the errors of these two coefficients depends chiefly upon $r_{x_1 x_2}$, whose measured value is 0.63. Using the full formula, and employing the measured values in default of the true ones, the correlation between $r_{x_1 a}$ and $r_{x_2 a}$ turns out to be .47. It is therefore (to an extent indicated by this value) probable that they are either both too large or both too small. The same argument holds, in varying degrees, for the other correlations all over Mr Wyatt's table, which are all positive. They have a tendency to be either all too large or all too small: in other words, the e 's tend to be all of the same sign. The relationship between the correlation coefficients of a column, and their errors, can therefore be summed up in the following table, in which the symbol $|e|$ denotes the magnitude of e regardless of sign.

TABLE I.

r'	$ e $	ρ'	ϵ or ϵ	$\rho'\epsilon$ or $\rho'\epsilon$
large	small	+	- +	- +
		+	- +	- +
		+	- +	- +
		-	- +	+ -
		-	+ -	- +
		-	+ -	- +
small	large	-	+ -	- +
$S(\rho'\epsilon) =$				- or +

The first column shows the true correlations r' arranged in order of magnitude. The second column expresses the fact that the sampling errors on any occasion will probably be arranged in the reverse order of magnitude, disregarding their signs. The third column shows the correlation coefficients measured from their mean. The upper ρ' 's are then positive, and the lower negative, and also, what is not shown in the table, the absolute values increase upwards and downwards from the point where the signs change. The fourth (double) column shows the probable arrangement of the signs of the quantities ϵ . If the e 's are all tending to be positive, then the left-hand member of the double column gives the arrangement, while if the e 's all tend to be negative, the other member of the double column does so. As shown in the last (double) column, therefore, the quantities $\rho'\epsilon$ tend either to be nearly all negative or nearly all positive. For a very small sample the signs of $\rho'\epsilon$ will no doubt be quite irregularly arranged. But with such a

small sample, even if ρ' and ϵ were really uncorrelated, it would be most unlikely for $S(\rho'\epsilon)$ to be negligible. As the sample increases the signs tend to settle down to the above arrangement, and $S(\rho'\epsilon)$ does not tend to disappear compared with $S(\epsilon\epsilon)$, but only to take on one or other of alternative values. It will only be zero when *all* the errors are zero, i.e. when *no* corrections are needed to R . The distribution of $S(\rho'\epsilon)$ about zero in a number of samples of the same size will not, that is, show a maximum at zero, but a minimum, as is shown qualitatively in Fig. 1.

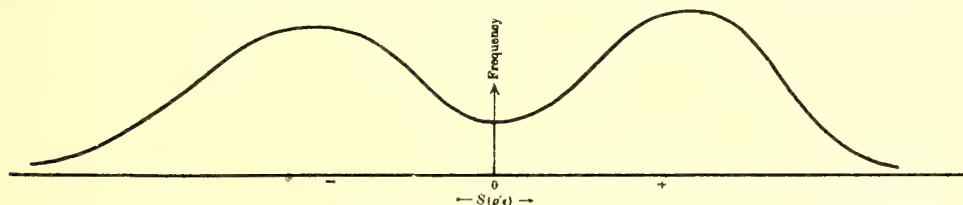


Fig. 1.

To show the order of magnitude of these neglected quantities, consider the following example, in which the true correlations are known *a priori*, and with their observed values were as follows:

$r'_{ca} = 0.730$,	$r_{ca} = 0.703$,	$e = -0.027$,
$r'_{da} = 0.593$,	$r_{da} = 0.708$,	$e = +0.115$,
$r'_{ea} = 0.356$,	$r_{ea} = 0.367$,	$e = +0.011$,
$r'_{fa} = 0.174$,	$r_{fa} = 0.337$,	$e = +0.163$,
$r'_{ga} = 0.167$,	$r_{ga} = 0.281$,	$e = +0.114$,
$r'_{ha} = 0.120$,	$r_{ha} = 0.371$,	$e = +0.251$,
$r'_{ka} = 0.116$,	$r_{ka} = 0.112$,	$e = -0.004$,
$r'_{la} = 0.112$,	$r_{la} = 0.133$,	$e = +0.021$.

The variates here were made up of dice throws, and the sample was one of 36 cases. Here, knowing as we do the actual true correlations* which would be given by the whole population or by a sufficiently large sample, we can form the quantities $S(\epsilon^2_{xa})$ and $2S(\rho'_{xa}\epsilon_{xa})$. They prove to be .064 and $-.116$. It is clearly unwise to neglect the latter of these in comparison with the former.

(4) *Experimental Demonstrations in Cases where the True Values of the Columnar Correlations are known a priori.*

The formula at which Dr Hart and Professor Spearman eventually arrive, after neglecting these quantities and making various other assumptions, is

$$R'_{ab} = \frac{S(\rho_{xa}\rho_{xb}) - (n-1)\overline{r_{ab}\sigma_{xa}\sigma_{xb}}}{\sqrt{\{S(\rho^2_{xa}) - (n-1)\overline{\sigma^2_{xa}}\}} \sqrt{\{S(\rho^2_{xb}) - (n-1)\overline{\sigma^2_{xb}}\}}} \dots\dots\dots(5),$$

where the σ 's are standard deviations of the correlation coefficients, the bar indicates mean values for the column, and n is the number of pairs of correlation

* G. H. Thomson, "A Hierarchy without a General Factor," *Brit. Journ. Psychol.* 1916, viii, p. 271.

coefficients concerned, in the two columns. In using their formula, its authors do not apply it to all the pairs of columns in the square table. They say: "In any case the correction must be kept within limits: as usual, the larger the correction the less it is to be trusted. If the sampling errors are large enough, they eventually will quite swamp the true differences of magnitude upon which the observed correlation should be based. In this case, the true correlation is beyond ascertainment; any attempt at correction is merely illusory. To avoid this, and at the same time to ensure impartial treatment of all data, it is necessary to fix beforehand some definite limit to the feasibility of correction. We have here adopted the following standard: in order to attempt to estimate the correct correlation between columns, *it is required that in each of these columns the mean square deviation should be at least double the correction to be applied to that deviation.*"

That is to say, the equation (5) is not to be used unless, in each factor of the denominator, $S(\rho^2)$ is at least double its correction $(n-1)\frac{\sigma^2}{8}$. This condition (the "correctional standard"), will be found to be important.

It is clear that the accuracy of this formula (5) could be conveniently tested were we in possession of material in which all the true correlations were known *a priori*, in addition to the observed correlations found in samples. Such material is supplied in perfection by correlated dice throws.

First Example. The first experiment with dice of the above nature which I carried out was described in the *Brit. Journ. Psychol.* 1916, VIII. There ten variates were artificially made up of group factors and specific factors, without any general factor, so as to make a very good hierarchy, which gave the following results when tested by the Hart and Spearman criterion.

TABLE II.

Columns passing standard	Observed columnar correlation R	True columnar correlation	The Hart and Spearman corrected columnar correlation R'
<i>ab</i>	0.95	1.00	1.04
<i>ac</i>	0.89	0.99	1.00
<i>bc</i>	0.91	1.00	1.01
<i>cd</i>	0.90	1.00	1.11
Means	0.91	1.00	1.04

Here the exaggeration of the Hart and Spearman R' is not very noticeable, for the hierarchy is in any case almost perfect. Indeed in this case I took some pains to make the arrangement of group factors imitate a perfect hierarchy very closely, for the sake of emphasising the point I then wished to make, viz., that such group factors can, unaided by any general factor, approach exceedingly close to perfection of hierarchical order. I did not then realise that the pains I took over this point were hardly necessary, for *random* sampling of the group factors gives good

hierarchies, though such perfection as the above would be unlikely to arise from chance.

Second Example. For a second example I have therefore chosen a hierarchy formed thus by the chance sampling of group factors, without any general factor, and moreover one which shows considerable departure from perfection of hierarchical order, it being the least perfect of those which I have up to the present formed in this way. The mode of construction of the variates is given in detail in *Roy. Soc. Proc. A.* xcv. (April 1st, 1919) on page 402, and the theoretical correlations on page 403 of that article. The latter show a certain degree of hierarchical order, though not very high, the true mean columnar correlation R for all pairs of columns being 0.59.

Dice were now thrown to form 20 measures of each of the ten variates, $x_1, x_2, x_3, \dots x_{10}$.

First the magnitudes of the group factors (which it will be recalled were in that article named after the cards of a playing pack) were decided by throwing dice, with the following results.

TABLE III.

Number identifying the subject	Name of Group Factor													
	Ace	2	3	4	5	6	7	8	9	10	Kn	Q	K	
1	6	1	6	6	5	5	4	4	2	3	1	1	4	
2	2	4	3	2	6	4	3	5	6	2	5	3	3	
3	5	1	1	4	6	3	1	4	3	6	4	3	2	
4	6	6	2	5	5	5	6	6	4	5	5	4	5	
5	5	5	4	2	2	5	2	4	1	1	4	4	5	
6	5	6	2	6	1	4	5	6	6	4	2	5	4	
7	1	1	3	4	6	4	6	6	2	3	2	5	4	
8	1	3	2	2	5	3	4	4	1	6	2	3	2	
9	1	6	1	5	3	2	2	1	3	5	4	1	5	
10	6	1	3	2	1	2	4	5	1	4	4	3	2	
11	2	4	4	2	4	3	2	1	3	3	2	3	2	
12	6	3	6	1	1	2	1	1	1	2	1	1	6	
13	6	1	6	5	5	1	6	1	3	4	4	3	1	
14	5	6	1	2	5	5	2	3	5	2	1	4	5	
15	5	2	6	2	2	5	5	1	6	6	1	2	5	
16	6	2	6	4	4	6	3	6	4	6	2	2	3	
17	2	1	6	3	3	1	2	6	2	6	1	2	5	
18	4	1	2	4	2	4	3	6	4	6	6	3	5	
19	6	2	6	3	1	2	2	5	4	5	3	1	4	
20	3	3	2	5	2	6	2	3	6	6	2	4	6	

Using these numbers, we can make up the scores for the group factor portion of each of the ten tests described in the article quoted. There results (see Table IV).

The proper number of dice, as described in the article quoted, were then thrown for each test and for each subject to represent the specific factors, and the scores of these dice added to the scores given in the last table, the resulting total being the complete score for each subject in each test (Table V).

From the dice scores the observed correlations between the variates can be

TABLE IV.

Number identifying the subject	Scores in the group factor portion of the tests									
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1	20	15	37	1	25	31	42	1	41	40
2	18	20	42	3	26	36	46	4	32	40
3	15	19	40	3	22	34	43	1	27	35
4	28	24	59	4	38	45	64	6	46	54
5	19	10	39	4	27	25	44	5	34	38
6	21	21	52	5	32	39	56	6	39	45
7	18	22	43	5	29	38	47	1	35	44
8	15	19	35	3	22	30	38	3	26	36
9	17	14	37	1	19	26	39	6	26	35
10	14	13	36	3	22	26	38	1	28	31
11	14	15	32	3	18	23	35	4	24	30
12	17	6	30	1	19	11	32	3	27	25
13	19	21	45	3	23	32	46	1	32	37
14	23	18	41	4	30	29	46	6	34	36
15	19	21	43	2	22	30	48	2	33	37
16	18	19	48	2	26	37	54	2	40	44
17	13	15	39	2	21	26	40	1	29	36
18	15	18	46	3	24	38	50	1	31	42
19	15	13	42	1	21	26	44	2	31	34
20	16	20	44	4	23	36	50	3	32	41

TABLE V.

Number identifying the subject	Total scores in the tests									
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1	36	26	70	9	47	49	92	10	54	51
2	33	28	77	14	53	55	82	11	42	56
3	22	31	77	11	47	51	93	9	41	58
4	44	34	104	15	57	59	114	13	60	65
5	35	21	72	17	47	45	83	11	46	58
6	41	37	79	17	50	55	105	12	49	61
7	36	33	84	17	54	52	93	4	43	62
8	31	30	83	7	40	56	83	12	41	49
9	40	23	92	16	38	44	81	10	39	54
10	36	24	90	14	47	46	89	10	34	52
11	29	19	67	18	43	40	72	9	35	49
12	36	18	68	16	36	33	69	9	37	43
13	40	39	76	18	46	45	77	8	48	53
14	37	23	80	16	44	41	90	11	43	61
15	35	29	89	10	41	44	104	8	44	54
16	38	30	80	14	47	54	93	7	53	63
17	30	27	83	12	46	43	91	9	40	56
18	38	28	92	12	43	54	104	8	38	63
19	34	22	91	10	37	41	97	8	43	54
20	35	35	82	16	48	52	101	7	40	61

calculated, just as the correlations between mental tests are calculated. Using the product-moment formula we obtain the set of values in Table VI, arranged in hierarchical order, only slightly different from the true hierarchical order, except that variate x has changed its position rather violently.

TABLE VI.
The Observed Hierarchy.

	x_{10}	x_7	x_9	x_5	x_2	x_6	x_1	x_3	x_4	x_8
x_{10}	•	·72	·47	·64	·53	·50	·34	·45	·21	·09
x_7	·72	•	·48	·43	·75	·48	·32	·67	—·26	·10
x_9	·47	·48	•	·51	·46	·45	·50	·46	—·02	·24
x_5	·64	·43	·51	•	·58	·60	·20	·15	·29	·08
x_2	·53	·75	·46	·58	•	·63	·26	·33	·05	—·11
x_6	·50	·48	·45	·60	·63	•	·22	·29	—·16	·18
x_1	·34	·32	·50	·20	·26	·22	•	·41	·38	·15
x_3	·45	·67	·46	·15	·33	·29	·41	•	—·20	·08
x_4	·21	—·26	—·02	·29	·05	—·16	·38	—·20	•	—·11
x_8	·09	·10	·24	·08	—·11	·18	·15	·08	—·11	•

The pairs of columns which pass the Hart and Spearman correctional standard give the following values :

TABLE VII.

Columns passing standard	Observed columnar correlation R	True columnar correlation	The Hart and Spearman corrected columnar correlation R'
2 & 7	0·73	0·75	0·76
6 & 7	0·63	0·89	1·15
2 & 3	0·70	0·60	1·01
2 & 6	0·81	0·88	1·06
3 & 6	0·66	0·83	1·04
Means	0·71	0·79	1·00
True mean columnar correlation of the whole table and not merely of the pairs of columns selected by the correctional standard			0·59

Dr Hart and Professor Spearman would therefore claim the hierarchy as being a sample of a perfect one. The true mean columnar correlation for the whole table is 0·59, the Hart and Spearman correctional standard selects pairs of columns whose true mean columnar correlation is 0·79, and the mean value of these when corrected according to their formula rises to unity. This example goes far, I think, towards shaking confidence in their criterion.

It must, I think, be partly chance which makes it so peculiarly unfavourable to their work : but I give it as it came. Really a very large number of such examples is necessary, and not all of these could be expected to be so unfavourable. The only other example which I have attempted I have carried far beyond 20 cases

without as yet reaching a point where any of the columns pass the correctional standard. I feel that working a large number of such examples is beyond the power of an individual, with other claims on his time, and rather a task for a statistical laboratory with experienced computers and mechanical aids.

(5) *The Effect of the Correctional Standard.*

Clearly the fact that the criterion is apparently too large in a majority of cases requires further explanation beyond the error already pointed out of neglecting the terms in ρ' .

The other approximations made in obtaining the criterion do not appear to be so erroneous as this one, though their cumulative effect may explain some anomalies. Leaving them on one side let us consider the "correctional standard" required by Dr Hart and Professor Spearman before they admit any pair of columns. It is this correctional standard, combined with the peculiar distribution of R' , which chiefly is responsible for the exaggeration of perfection produced by this criterion, and for the regularity with which an average value of unity is arrived at.

Let us examine first the actual distribution of the Hart and Spearman R' in a psychological hierarchy, viz., that of Wyatt already referred to, and calculate R' not only for those columns which pass the correctional standard, but also for other pairs of columns. What we find is that its value rises as we descend the hierarchy, rushing asymptotically to infinity, remaining for a time imaginary, and then returning. The value reaches infinity when one of the corrections in the denominator becomes as large as the term to be corrected, and remains imaginary until the other term is likewise passed by its correction, when both quantities under the square root are negative and an arithmetically possible but meaningless value is again calculable. Specimen values from Mr Wyatt's hierarchy are given in this Table.

TABLE VIII.

<i>Pairs of Columns</i>	<i>Values of the Hart and Spearman R'</i>
Analogies and Wordbuilding	0.93
Completion and Wordbuilding	0.97
Completion and Part-wholes	1.05
Wordbuilding and Part-wholes	0.99
Part-wholes and Memory (delayed)	0.92
Rearranged letters and Missing digits	1.17
Wordbuilding and $E R$ Test	1.26
Sentence construction and Fables	1.33
Rearranged letters and $E R$ Test	Practically infinity
Nonsense syllables and Dissected pictures	Imaginary
Crossline test and Letter squares	0.35, both factors in the denominator being now negative.

Expressed in diagrammatic form this and similar calculations lead to the conclusion that in actual practice the criterion is distributed as in Fig. 2, where the curve is to be understood as a "best fitting" curve among the values of R' scattered, with a very considerable dispersion, on both sides of it. The line, in fact, ought to be a broad smudge.

Now clearly, with a distribution of this sort, it is very important that the boundary between the values that are to be rejected and those that are to be accepted should be chosen with the greatest care, and not arbitrarily but scientifically. Either sound theoretical reasons should be given for the choice of the correctional standard, or the choice should be based empirically on experiments in

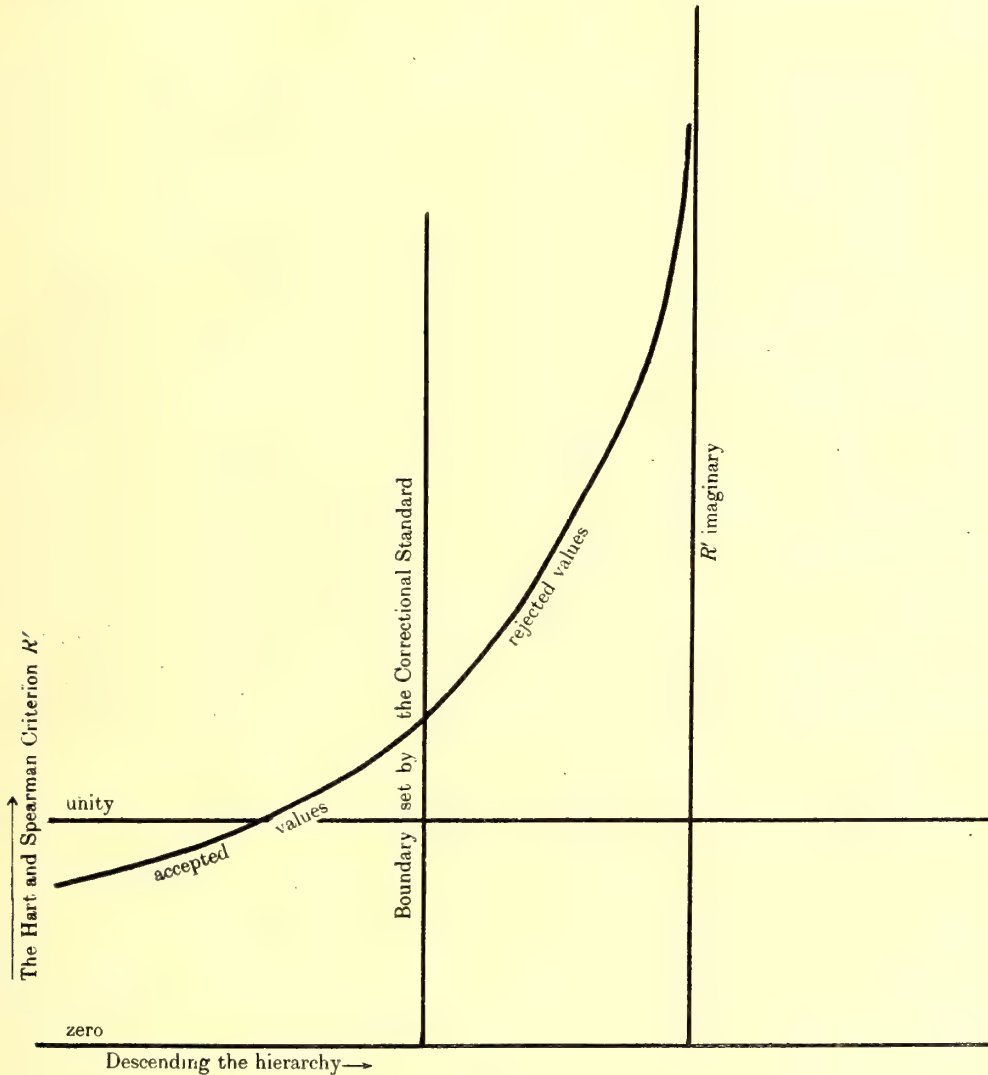


Fig. 2.

material where the truth is known *a priori*, as in the above dice experiments. For obviously, by moving this boundary, we can make the final average take on almost any value. Another point is that the criterion rushes to infinity at such speed that its probable error must be enormous. Dr Hart and Professor Spearman,

however, give no reasons for their choice of this particular standard, upon which depends so much the values they obtain. The standard which they thus arbitrarily adopt begins admitting the criteria at just such a distance above unity as to balance the cases which give a criterion below unity, and entirely explains the remarkable unanimity with which this average value unity is obtained by them in their calculations.

(6) *Conclusion.*

A criterion suggested by Dr Hart and Professor Spearman has been widely used by psychologists for the purpose of ascertaining the degree of "hierarchical" order among theoretical correlation coefficients of which only experimental values are known, and a Theory of General Ability has been based on the results. In the present paper it is however shown theoretically that an assumption made in deducing this criterion, namely that ρ' and ϵ are uncorrelated and the sums $S(\rho'\epsilon)$ negligible, is incorrect. The quantity ϵ taken regardless of sign is strongly correlated with ρ' , and its signs tend to be either all the same as, or all different from, those of ρ' . The distribution of the sums $S(\rho'\epsilon)$ shows a minimum, not a maximum, at zero.

Otherwise the paper is empirical, and applies the criterion in question to correlated dice throws. In the cases tried, this criterion exaggerates the perfection of the hierarchy considerably, claiming a quite poor hierarchy formed by random group factors as being perfect (true mean columnar correlation 0.59, the Hart and Spearman $R' = 1.00$). The reason for this exaggeration, and for the unanimity with which in so many experiments the average value unity has been found for the Hart and Spearman criterion, appears to be mainly the peculiar distribution of this quantity, combined with the action of the "correctional standard" adopted, which commences admitting the criteria at such a distance above unity as to balance those which are less than unity.

MISCELLANEA.

I. Inheritance of Psychical Characters.

BY KARL PEARSON, F.R.S.

In view of the papers that have been published on the inheritance of intelligence, it is strange that there should still remain any doubt that psychical characters are inherited at the same rate as physical characters. But having regard to the existence of that doubt any material bearing on the point deserves special recognition and emphasis.

In a recent contribution to the *Journal of Delinquency*, Vol. iv. p. 46, Dr Kate Gordon gives the results of her tests by the Binet-Simon method of the intelligence of the children in three orphanages in California. Among other data she gives, almost as an aside, a small table for the correlation in intelligence-quotients of 91 pairs of siblings. This table appears to me of very considerable interest and supplies what is occasionally lacking, a *nearly uniform environment** both in training and in nourishment to the pairs dealt with. Those who dislike the idea that the mental as well as the physical characters are largely fixed for us by our ancestry are apt to attribute—regardless of known measurements of the intensity of environmental influence—the correlation of pairs of siblings for mental characters to a differential environment of the pairs, i.e. to differential family or home training. Hence the value of data obtained within the walls of an orphanage, as tending to minimise this differentiation.

The Intelligence Quotient, it will be remembered, is the ratio of the mental age as given by an intelligence test of the Binet-Simon type to the actual age. The accompanying correlation table is the 'scatter' table of Dr Gordon rendered symmetrical, so that we can enter with either member of the pair. The probable error must, of course, be calculated for the correlation on the basis of 91 pairs, but for the mean and standard-deviation on 182 individuals. We find :

Mean Intelligence Quotient	...	...	=92·857 ± '836,
Variability in Intelligence, s.d.	...	...	=16·727 ± '591,
Coefficient of Variation	...	...	=18·014 ± '657,
Correlation of Intelligence between Siblings	r		= .5082 ± '0524.

At first sight it might seem as if the mean Intelligence Quotient was somewhat low. For a normal child it should be theoretically 100, but so much depends on the nature of the tests used and also on the manner in which they are applied that we cannot dogmatise on this point. In some recent American data we found a very low intelligence quotient among literate adults, and the result was clearly due to the nature and method of applying the test. The coefficient of variation in this case rose to the high value of 38·52, fully double the value we have found in other cases. We may note that the coefficient of variation is also large in the present case, which is distinctly against intelligence being much influenced by environmental conditions—

* The ideal method would be to take all the siblings in a very large orphanage, such for example as the Reedham asylum, and select if the numbers should prove adequate only the children who had entered the orphanage at an early age.

for in this instance we have considerable approximation to uniformity of environment. For 261 normal children examined by the Binet-Simon method by Dr Jaederholm, I find the coefficient of variation in intelligence as measured in mental years to be 19·476. For 420 children in two schools I find a coefficient of variation in general intelligence of 21·986, and for 1725 children in eight schools I find a coefficient of variation in terms' marks of 23·133. These are somewhat greater than the variability obtained for the orphanage children, but do not show the great increase some might anticipate from variety in home and school training, and the increase of the last two results may be solely due to the different standards imposed by the judgments of a variety of teachers instead of, as in Jaederholm's and our present cases, an identical series of tests made by a single psychologist. The noteworthy value, however, lies in the correlation of ·508. The values obtained for 12 cases of physical characters in siblings (*Biometrika*, Vol. II. p. 387) have exactly this value for their mean. No stress can, of course, be laid on the absolute identity considering the smallness of the present series, but much stress may be laid on the approximation of the two results.

But the present data are of further interest—although they are so slender—when we compare the results to be obtained from them with those for a far longer series of pairs of siblings obtained by the method of “broad-categories.” This series is also formed from pairs of siblings who are children. They belonged to a great variety of schools taken throughout Great Britain. Every variety of environment, every variety of educational and home training is therefore included. Accordingly if the intellectual resemblance of siblings were the result or largely the result of differential treatment, we ought to anticipate a great increase of correlation in this material over that of the material drawn from the Californian orphanages. We have also the possibility of obtaining light on two further problems :

- (i) Whether the method of “broad-categories” really does give results markedly inferior to the Binet-Simon method of direct quantitative measurement.
- (ii) What is the approximate value of the “mentace” or unit of intelligence in terms of a unit obtained from a Binet-Simon test.

The definitions of the “broad-categories” used by the Galton Laboratory in its intelligence investigations have already been published in this journal*, and a “mentace” has been defined as the $\frac{1}{160}$ part of the range which limits the category “Intelligent†.” Now if we compare the two series, the one determined by “broad-categories” and the other by the Binet-Simon test for the total frequencies up to the beginning and up to the end of the range “Intelligent,” we shall have a first approximation—on the assumption that both series are measuring the same general intelligence character and both approximate to normal distributions—to the absolute value of a “mentace.” I find that my mentace is equal to ·1604 of Dr Gordon's intelligence quotient units, or with the average age of 10·2 (which appears to have been that of her children) it equals six days about of mental growth of children at this age. Roughly we might say that a mentace is equal to about a week's mental growth at the age of ten years. In estimating the meaning of this statement we must remember that mental growth is very rapid at this age‡.

As the American data pool children of both sexes I have for purposes of comparison done the same. The following table represents my material for 5602 children in 2801 pairs, each pair being entered either way so as to produce a symmetrical table.

* *Biometrika*, Vol. VIII. p. 93.

† *Biometrika*, Vol. V. p. 109.

‡ The reader will of course avoid the conclusion that the mentace is an intelligence unit varying with age. It is the time rate of growth of intelligence which varies with age, and we must state a particular age in evaluating the mentace in terms of growth of intelligence.

Contingency Table for General Intelligence in Siblings.

Category of Intelligence of Second Sibling.	Category of Intelligence of First Sibling.						Totals
	Quick Intelligent	Intelligent	Slow Intelligent	Slow	Slow Dull	Very Dull	
Quick Intelligent	312	263·25	131·75	40	16·25	4·25	767·5
Intelligent ...	263·25	876·5	564·25	172·5	36·25	14·75	1927·5
Slow Intelligent	131·75	564·25	697·5	249·5	72·5	27	1742·5
Slow ...	40	172·5	249·5	219·5	80	12	773·5
Slow Dull ...	16·25	36·25	72·5	80	68	18	291
Very Dull ...	4·25	14·75	27	12	18	24	100
Totals	767·5	1927·5	1742·5	773·5	291	100	5602

The first question that arises is that of the method to be employed in the reduction of this table. The answer is fairly straightforward. The only legitimate method is that of corrected contingency. The mean square contingency of this 6 × 6 fold table is

$$\phi^2 = .293,1833.$$

Of the two methods of correcting this raw mean square contingency* for number of cells one leads to

$$\phi^2 = .286,9355$$

and the other to

$$\phi^2 = .288,5659$$

giving the nearly equivalent results for the contingency coefficient C_2 ,

$$C_2 = .4722 \text{ and } C_2 = .4732.$$

The class-index correlations are the same in both directions and we find :

$$r_{c_x x} r_{c_y y} = r_{c_x x}^2 = .91738.$$

Hence finally we have for the correlation

$$r = .5147 \text{ and } r = .5158,$$

and accordingly it is amply adequate to take the correlation of siblings in general intelligence be .515, which agrees excellently with the value .508 found from Dr Gordon's data.

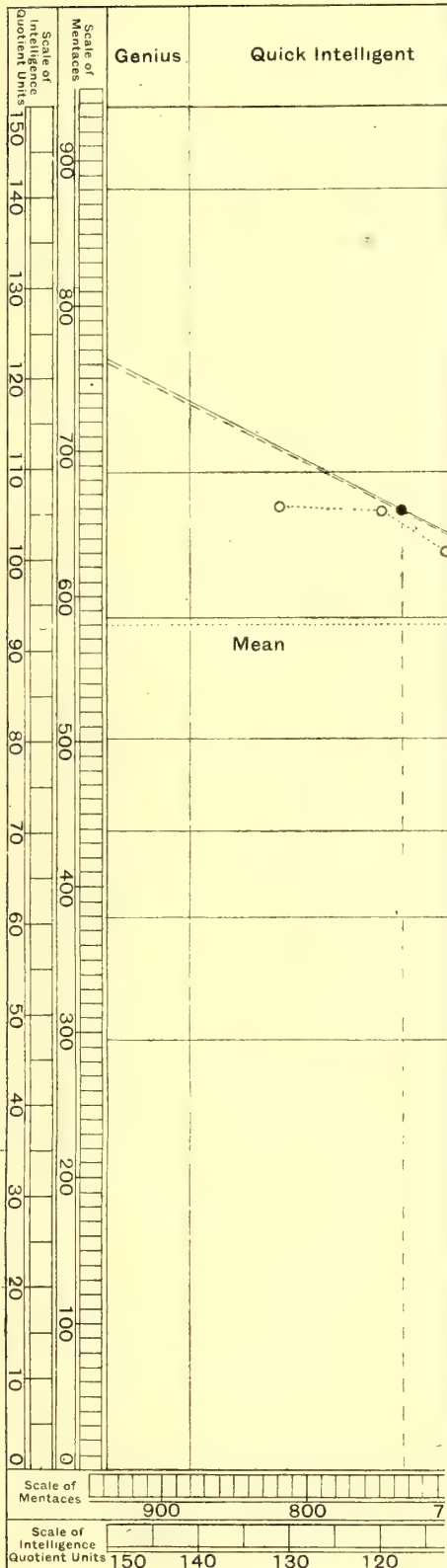
But as the bald figures .508 and .515 convey little to the mind untrained to statistical appreciations, I have attempted to provide an illustrative diagram : see Plate VII.

Assuming normal distribution for the marginal totals and the arrays, I have superposed the means of the two systems (General Intelligence and Dr Gordon's Stanford Revision of the Binet-Simon tests) and equated their s.d.'s. Using I.Q.U. for an intelligence quotient unit, i.e. a change of one digit in the intelligence quotient (or 100 mental age/physical age), we find :

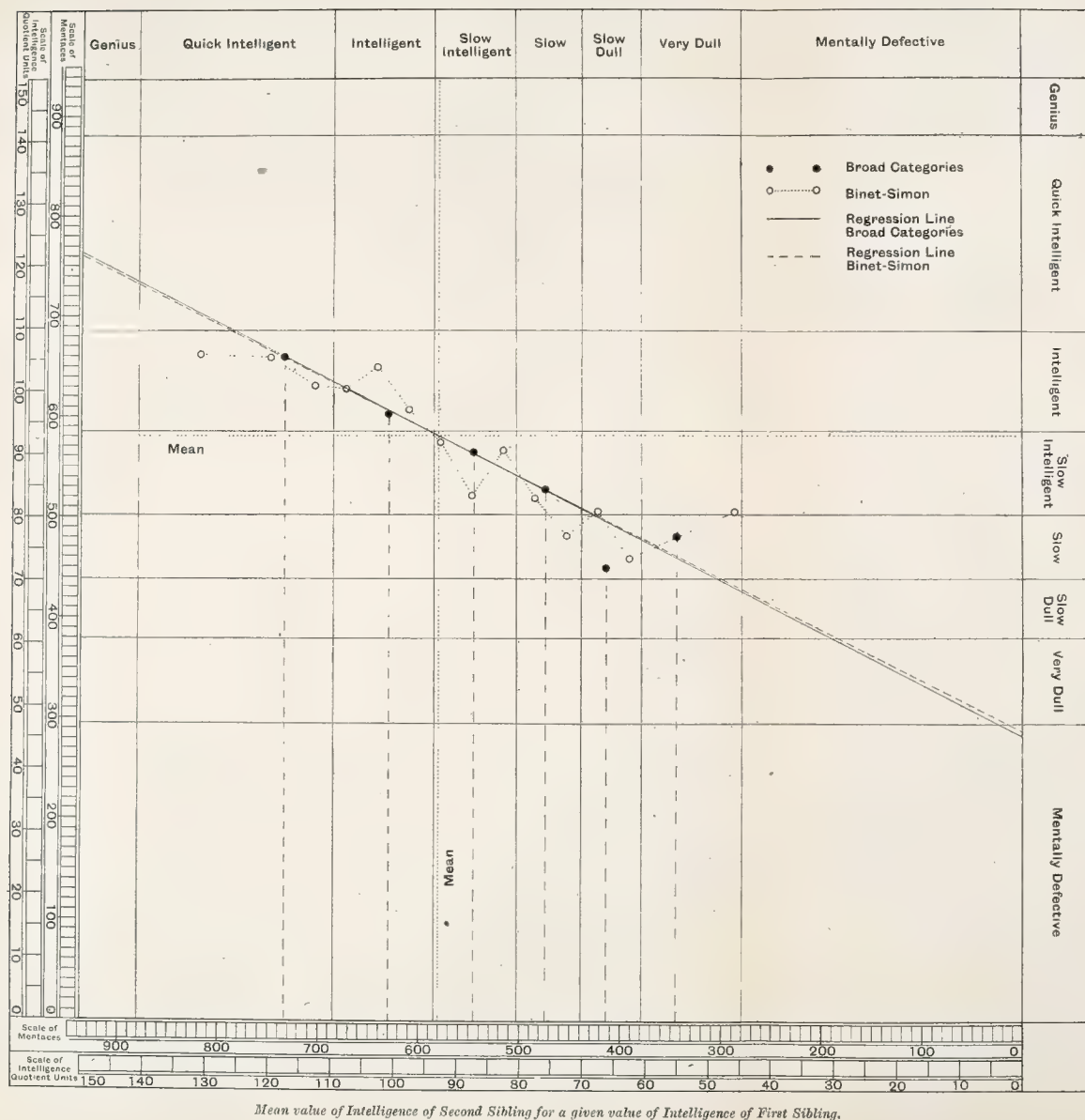
Mean = 578·909 mentaces = 92·857 I.Q.U.'s, Standard Deviation = 95·5566 mentaces = 15·3215 I.Q.U.'s.

Thus a mentace = .1604 I.Q.U.

* It is hoped to publish shortly the long-delayed memoir on contingency corrections. The delay has largely arisen from the labour involved in reducing adequate material by way of illustration.



Comparison of Association in Intelligence of Pairs of Siblings as determined by Broad Categories and Binet-Simon Tests.



Boundaries between Categories and Means of Categories measured from the Mean of Intelligence.

Quick Intelligent and Intelligent	104·5294	mentaces	or	16·7602	I.Q.U.'s.
Intelligent and Slow Intelligent	4·5294	"	"	·7262	"
Slow Intelligent and Slow	- 77·7640	"	"	- 12·4686	"
Slow and Slow Dull	- 141·1658	"	"	- 22·6345	"
Slow Dull and Very Dull	- 200·6975	"	"	- 32·1797	"
Mean of Quick Intelligent	152·967	"	"	24·527	"
" Intelligent	49·756	"	"	7·978	"
" Slow Intelligent	- 34·410	"	"	- 5·517	"
" Slow	- 105·552	"	"	- 16·924	"
" Slow Dull	- 165·590	"	"	- 26·551	"
" Very Dull	- 235·299	"	"	- 37·728	"

After careful consideration of a number of factors we divided* our "Intelligent" category of 100 mentaces range into "Fair intelligence" for the first 45 mentaces and "Capable" for the remaining 55 mentaces. Our "Quick Intelligent" category was again subdivided into a range of 200 mentaces corresponding to "Specially Able" and to "Genius" or the 1·4 per mille who exceed the mental type by more than 300 mentaces. The "Very Dull" were again subdivided at 300 mentaces less than the mean and the 1·4 per mille beyond this may be looked upon as mentally defective. This per mille of mental defectives corresponds fairly well with the primary school returns. Thus the average 'genius' will have 312 mentaces or be almost exactly 50 I.Q.U.'s above mediocrity, i.e. with a mean of 143 I.Q.U.'s, and the average mentally defective 312 mentaces or 50 I.Q.U.'s below the type, i.e. will have about 43 instead of 93 for intelligence quotient†. These limits are marked on our diagram.

Dr Gordon's results therefore bring out a point that was not correct in my diagram of 1906. The zero of intelligence is not about 300 mentaces below mediocrity, but nearer 600! Even an "imbecile" girl has an intelligence quotient of 29, or some 180 mentaces, where I in 1906 assumed she should be credited with none. I still think *complete* imbecility should be marked by a total absence of mentaces or by a zero intelligence quotient. It appears better therefore to talk of those with intelligence less by 300 mentaces than the mean as mental defectives‡. The problem is rather theoretical than practical, depending not so much on the existence of zero intelligence, as on the limen or threshold value at which we are able to realise its existence. Anyhow the conclusion seems to be that we must search a large number of millions if we wish to find an individual absolutely without intelligence.

Examining our diagram we note how extremely closely the black points which represent the means of the general intelligence categories lie on their regression line. They lie so closely that we might almost feel disappointed that the means for the Slow Dull and Very Dull categories are not equally close to the regression line. But here regard must be paid to the fact that these are the smallest of the categories in size; and further to disturbing factors arising from the

* See *Biometrika*, Vol. v. p. 110.

† Dr Gordon notes a very able girl with 137 I.Q.U.'s and an *imbecile* girl with only 29 I.Q.U.'s in a total of 335 cases.

‡ I wrote in 1906 (*Biometrika*, Vol. v. p. 111 ft.) that: "He [the median individual] can hardly have more than 350 to 400 mentaces, for at a negative position of - 350 to - 400 on the scale we have passed through the very dull group into imbecility and complete absence of reasoning power. The child whose low grade of intelligence occurs only 3 or 4 times in 100,000 cases must be sought in the idiot asylum." I was probably wrong in assuming the worst type of idiot had zero intelligence. Dr Gordon's mean is 6·06 times her S.D., or the absolute zero of intelligence would only occur 1 in 100,000,000. This is probably excessive. Dr Jaederholm's data appear to indicate 5·5 times as the ratio or 1 in 12,500,000 as the occurrence.

probability that dull children remain longer at school than very intelligent ones, and start later. These factors act in a not easily interpretable way on the number of pairs of dull and very dull children.

On the other hand Dr Gordon's plotted observations show little more than the variations due to random sampling in such a slender series. Both sets of observations together undoubtedly indicate within the limits of error one and the same law of relationship*. It is almost impossible to conceive that such diverse environmental conditions rather than a fundamental germinal relation could produce such concordance. The conclusion which is emphasised by material drawn by such different methods from such very different environments is that the relation of intelligence between siblings is fixed by something more innate than environment. That something more innate, more constant and more universal in its domination can only be the hereditary factor.

Of course the results in the present paper for the relation between Intelligence Quotient and Mentace can only be considered as suggestions until we have far longer series of pairs of siblings tested by the Binet-Simon or allied methods. But they serve to indicate that very fruitful work can be achieved in this direction, and even the present data owing to their relatively limited environmental conditions may help to dispel the notion—largely based on prejudice, not on acquaintance with actual measurements—that differential environment is the source of resemblance between siblings.

II. Variation and Distribution of Leaves in *Sassafras*.

By N. M. GRIER.

The following note is made on the basis of examination of ten *sassafras* trees and 102 seedlings near Pittsburg, Pa., and eight trees near St Louis, Mo. Only three kinds of leaves were met with, three-lobed, two-lobed and single-lobed, but it may be inferred that the same laws will govern the distribution of the four- five- and six-lobed forms described by Berry some years ago in the *Botanical Gazette*.

The single-lobed leaves are in great preponderance, constituting two-thirds of the foliage in Pittsburg specimens, while, near St Louis, three trees were observed in which other than single-lobed leaves were wanting. In these an extensive self-pruning had taken place. The terminal leaves of young branches are single-lobed, although there may be an occasional two-lobed leaf. Tops of trees are usually composed almost entirely of single-lobed leaves.

The dissected forms of leaves appear to be most plentifully developed under the influence of shade. In such cases they were most thickly distributed at the middle of the tree (as has been noted for three-lobed leaves in the Britton and Brown Flora), on young twigs whose terminal leaves were dissected, and toward the bottom on older twigs. There was a tendency for more three-lobed and less one-lobed leaves to be found on smaller twigs growing near the trunk, but occasionally on larger twigs, or smaller boughs growing among the larger boughs.

No transitional forms between the three-lobed and two-lobed leaves were noted on the same tree. The latter apparently increase in number as the three-lobed forms decrease, and are associated mostly with the single-lobed leaves, being about equally distributed between the younger and older twigs. They are rarely found at the top of the tree. Evidence that the available amount of light may play some part in the distribution of leaves is found in the fact that the great majority of observed seedlings growing in the shade develop the three-lobed or two-lobed leaves in combination. Contrast is offered by a statement made in a standard American textbook of botany—"In *Sassafras*, almost any leaf may be entire or variously lobed,

* Both series of observations also indicated how satisfactorily the normal law of distribution may be applied to material of this kind.

apparently without relation to transpiration, nutrition, etc." It will be observed that these findings substantiate in general those of Fry made in 1902*.

Bearing in mind the foregoing statement, an attempt was made to ascertain experimentally the relation of amount of light as to kind of leaves developed. This year's twigs bearing only one-lobed forms were tied back into shaded positions. Of ten such cases, three twigs produced isolated, three-lobed leaves. In another lot of the younger twigs bearing only one-lobed forms, the leaves were stripped from the twigs, and these too tied back in the shade. Only one twig of this lot responded, producing two two-lobed and one three-lobed leaf. A consistent explanation of this fragmentary evidence would be that the formative elements for three-lobed leaves in the twigs are stimulated to produce those forms. A more positive point brought out is the lack of proliferating power in the trees under the condition of the experiments—when compared with other forms possessing divided leaves as the mulberry—the majority of mutilated twigs at this season, early August, not renewing their leaves. The writer is indebted for use of material to Mrs W. G. Gibson of Avalon, Pa., and Prof. W. J. Stevens, Field School, St Louis, Mo.

III. Life-History Albums.

By ETHEL M. ELDERTON.

The Personal and Family History Register† compiled by Dr Taylor is extremely interesting, and if people could be persuaded to keep the records asked for and to forward the book when completed to some central agency such as is intended, the statistical data then available should be most useful. In this register under one cover all the children of one family have their life histories recorded, and if the individuals are to be studied only in their childhood this is an advantage, but if it is hoped by means of a register to provide the life history rather than the child history a separate volume for each child would be preferable; then as each child left the home the book could go with him to be continued and completed. Francis Galton in the *Life-History Album* issued years ago preferred this second plan and arranged that each child in the family should have its own album‡.

To the statistical worker in Eugenics so many problems in heredity are still unsolved, problems dealing with fertility, with inheritance of disease, with age at death, etc. that no record of personal history seems adequate which does not provide the data from which such problems can be attacked. In the Personal and Family History Register information as to date of birth and date of death is sought for parents, grandparents, great-grandparents, etc. up to the sixty-four ancestors in the seventh generation, and such a record is interesting, but one feels that cause of death and some information as to general health, if obtainable, would make the data more useful. Further there is no space assigned for collaterals. In the introduction the following occurs: "It is of interest to obtain data also on collaterals (uncles, aunts, cousins, etc.) and alliants (members by marriage). These extras can be inscribed on a page marked 'Special Happenings' or on separate sheets or cards, and placed in the pocket at the end." Our experience is that even when a special space is provided for an entry the information required is not always given, and I think that except in a very few cases extra data of this kind will not be given, and I am inclined to think that knowledge of the brothers and sisters of the parents is of more importance for determining the hereditary characteristics of an individual than knowledge of the great-grandparents. Cousins, we found, were as closely related to one another as grandparents to their grandchildren, and the data concerning them could be more

* *Biometrika*, I. 258, Jan.

† *Ourselves. A Personal and Family History Register*, by John Madison Taylor, A.B., M.D., published by F. A. Davis Company, Philadelphia, 1917.

‡ This Album is now re-issued by the Galton Laboratory through the Cambridge University Press. Price 9s. net.

easily obtained and would be more reliable than those concerning individuals who lived perhaps 100 years ago.

Personally I feel that careful details concerning the life history of a baby, interesting as they may be, are of little value to the student of Eugenics, unless the hereditary history is fully given. The old difficulty of deciding the relative importance of eugenics or euthenics (the word used by Dr Taylor to describe the science of right living) is impossible of solution if the facts concerning any individual are restricted entirely to one side or the other. In Dr Taylor's Register the family is crowded out by the personal element. Dr Taylor fully recognizes our ignorance of the laws of heredity and of the question of how far "pronouncedly unfavourable heredity" can be influenced by euthenics, but I think he assumes that the race can be improved through a better environment to an extent which I venture to think is unproven.

Both the *Personal and Family History Register* and the *Life-History Album* are rather large volumes, somewhat alarming to the busy parent from the very size of them. But those who are keenly interested in the well-being of the race will be induced to keep the record; they will be limited in number and may belong to a rather narrow circle, at least at the present time when the science of eugenics is still regarded as the fad of a few individuals.

I believe that *The Record of Family Faculties* issued by Francis Galton in 1884 would prove far more convenient both for the recorder and for the statistical worker than either of the two more bulky registers, the *Life-History Album* and the *Personal and Family History Register*. It is thirty-five years since this volume was first published and one marvels at the genius of the man who then saw what data would be needed to solve the problems of the present day. The introduction to this book supplies in a few words the justification for requiring the data and indicates the reason for the questions asked. Thus:

3. Age at marriage	Total {sons	No. of sons deceased	Ages
4. Age of husband	No. of {daughters	No. of daughters deceased	Ages

In the introduction Francis Galton writes "The ages at marriage of the two parents, the number and the duration of life of the children, would enable inquiry to be made into fertility as associated with different admixtures of race or of disease tendencies. We have yet to learn the conditions under which some families are prolific in their various branches, and others die out." Further Question 5 is, Mode of life so far as affecting growth or health, and the justification for asking this question is as follows: "The mode of life, so far as it affects growth or health, would, if known, throw light on the effect of nurture over nature. We require to select the families in each of which there had been a noticeable difference in the mode of life of two or more of its members, and to cross divide those members into two groups, in one of which the mode of life had been healthy, the other in which it had been the reverse. Then by contrasting these groups we should see the relative effects of good and bad nurture on the development of body and mind, and on the health, fertility, and duration of life."

According to the problems with which one has come in contact, each investigator would desire certain modifications in the questions asked, but on the whole, I believe that a collection of *Records of Family Faculties* would enable one to determine "many vital questions in domestic economics," and it is very desirable that this book of Galton's should be reissued.

IV. The Check to the Fall in the Phthisis Death-rate since the Discovery of the Tubercle Bacillus and the Adoption of Modern Treatment.

By KARL PEARSON, F.R.S.

In 1911* I pointed out that from '65 to about '95 there was a continuous and rapid fall in the *corrected* phthisis death-rate, and also in the percentage which the deaths from phthisis were of all deaths. I further indicated that from 1895 onwards there had been a check to this

* *The Fight against Tuberculosis and the Death-rate from Phthisis*, Cambridge University Press, 1911.

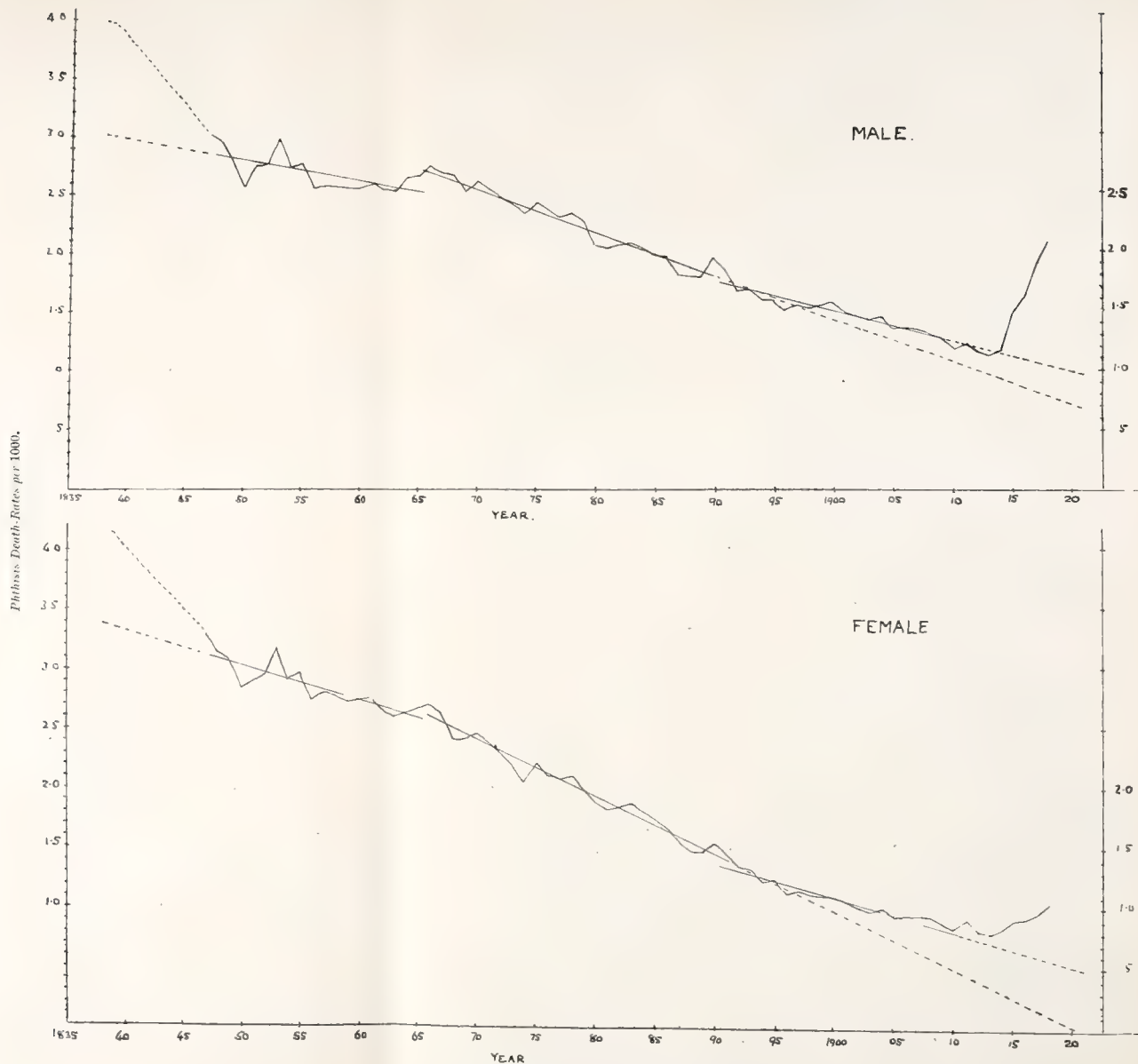


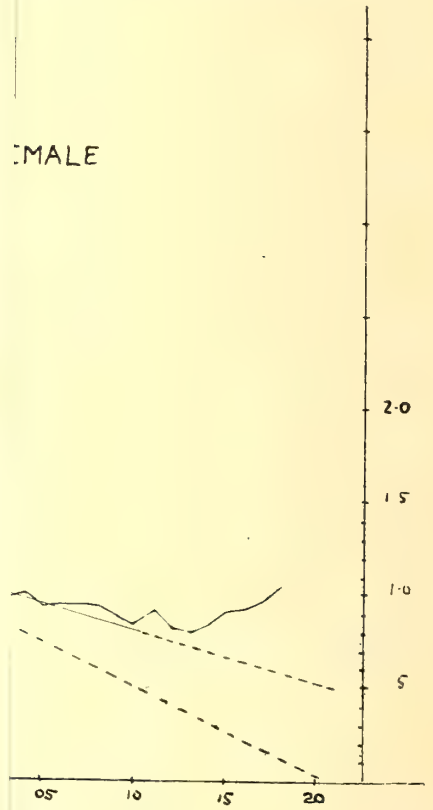
Diagram i. The Fall in the corrected Phthisis Death-Rate.

Plate VIII

MALE.



MALE



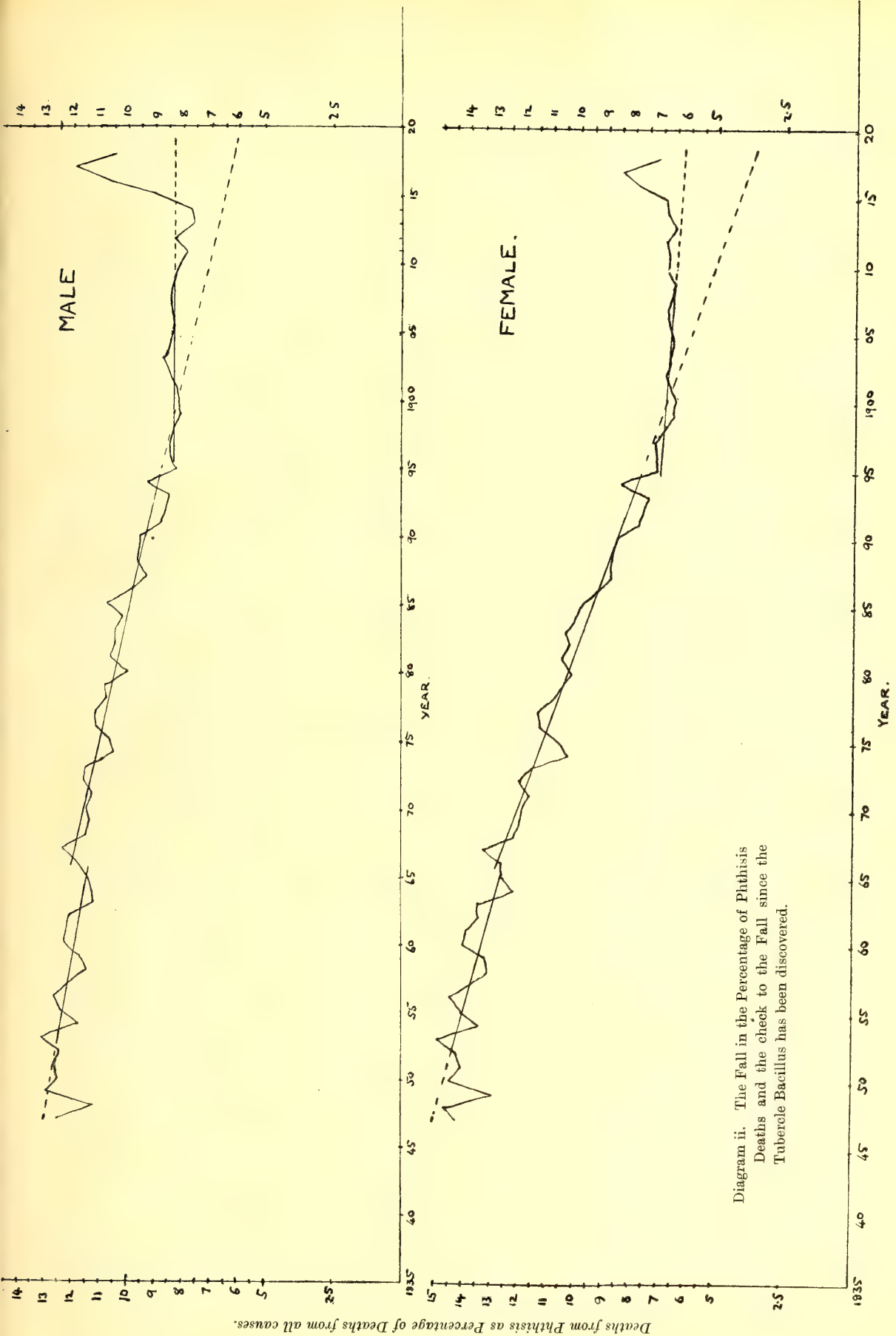


Diagram ii. The Fall in the Percentage of Phthisis Deaths and the check to the Fall since the Tubercle Bacillus has been discovered.

Deaths from Phthisis as Percentage of Deaths from all causes.

rapid fall and that the curves seemed to indicate that an actual rise in the phthisis death-rate might in the near future be reasonably anticipated. This view was rendered still more probable when I plotted the returns for 1910 to 1914. Since then the Great War has rendered it almost impossible for us so to feel our way in mortality statistics, that we can get returns comparable with the pre-war data. It seems to me, however, just worth while to see what our graphs will look like with the war years added to them. I must thank Dr Stevenson of the General Register Office for a renewal of his unfailing courtesy in providing me with the required data, and furthermore for several valuable suggestions as to the source of the remarkable results manifested.

If we could trust the accompanying diagrams the anticipated rise in the phthisis death-rate has already occurred. But complete trust would be very much misplaced. In the first place our phthisis death-rate is for civilians, and since able-bodied civilians have been largely drawn into the army, there has naturally been a heavier death-rate of all kinds, and therefore a heavier phthisis death-rate than in pre-war times. There might therefore be nothing really significant in the marked male death-rate rise. On the other hand this explanation hardly applies to the rise—it is true not so marked—in the female death-rate. At the same time the whole nation, male and female, has been more crowded together in factories and subject to far greater strain than in pre-war days. This would naturally tend to emphasise the death-rate of women as well as of men. If we turn, however, to our second diagram we see that not only has the phthisis death-rate increased like the general death-rate, but it has been increasing at a more rapid rate than the general death-rate. This can only be accounted for on the assumption that phthisis more than all other diseases will be emphasised by war-strain. It can hardly be said that we were relieved of war-strain during 1918, indeed some of the hardest months of work and some of the periods of heaviest depression occurred in that year; there was further a most severe epidemic of influenza, and many deaths, Dr Stevenson tells me, recorded as influenza and phthisis were tabulated under the latter. Yet notwithstanding strain and influenza the proportion of phthisis deaths to deaths in general *fell* (see Diagram ii).

A noteworthy feature is that the tuberculous mortality in lunatic asylums increased in an extraordinary manner from an average of 1800 deaths in 1912–14 to 5605 deaths in 1918. Dr Stevenson tells me that this will practically account for half the increase in tuberculous deaths for the total population in that time. Now this raises very important questions which ought to be answered. Were the lunatics who died of tuberculosis lunatics before the war, and again were they tuberculous before the war? Or did more lunatics become tuberculous owing to bad conditions—removal of much nursing and medical supervision—during the war? Or again did the tuberculous lunatics enter the asylum during the war? That is: Were the phthisical, simply because of their phthisis, less able to avoid mental breakdown under the severe war conditions? If so they would probably have died of phthisis outside the asylum in non-war conditions, and it would not be legitimate to cite the increased tuberculous deaths in asylums as something anomalous.

On the whole it is risky to form a very definite judgment, but having regard to the female phthisis death-rate and to the percentage of the phthisis death-rate on the general death-rate, war difficulties do not seem to me sufficient to obscure the general trend of our graphs (as indicated before the war), namely that somewhere about 1915 the fall in the phthisis rate which had been less rapid since 1895 would cease altogether and probably be followed by a *rise*. The next five years will show whether this be true or not. We should expect a fall in the phthisis death-rate immediately, but on the average the value will remain higher than that of 1915.

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